Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 710150, 9 pages
http://dx.doi.org/10.1155/2014/710150

*Research Article*

# Stability Analysis of Impulsive Stochastic Functional Differential Equations with Delayed Impulses via Comparison Principle and Impulsive Delay Differential Inequality

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Received 25 July 2013; Revised 26 November 2013; Accepted 10 December 2013; Published 16 February 2014

Academic Editor: Yuriy Rogovchenko



The problem of stability for nonlinear impulsive stochastic functional differential equations with delayed impulses is addressed in this paper. Based on the comparison principle and an impulsive delay differential inequality, some exponential stability and asymptotical stability criteria are derived, which show that the system will be stable if the impulses' frequency and amplitude are suitably related to the increase or decrease of the continuous stochastic flows. The obtained results complement ones from some recent works. Two examples are discussed to illustrate the effectiveness and advantages of our results.

## 1. Introduction

Impulsive dynamical equations have received considerable attention during the recent decades since they provide a natural framework for mathematical modeling of many real world evolutionary processes where the states undergo abrupt changes at certain instants (see [1–7]). In particular, more researchers have given special interests to the stability and stabilization analysis of impulsive functional differential equations (IFDEs) and there are extensive literatures in this field (see [8–14] and reference therein).

In the current literature concerning IFDEs, the impulses are assumed to take the form $\Delta x(t_k) = I_k(t_k, x(t_k^-))$, which indicates that the state "jump" at the impulse times $t_k$ is only related to the present state variables. But in most cases, it is more applicable that the state variables on the impulses that we add are also related to the past ones. For example, in the transmission of the impulse information, input delays are often encountered (see, e.g., [15, 16]). So, it is more meaningful if the above impulses are modified as $\Delta x(t_k) = x(t_k) - x(t_k^-) = I_k(t_k, x((t_k - d_k)^-))$. Recently, there have been several attempts in the literature to study the stability and control problems of IFDEs with delayed impulse (IFDEs-DI). For example, by using Lyapunov functions couples with Razumikhin techniques, some Razumikhin-type asymptotic stability and exponential stability criteria for IFDEs-DI were established in [17–19], and some Lyapunov-based sufficient conditions for the exponential stability of the equations were derived in [20].

On the other hand, stochastic perturbations are unavoidable in real equations (see [21, 22] and reference therein). In recent years, the stability analysis of impulsive stochastic functional equations which include delay equations is interesting to many investigators, and many results of stability criteria of these equations have been reported (see, e.g., [23–29]). Very recently, [30, 31] took environment noise into account and generalized delayed impulses to stochastic equations. In particular, applying the Lyapunov functions couples with Razumikhin techniques, [30] investigates both moment and almost sure exponential stability of impulsive stochastic functional differential equations with delayed impulses (ISFDEs-DI), and several Razumikhin-type criteria on the exponential stability and uniform stability in terms of two measures for the equations were established in [31].

But it is worth noting that the stability analysis in [30] and the effects of time delay on the impulses have been ignored. And in [30, 31], the authors only consider the case that the impulsive stabilization. Moreover, it is well known that the Razumikhin techniques are very effective in the study of stability problems for ordinary and functional differential equations. However, when we use the Razumikhin techniques, we need to choose an appropriate minimal class of functionals relative to which the derivative of the Lyapunov function or Lyapunov functional is estimated, which is not entirely convenient.

Motivated by the above discussion, in this paper, we will further investigate the stability of ISFDEs-DI. By using the comparison principle and an impulsive delay differential inequality, some exponential and asymptotical stability criteria are derived, which are more convenient to be applied than those Razumikhin-type conditions. Our results complement ones from some recent works and show that the ISFDE-ID will be stable if the impulses' frequency and amplitude are suitably related to the increase or decrease of the corresponding continuous stochastic flows. The rest of the paper is organized as follows. In Section 2, some relevant notations and definitions are presented. In Section 3, the comparison principle, an impulsive delay differential inequality, and several criteria on the exponential stability and asymptotical stability are established. Section 4 provides two illustrative examples to demonstrate the applications of the obtained results. Finally, conclusions are drawn in Section 5.

## 2. Preliminaries

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions; that is, it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t) = (w_1(t), \ldots, w_d(t))^T$ be a $d$-dimensional Brownian motion defined on the probability space. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, and $\mathbb{R}^{n\times d}$ the space of $n \times d$ real matrices. $I$ stands for the identity matrix of appropriate dimensions. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. For $A \in \mathbb{R}^{n\times d}$, $\|A\|$ denotes spectral norm of the matrix $A$. Denote by $\lambda_{\min}(\cdot)$ the minimum eigenvalue of a matrix. If $A$ is a vector or matrix, its transpose is denoted by $A^T$.

Let $\tau > 0$ and $\mathrm{PC}([-\tau, 0]; \mathbb{R}^n) = \{\varphi : [-\tau, 0] \rightarrow \mathbb{R}^n \mid \varphi(t^+) = \varphi(t)$ for all $t \in [-\tau, 0)$, $\varphi(t^-)$ exist and let $\varphi(t^-) = \varphi(t)$ for all but at most a finite number of points $t \in (-\tau, 0]\}$ be with the norm $\| \varphi \| = \sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$, where $\varphi(t^+)$ and $\varphi(t^-)$ denote the right-hand and left-hand limits of function $\varphi(t)$ at $t$, respectively. Denote $\mathrm{PC}([t_0 - \tau, \infty); \mathbb{R}) = \{\varphi \mid \varphi|_{[t_0-\tau,b]} \in \mathrm{PC}([t_0 - \tau, b]; \mathbb{R})$ for all $b > t_0 - \tau\}$.

For $p > 0$ and $t \geqslant 0$, let $\mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$-measurable $\mathrm{PC}([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\varphi$ such that $\sup_{-\tau\leqslant\theta\leqslant 0}\mathbb{E}|\varphi(\theta)|^p < \infty$, where $\mathbb{E}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathbb{P}$. And $L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$ measurable $\mathbb{R}^n$-valued random variables $X$, such that $\mathbb{E}|X|^p < \infty$. Let $\mathrm{PC}^b([-\tau, 0]; \mathbb{R}^n)$ be the family of all bounded $\mathrm{PC}([-\tau, 0]; \mathbb{R}^n)$-valued functions, and let $\mathrm{PC}^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_{t_0}$ measurable $\mathrm{PC}^b([-\tau, 0]; \mathbb{R}^n)$-valued functions.

Consider the following ISFDE-DI:

$$\begin{aligned} \mathrm{d}x(t) &= f(t, x_t)\,\mathrm{d}t + g(t, x_t)\,\mathrm{d}w(t), \quad t \neq t_k,\ t \geqslant t_0, \\ x(t_k) &= I_k\left(t_k, x(t_k^-), x\left((t_k - d_k)^-\right)\right), \quad k \in \mathbb{N}, \\ x_{t_0}(\theta) &= \xi(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{1}$$

where the initial value $\xi \in \mathrm{PC}^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$, $x(t) = ((x_1(t), \ldots, x_n(t))^T$, $x_t = x(t + \theta) \in \mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$. Both $f : \mathbb{R}_+ \times \mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathrm{PC}^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^{n\times d}$ are Borel measurable. $I_k : \mathbb{R}_+ \times L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \times L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ represents the impulsive perturbation of $x$ at time $t_k$. The fixed moments of impulse times $\{t_k, k \in \mathbb{N}\}$ satisfy $0 \leqslant t_0 < t_1 < \cdots < t_k < \cdots, t_k \rightarrow \infty$ (as $k \rightarrow \infty$). $\{d_k \geqslant 0, k \in \mathbb{N}\}$ are the impulse input delays satisfying $d = \sup_{k\in\mathbb{N}} d_k < \infty$.

As a standing hypothesis, we assume that for any $\xi \in \mathrm{PC}^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$ there exists a unique stochastic process satisfying (1) denoted by $x(t; t_0, \xi)$, which is continuous on the right-hand side and limitable on the left-hand side (see [32]). Moreover, we assume that $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, and $I_k(t, 0, 0) \equiv 0$ for all $t \geqslant t_0$, $k \in \mathbb{N}$; then (1) admits a trivial solution $x(t) \equiv 0$.

We introduce the following scalar IFDE-DI as the comparison system:

$$\begin{aligned} \dot{u}(t) &= h(t, u(t), u_t), \quad t \neq t_k,\ t \geqslant t_0, \\ u(t_k) &= \Psi_{1k}(u(t_k^-)) + \Psi_{2k}\left(u(t_k - d_k)^-\right), \quad k \in \mathbb{N}, \\ u_{t_0}(\theta) &= \zeta(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{2}$$

where the initial value $\zeta \in \mathrm{PC}([-\tau, 0]; \mathbb{R}_+)$; $u_t \in \mathrm{PC}([-\tau, 0]; \mathbb{R}_+)$ is defined as $u_t = u(t + \theta)$, $\theta \in [-\tau, 0]$. $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathrm{PC}([-\tau, 0]; \mathbb{R}_+) \rightarrow \mathbb{R}_+$ is continuous, Lebesgue measurable, and nondecreasing with respect to the last argument; $\Psi_{1k}$, $\Psi_{2k} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous and nondecreasing. Assume that $h(t, 0, 0) \equiv 0$, $\Psi_{1k}(0) \equiv 0$, and $\Psi_{2k}(0) \equiv 0$; then system (2) admits a trivial solution $u(t) \equiv 0$. We further assume that for any $\zeta \in \mathrm{PC}^b([-\tau, 0]; \mathbb{R}_+)$, there exists a unique solution to system (2) on $[t_0 - \tau, \infty)$ denoted by $u(t; t_0, \zeta)$ (see [5, 6]) which is continuous on the right-handside and limitable on the left-hand side.

For convenience, we introduce the following function classes:

$\mathcal{K} = \{\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous and strictly increasing, $\phi(0) = 0\}$.

$\mathcal{K}_\infty = \{\phi \in \mathcal{K}, \phi(s) \rightarrow \infty$ as $s \rightarrow \infty\}$.

$C\mathcal{K} = \{\phi \in \mathcal{K}, \phi$ is concave$\}$.

$V\mathcal{K}_\infty = \{\phi \in \mathcal{K}_\infty,\ \phi$ is convex$\}$.

At the end of this section, let us introduce the following definitions.

*Definition 1* (see [23, 26]). The trivial solution of (1) is said to be as follows.

(i) $p$th moment stable if, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\mathbb{E}\left|x(t; t_0, \xi)\right|^p \leqslant \varepsilon, \quad t \geqslant t_0, \tag{3}$$

whenever $\mathbb{E}\|\xi\|^p < \delta$.

(ii) $p$th moment asymptotically stable if it is $p$th moment stable and there exists $\delta_0 = \delta_0(t_0)$ such that

$$\lim_{t\to\infty} \mathbb{E}\left|x\left(t; t_0, \xi\right)\right|^p = 0, \quad t \geqslant t_0, \tag{4}$$

whenever $\mathbb{E}\|\xi\|^p < \delta_0$.

(iii) $p$th moment globally exponentially stable if there is a pair of positive constants $\lambda, C$ such that

$$\mathbb{E}\left|x\left(t; t_0, \xi\right)\right|^p \leqslant C\mathbb{E}\|\xi\|^p e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 \tag{5}$$

for all $\xi \in \mathrm{PC}^b_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^n)$. When $p = 2$, it is usually said to be globally exponentially stable in mean square.

*Definition 2* (see [26]). A function $V : [t_0-\tau, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to class $\nu_0$ if

(i) $V$ is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and for each $x, y \in \mathbb{R}^n$, $t \in [t_{k-1}, t_k)$, and $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists;

(ii) $V(t, x)$ is continuously once differentiable in $t$ and twice in $x$ in each of the sets $(t_{k-1}, t_k) \times \mathbb{R}^n$, $k \in \mathbb{N}$.

If $V \in \nu_0$, define an operator $\mathcal{L}V$ from $[t_0, \infty) \times \mathrm{PC}([-\tau, 0]; \mathbb{R}^n)$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(t, \varphi) &= V_t(t, \varphi(0)) + V_x(t, \varphi(0)) f(t, \varphi) \\ &\quad + \frac{1}{2}\operatorname{trace}\left[g^T(t, \varphi) V_{xx}(t, \varphi(0)) g(t, \varphi)\right],\end{aligned} \tag{6}$$

where

$$\begin{aligned} V_t(t, x) &= \frac{\partial V(t, x)}{\partial t}, \\ V_x(t, x) &= \left(\frac{\partial V(t, x)}{\partial x_1}, \ldots, \frac{\partial V(t, x)}{\partial x_n}\right), \\ V_{xx}(t, x) &= \left(\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j}\right)_{n\times n}. \end{aligned} \tag{7}$$

## 3. Main results

In this section, we will develop an impulsive delay differential inequality and comparison principles and establish some criteria on $p$th moment exponential stability and asymptotical stability for (1).

**Lemma 3** (impulsive delay differential inequality). *Assume that $c \in \mathbb{R}$, $\delta \in \mathbb{R}$, $q \in \mathbb{R}_+$, $a_k > 0$, $b_k \geqslant 0$, $k \in \mathbb{N}$, $\overline{u}(t) := \sup_{\theta\in[-\tau,0]} u(t + \theta)$, and*

(i) $\ln(a_k + b_k e^{cd_k}) \leqslant \delta(t_k - t_{k-1})$ *for each* $k \in \mathbb{N}$;

(ii) $\delta + c + q\gamma < 0$, *where* $\gamma = \sup_{k\in\mathbb{N}}\{e^{\delta(t_k-t_{k-1})}, 1/e^{\delta(t_k-t_{k-1})}\}$.

*Then any solution $u \in PC([t_0 - \tau, \infty); \mathbb{R}_+)$ of the scalar impulsive delay differential inequality problem*

$$\begin{aligned} D^+u(t) &\leqslant cu(t) + q\overline{u}(t), \quad t \neq t_k,\ t \geqslant t_0, \\ u(t_k) &\leqslant a_k u(t_k^-) + b_k u\left((t_k - d_k)^-\right), \quad k \in \mathbb{N} \end{aligned} \tag{8}$$

*satisfies*

$$u(t) \leqslant \gamma\overline{u}(t_0) e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 - \tau, \tag{9}$$

*where $\lambda$ is the unique positive solution of $\lambda + \delta + c + q\gamma e^{\lambda\tau} = 0$.*

*Proof*. Set $v(t) = e^{-c(t-t_0)}u(t)$, $t \in [t_0 - \tau, \infty)$. For each $k \in \mathbb{N}$, by the second inequality of (8), we have

$$\begin{aligned} v(t_k) &= e^{-c(t_k-t_0)} u(t_k) \\ &\leqslant e^{-c(t_k-t_0)}\left[a_k u(t_k^-) + b_k u\left((t_k - d_k)^-\right)\right] \\ &= a_k e^{-c(t_k-t_0)} u(t_k^-) + \beta_k b_k u\left((t_k - d_k)^-\right) e^{-c(t_k-d_k-t_0)} \\ &= a_k v(t_k^-) + \beta_k b_k v\left((t_k - d_k)^-\right), \end{aligned} \tag{10}$$

where $\beta_k = e^{cd_k}$.

On the other hand, for any $t \neq t_k$, $k \in \mathbb{N}$,

$$D^+v(t) = e^{-c(t-t_0)}\left[-cu(t) + D^+u(t)\right] \leqslant qe^{-c(t-t_0)}\overline{u}(t). \tag{11}$$

For $t \in [t_0, t_1)$, integrating inequality (11) from $t_0$ to $t$, we obtain

$$v(t) \leqslant v(t_0) + \int_{t_0}^{t} qe^{-c(s-t_0)}\overline{u}(s)\,\mathrm{d}s; \tag{12}$$

this implies that

$$v(t_1^-) \leqslant v(t_0) + \int_{t_0}^{t_1} qe^{-c(s-t_0)}\overline{u}(s)\,\mathrm{d}s. \tag{13}$$

For $t \in [t_1, t_2)$, by the same method, together with (10), (11), and (13), we have

$$\begin{aligned} v(t) &\leqslant v(t_1) + \int_{t_1}^{t} q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s \\ &\leqslant a_1 v(t_1^-) + \beta_1 b_1 v\left((t_1 - d_1)^-\right) + \int_{t_1}^{t} q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s \\ &\leqslant a_1 \left[ v(t_0) + \int_{t_0}^{t_1} q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s \right] \\ &\quad + \beta_1 b_1 \left[ v(t_0) + \int_{t_0}^{t_1 - d_1} q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s \right] \\ &\quad + \int_{t_1}^{t} q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s \\ &\leqslant (a_1 + \beta_1 b_1) v(t_0) + (a_1 + \beta_1 b_1) \\ &\quad \times \int_{t_0}^{t_1} q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s + \int_{t_1}^{t} q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s. \end{aligned} \tag{14}$$

By induction, we have, for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$,

$$\begin{aligned} v(t) &\leqslant v(t_0) \prod_{t_0 < t_j \leqslant t} \left(a_j + \beta_j b_j\right) \\ &\quad + \int_{t_0}^{t} \prod_{s < t_j \leqslant t} \left(a_j + \beta_j b_j\right) q e^{-c(s-t_0)} \overline{u}(s)\,\mathrm{d}s. \end{aligned} \tag{15}$$

Thus, for $t > t_0$, we get

$$\begin{aligned} u(t) &\leqslant u(t_0) e^{c(t-t_0)} \prod_{t_0 < t_j \leqslant t} \left(a_j + \beta_j b_j\right) \\ &\quad + \int_{t_0}^{t} \prod_{s < t_j \leqslant t} \left(a_j + \beta_j b_j\right) q e^{c(t-s)} \overline{u}(s)\,\mathrm{d}s. \end{aligned} \tag{16}$$

Let $t_{j_1}, t_{j_2}, \ldots, t_{j_m}$ be impulse points in $(s, t]$, $t > s$. In view of condition (i), we get

$$\begin{aligned} \prod_{s < t_j \leqslant t} \left(a_j + \beta_j b_j\right) &= \left(a_{j_1} + \beta_{j_1} b_{j_1}\right) \\ &\quad \times \left(a_{j_2} + \beta_{j_2} b_{j_2}\right) \cdots \left(a_{j_m} + \beta_{j_m} b_{j_m}\right) \\ &\leqslant e^{\delta(t_{j_1} - t_{j_1 - 1})} e^{\delta(t_{j_2} - t_{j_1})} \cdots e^{\delta(t_{j_m} - t_{j_{m-1}})} \\ &= e^{\delta(t_{j_m} - t_{j_1 - 1})} = e^{\delta(t-s)} e^{\delta(t_{j_m} - t)} e^{\delta(s - t_{j_1 - 1})} \\ &\leqslant \gamma e^{\delta(t-s)}, \end{aligned} \tag{17}$$

where $t_{j_1 - 1}$ is the first impulsive point before $t_{j_1}$ and satisfies $t_{j_1 - 1} < s$. Submitting this into inequality (16), then, for $t > t_0$,

$$u(t) \leqslant \gamma e^{(c+\delta)(t-t_0)} u(t_0) + \int_{t_0}^{t} \gamma q e^{(c+\delta)(t-s)} \overline{u}(s)\,\mathrm{d}s. \tag{18}$$

Let $\Phi(\lambda) = \lambda + c + \delta + \gamma q e^{\lambda\tau}$. Then condition (ii) implies $\Phi(0) < 0$. Moreover, $\Phi(+\infty) = +\infty$ and $\Phi'(\lambda) = 1 + \tau\gamma q e^{\lambda\tau} > 0$. Hence $\Phi(\lambda) = 0$ has a unique positive solution $\lambda$. Next, we claim that

$$u(t) \leqslant \gamma \overline{u}(t_0) e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 - \tau. \tag{19}$$

Since

$$u(t) \leqslant \overline{u}(t_0) \leqslant \gamma \overline{u}(t_0) e^{-\lambda(t-t_0)}, \quad t \in [t_0 - \tau, t_0]. \tag{20}$$

So we only need to prove (19) for $t > t_0$. Suppose not, then there exists a $t^* \in (t_0, +\infty)$ such that

$$u(t^*) > \gamma \overline{u}(t_0) e^{-\lambda(t^* - t_0)}, \tag{21}$$

$$u(t) \leqslant \gamma \overline{u}(t_0) e^{-\lambda(t-t_0)}, \quad t \in [t_0 - \tau, t^*). \tag{22}$$

Thus from (18), (22), and $\Phi(\lambda) = 0$, we see that

$$\begin{aligned} u(t^*) &\leqslant \gamma \overline{u}(t_0) e^{(c+\delta)(t^* - t_0)} + \gamma \int_{t_0}^{t^*} q e^{(c+\delta)(t^* - s)} \overline{u}(s)\,\mathrm{d}s \\ &\leqslant \gamma \overline{u}(t_0) e^{(c+\delta)(t^* - t_0)} \\ &\quad + \gamma \int_{t_0}^{t^*} \gamma q e^{\lambda\tau} e^{(c+\delta)(t^* - s)} e^{-\lambda(s-t_0)} \overline{u}(t_0)\,\mathrm{d}s \\ &= \gamma \overline{u}(t_0) e^{-\lambda(t^* - t_0)}, \end{aligned} \tag{23}$$

which is a contradiction. Therefore, (19) holds. This completes the proof. □

**Lemma 4** (comparison principle). *Assume that there exists a function $V \in v_0$ such that*

(i) $\mathbb{E}\mathscr{L}V(t, \varphi) \leqslant h(t, \mathbb{E}V(t, \varphi(0)), \mathbb{E}V(t + \theta, \varphi))$ *for any* $(t, \varphi) \in [t_{k-1}, t_k) \times PC^p_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$, $k \in \mathbb{N}$;

(ii) $\mathbb{E}V(t_k, I_k(t_k, X, Y)) \leqslant \Psi_{1k}(\mathbb{E}V(t_k^-, X)) + \Psi_{2k}(\mathbb{E}V((t_k - d_k)^-, Y))$ *for all* $X, Y \in L^p_{\mathscr{F}_t}(\Omega; \mathbb{R}^n)$, $k \in \mathbb{N}$.

*Then,*

$$\mathbb{E}V(t, x(t)) \leqslant u(t; t_0, \zeta), \quad t \geqslant t_0 \tag{24}$$

*provided* $\mathbb{E}V(t_0 + \theta, x(t_0 + \theta)) \leqslant \zeta(\theta)$, $\theta \in [-\tau, 0]$, *where* $x(t) = x(t; t_0, \xi)$ *is the solution process to* (1).

*Proof.* For any $t \in [t_{k-1}, t_k)$ and $\alpha > 0$ sufficiently small satisfying $t + \alpha < t_k$, by the Itô formula together with condition (i), we have

$$\begin{aligned} &\mathbb{E}V(t + \alpha, x(t + \alpha)) - \mathbb{E}V(t, x(t)) \\ &\quad = \int_{t}^{t+\alpha} \mathbb{E}\mathscr{L}V(s, x_s)\,\mathrm{d}s \\ &\quad \leqslant \int_{t}^{t+\alpha} h(s, \mathbb{E}V(s, x(s)), \mathbb{E}V(s + \theta, x_s))\,\mathrm{d}s; \end{aligned} \tag{25}$$

this implies that

$$\begin{aligned}
&D^{+}\mathbb{E}V(t,x(t))\\
&\quad := \limsup_{\alpha\to 0^{+}} \frac{\mathbb{E}V(t+\alpha,x(t+\alpha)) - \mathbb{E}V(t,x(t))}{\alpha}\\
&\quad \leqslant \limsup_{\alpha\to 0^{+}} \frac{1}{\alpha}\int_{t}^{t+\alpha} h\left(s,\mathbb{E}V(s,x(s)),\mathbb{E}V\left(s+\theta,x_{s}\right)\right)\mathrm{d}s\\
&\quad = h\left(t,\mathbb{E}V(t,x(t)),\mathbb{E}V\left(t+\theta,x_{t}\right)\right).
\end{aligned} \tag{26}$$

Write $u(t;t_0,\zeta) = u(t)$ simply. Now supposing that for each $\theta\in[-\tau,0]$, $\mathbb{E}V(t_0+\theta,x(t_0+\theta))\leqslant\zeta(\theta)$, we claim that

$$\mathbb{E}V(t,x(t)) \leqslant u(t), \quad t\in[t_0-\tau,t_1). \tag{27}$$

Consider the system

$$\begin{aligned}
&\dot{U}(t) = h\left(t,U(t),U_t\right)+\varepsilon, \quad t\in[t_0,t_1),\\
&U(\theta) = \zeta(\theta)+\varepsilon, \quad \theta\in[t_0-\tau,t_0],
\end{aligned} \tag{28}$$

where $\varepsilon > 0$ is a constant. We claim that $U(t)\geqslant\mathbb{E}V(t,x(t))$ for $t\in[t_0-\tau,t_1)$.

In fact, if this is not true, then from the continuity of $U(t)$ and $\mathbb{E}V(t,x(t))$ in $t\in[t_0,t_1)$, we know that there exist a $t^*\in(t_0,t_1)$ and a sufficiently small constant $\alpha>0$ such that $t^*+\alpha<t_1$ and

$$\begin{aligned}
&\mathbb{E}V(t,x(t)) \leqslant U(t), \quad t\in[t_0-\tau,t^*),\\
&\mathbb{E}V(t^*,x(t^*)) = U(t^*),\\
&\mathbb{E}V(t,x(t)) > U(t), \quad t\in(t^*,t^*+\alpha).
\end{aligned} \tag{29}$$

Thus $\dot{U}(t^*) = D^{+}U(t^*) \leqslant D^{+}\mathbb{E}V(t^*,x(t^*))$. On the other hand, by condition (i), we obtain that

$$\begin{aligned}
\dot{U}(t^*) &= h\left(t^*,U(t^*),U_{t^*}\right)+\varepsilon\\
&\geqslant h\left(t^*,V(t^*,x(t^*)),\mathbb{E}V\left(t^*+\theta,x_{t^*}\right)\right)+\varepsilon\\
&> h\left(t^*,V(t^*,x(t^*)),\mathbb{E}V\left(t^*+\theta,x_{t^*}\right)\right)\\
&\geqslant D^{+}\mathbb{E}V(t^*,x(t^*)).
\end{aligned} \tag{30}$$

This is a contradiction. So $U(t)\geqslant\mathbb{E}V(t,x(t))$ holds for all $t\in[t_0-\tau,t_1)$. Let $\varepsilon\to 0$; then $U(t)\to u(t)$, and hence inequality (27) holds.

Noting that $\Psi_{1k}(\cdot)$ and $\Psi_{2k}(\cdot)$ are nondecreasing, by (27) and condition (ii), we get

$$\begin{aligned}
&\mathbb{E}V(t_1,x(t_1))\\
&\quad = \mathbb{E}V\left(t_1,I_1\left(t_1,x(t_1^-),x(t_1-d_1)^-\right)\right)\\
&\quad \leqslant \Psi_{11}\left(\mathbb{E}V\left(t_1^-,x(t_1^-)\right)\right)\\
&\qquad + \Psi_{21}\left(\mathbb{E}V\left((t_1-d_1)^-,x(t_1-d_1)^-\right)\right)\\
&\quad \leqslant \Psi_{11}\left(u(t_1^-)\right)+\Psi_{21}\left(u(t_1-d_1)^-\right) = u(t_1).
\end{aligned} \tag{31}$$

Thus, it follows from (27) and (31) that

$$\mathbb{E}V\left(t_1+\theta,x(t_1+\theta)\right) \leqslant u(t_1+\theta), \quad \theta\in[-\tau,0]. \tag{32}$$

Similar to the previous process, we have $\mathbb{E}V(t,x(t))\leqslant u(t)$ when $t\in[t_0-\tau,t_2)$. By induction, it follows that $\mathbb{E}V(t,x(t))\leqslant u(t)$, $t\in[t_0-\tau,\infty)$. The proof is complete. □

**Theorem 5.** *Assume that there exist functions $V\in\nu_0$, $\phi_1\in V\mathcal{K}_\infty$, and $\phi_2\in C\mathcal{K}$ such that*

(i) $\phi_1(|x|^p)\leqslant V(t,x)\leqslant\phi_2(|x|^p)$ *for any* $(t,x)\in[t_0-\tau,\infty)\times\mathbb{R}^n$;

(ii) $\mathbb{E}\mathscr{L}V(t,\varphi)\leqslant h(t,\mathbb{E}V(t,\varphi(0)),\mathbb{E}V(t+\theta,\varphi))$ *for any* $(t,\varphi)\in[t_{k-1},t_k)\times PC^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$, $k\in\mathbb{N}$;

(iii) $\mathbb{E}V(t_k,I_k(t_k,X,Y))\leqslant\Psi_{1k}(\mathbb{E}V(t_k^-,X))+\Psi_{2k}(\mathbb{E}V((t_k-d_k)^-,Y))$ *for all* $X,Y\in L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, $k\in\mathbb{N}$.

*Then the stability properties of the trivial solution of IFDE-DI (2) imply the corresponding stability properties of the trivial solution of ISFDE-DI (1). Moreover, if condition (i) is replaced by*

($\mathrm{i}^*$) *there exist positive constants $p$, $c_1$, and $c_2$ such that for all $(t,x)\in[t_0-\tau,\infty)\times\mathbb{R}^n$*

$$c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p \tag{33}$$

*then the global exponential stability of the trivial solution of IFDE-DI (2) implies that pth moment global exponential stability of ISFDE-DI (1).*

*Proof.* Firstly, assume that the trivial solution of IFDE-DI (2) is stable. Let $\varepsilon>0$; then for given $\phi_1(\varepsilon)>0$, there exists $\delta_1=\delta_1(t_0,\varepsilon)>0$ such that $\delta_1<\phi_1(\varepsilon)$ and

$$\|\zeta\|^p<\delta_1 \text{ implies } u(t;t_0,\zeta)<\phi_1(\varepsilon), \quad t\geqslant t_0. \tag{34}$$

Let $\zeta(\theta)=\mathbb{E}V(t_0+\theta,x(t_0+\theta))$, $\theta\in[-\tau,0]$. From conditions (ii) and (iii) and Lemma 4, we get that

$$\mathbb{E}V(t,x(t)) \leqslant u(t;t_0,\zeta), \quad t\geqslant t_0. \tag{35}$$

Let $\delta\leqslant\phi_2^{-1}(\delta_1)$ and $\mathbb{E}\|\xi\|^p<\delta$; then by condition (i) and $\phi_2\in C\mathcal{K}$, we have $\|\zeta\|^p\leqslant\mathbb{E}\phi_2(\|\xi\|^p)\leqslant\phi_2(\mathbb{E}\|\xi\|^p)<\phi_2(\delta)\leqslant\delta_1$. Hence, by (34) and (35), we have

$$\mathbb{E}V(t,x(t)) < \phi_1(\varepsilon), \quad t\geqslant t_0. \tag{36}$$

If $\mathbb{E}\|\xi\|^p<\delta$, then by conditions (i) and (36), we have

$$\mathbb{E}|x(t)|^p \leqslant \phi_1^{-1}\left(\mathbb{E}V(t,x(t))\right) < \varepsilon, \quad t\geqslant t_0; \tag{37}$$

that is, the trivial solution of ISFDE-DI (1) is stable.

Next, let us suppose that the trivial solution of IFDE-DI (2) is asymptotically stable. This implies that the trivial solution of ISFDE-DI (1) is stable. Let $\zeta(\theta)=\mathbb{E}V(t_0+\theta,x(t_0+\theta))$, $\theta\in[-\tau,0]$. Since $u=0$ is attractive, for any $\varepsilon>0$, there exist $\delta_0=\delta_0(t_0)>0$ and $T=T(t_0,\delta_0)$ such that

$$\|\zeta\|^p<\delta_0, \text{ implies } u(t;t_0,\zeta)<\phi_1(\varepsilon), \quad t\geqslant t_0+T. \tag{38}$$

Choose $\mathbb{E}\|\xi\|^p < \delta_0$. Note the fact that $\phi \in V\mathscr{K}$ implies $\phi^{-1} \in C\mathscr{K}$. Then by (35) and (37), we get

$$\mathbb{E}|x(t)|^p \leqslant \phi_1^{-1}(\mathbb{E}V(t,x(t))) < \varepsilon, \quad t \geqslant t_0 + T, \tag{39}$$

which implies that the trivial solution of ISFDE-DI (1) is asymptotically stable.

Thirdly, let us suppose that the trivial solution of IFDE-DI (2) is globally exponentially stable and condition (i$^*$) holds. Then, there exists a couple of positive constants $\gamma$ and $K$ such that

$$u(t) \leqslant K\|\zeta\| e^{-\gamma(t-t_0)}, \quad t \geqslant t_0. \tag{40}$$

Let $\zeta(\theta) = V(t_0+\theta, x(t_0+\theta))$, $\theta \in [-\tau, 0]$. Then by (35) and (40), we get $\mathbb{E}V(t,x(t)) \leqslant u(t) \leqslant K\mathbb{E}\|\xi\|^p e^{-\gamma(t-t_0)}$ for all $t \geqslant t_0$. Thus, by condition (i$^*$), it yields that

$$\mathbb{E}|x(t)|^p \leqslant \frac{Kc_2}{c_1}\mathbb{E}\|\xi\|^p e^{-\gamma(t-t_0)}, \quad t \geqslant t_0. \tag{41}$$

Hence, the trivial solution of ISFDE-DI (1) is $p$th moment globally exponentially stable. The proof is complete. □

**Theorem 6.** *Assume that there exist a function $V \in v_0$, positive constants $c_1$, $c_2$, $q$, and $a_k$, constants $c$ and $\delta$, and $b_k \geqslant 0$ such that*

(i) $c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p$ *for any* $(t,x) \in [t_0-\tau,\infty) \times \mathbb{R}^n$;

(ii) $\mathbb{E}\mathscr{L}V(t,\varphi) \leqslant c\mathbb{E}V(t,\varphi(0)) + q\mathbb{E}V(t+\theta,\varphi)$ *for any* $(t,\varphi) \in [t_{k-1},t_k) \times PC^p_{\mathscr{F}_t}([-\tau,0];\mathbb{R}^n)$, $k \in \mathbb{N}$;

(iii) $\mathbb{E}V(t_k, I_k(t_k,X,Y)) \leqslant a_k\mathbb{E}V(t_k^-,X) + b_k\mathbb{E}V((t_k-d_k)^-,Y)$ *for all* $X, Y \in L^p_{\mathscr{F}_t}(\Omega;\mathbb{R}^n)$, $k \in \mathbb{N}$;

(iv) $\ln(a_k + b_k e^{cd_k}) \leqslant \delta(t_k - t_{k-1})$ *for each* $k \in \mathbb{N}$;

(v) $\delta + c + q\gamma < 0$ *where* $\gamma = \sup_{k\in\mathbb{N}}\{e^{\delta(t_k-t_{k-1})}, 1/e^{\delta(t_k-t_{k-1})}\}$.

*Then the trivial solution of ISFDE-DI (1) is pth moment globally exponentially stable.*

*Proof.* Let $u(t) = \mathbb{E}V(t,\varphi(0))$, $h(t,u(t),u_t) = cu(t) + qu_t$, $\Psi_{1k}(u(t_k^-)) = a_k u(t_k^-)$, and $\Psi_{2k}(u((t_k-d_k)^-)) = b_k u((t_k-d_k)^-)$. We obtain the comparison system (2). It is easy to verify that all conditions of Theorem 5 are satisfied and so the global exponential stability of the trivial solution of IFDE-DI (2) implies that $p$th moment global exponential stability of ISFDE-DI (1).

Furthermore, let $\lambda$ be the unique positive solution of $\lambda + \delta + p + q\gamma e^{\lambda\tau} = 0$. Using conditions (ii) and (iii), we find

$$\begin{aligned} &D^+u(t) \leqslant cu(t) + q\overline{u}(t), \quad t \neq t_k,\ t \geqslant t_0, \\ &u(t_k) \leqslant a_k u(t_k^-) + b_k u\left((t_k-d_k)^-\right), \quad k \in \mathbb{N}. \end{aligned} \tag{42}$$

Thus from conditions (iv) and (v) and Lemma 3, we obtain that

$$u(t) \leqslant \gamma\overline{u}(t_0) e^{-\lambda(t-t_0)}, \quad t \geqslant t_0 - \tau, \tag{43}$$

which implies that the trivial solution of IFDE-DI (2) is globally exponentially stable. The proof of Theorem 6 is complete. □

*Remark 7.* An impulsive stochastic dynamical system can be viewed as a hybrid one comprised of two components: a continuous stochastic dynamic and a discrete dynamic. Theorem 6 can be used to deal will all three cases: the system with stable continuous stochastic dynamic and unstable discrete dynamic, the system with unstable continuous stochastic dynamic and stable discrete dynamic, and the system with stable continuous stochastic dynamic and stable discrete dynamic. When $c < 0$, the continuous stochastic dynamic of (1) may be stable. In this case, in order to ensure the stability of the entire system, the delayed impulses' frequency $\{t_k - t_{k-1}, k \in \mathbb{N}\}$ and amplitude $a_k$, $b_k$ should be suitably related to the decrease of continuous flows; that is, conditions (iv) and (v) hold. In this sense, Theorem 6 can be used to deal with the robust stabling of continuous stochastic dynamic subject to delayed impulsive perturbations. When $c \geqslant 0$, the continuous stochastic dynamic of (1) may be unstable and the stability of the entire system is determined by the delayed impulse effects. In this case, we need to require that the delayed impulses' frequency and amplitude should be suitablly related to the decrease of of continuous flows.

*Remark 8.* It is noted that the exponential stability analysis in [30, 31] only considers the case of impulsive stabilization. In this sense, Theorem 6 has a wider adaptive range.

## 4. Examples

In this section, the effectiveness and advantages of the results derived in the preceding section will be illustrated by two examples.

*Example 1.* Consider the two-dimensional nonlinear impulsive stochastic delay equation in the form

$$\begin{aligned} \mathrm{d}x_1(t) &= \left[-2x_2(t)\sin\left(x_1(t-\tau)\right) - 5x_1(t) \right. \\ &\quad \left. + 0.5x_2(t-\tau)\right]\mathrm{d}t + 0.2x_1(t-\tau)\,\mathrm{d}w(t), \\ &\qquad t \neq t_k, \\ \mathrm{d}x_2(t) &= \left[x_1(t)\sin\left(x_1(t-\tau)\right) - 5x_2(t) \right. \\ &\quad \left. + 0.4x_2(t-\tau)\right]\mathrm{d}t \\ &\quad + 0.4x_2(t-\tau)\,\mathrm{d}w(t), \quad t \neq t_k, \\ x_1(t_k) &= x_1(t_k^-) + \alpha x_1\left((t_k-d_k)^-\right), \quad k \in \mathbb{N}, \\ x_2(t_k) &= x_2(t_k^-) + \alpha x_2\left((t_k-d_k)^-\right), \quad k \in \mathbb{N}, \end{aligned} \tag{44}$$

where $\tau > 0$, $d_k \in [0,d]$, $\alpha \geqslant 0$. If there exists a positive constant $\varepsilon > 0$ such that

$$\begin{aligned} &\alpha < \sqrt{\frac{9/0.445 - 1 - \varepsilon}{1 + 1/\varepsilon}}, \\ &\varrho = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} > \frac{\ln\left[1 + \varepsilon + (1+1/\varepsilon)\alpha^2\right]}{9 - 0.445\left[1 + \varepsilon + (1+1/\varepsilon)\alpha^2\right]}, \end{aligned} \tag{45}$$

then (44) is globally exponentially stable for any bounded impulsive input delays $\{d_k\}$.

Denote $I_k(t_k, X, Y) = X + \alpha Y$. Choose the Lyapunov function $V(t,x) = (1/4)x_1^2 + (1/2)x_2^2$; then for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{E}V(t_k, I_k(t_k, X, Y)) \\
&\quad = \frac{1}{4}\left|X_1 + \alpha Y_1\right|^2 + \frac{1}{2}\left|X_2 + \alpha Y_2\right|^2 \\
&\quad = \mathbb{E}V(t_k^-, X) + \alpha^2 \mathbb{E}V\left((t_k - d_k)^-, Y\right) \\
&\qquad + \frac{\alpha}{2}\mathbb{E}(X_1 Y_1) + \alpha\mathbb{E}(X_2 Y_2) \\
&\quad \leqslant (1+\varepsilon)\mathbb{E}V(t_k^-, X) + \left(1 + \frac{1}{\varepsilon}\right)\alpha^2 \mathbb{E}V\left((t_k - d_k)^-, Y\right), \\
&\mathbb{E}\mathscr{L}V(t,\varphi) \\
&\quad = -10\mathbb{E}V(t, \varphi(0)) \\
&\qquad + \mathbb{E}\Big[0.25\varphi_1(0)\varphi_2(-\tau) + 0.4\varphi_2(0)\varphi_2(-\tau) \\
&\qquad\quad + 0.01\varphi_1^2(-\tau) + 0.08\varphi_2^2(-\tau)\Big] \\
&\quad \leqslant -10\mathbb{E}V(t,\varphi(0)) + \mathbb{E}\left[0.01\varphi_1^2(-\tau) + 0.08\varphi_2^2(-\tau)\right] \\
&\qquad + \mathbb{E}\Big[0.25\varphi_1^2(0) + 0.0625\varphi_2^2(-\tau) \\
&\qquad\quad + 0.5\varphi_2^2(0) + 0.08\varphi_2^2(-\tau)\Big] \\
&\quad = -9\mathbb{E}V(t,\varphi(0)) + \mathbb{E}\left[0.01\varphi_1^2(-\tau) + 0.2225\varphi_2^2(-\tau)\right] \\
&\quad \leqslant -9\mathbb{E}V(t,\varphi(0)) + 0.445\mathbb{E}V(t-\tau, \varphi(-\tau)),
\end{aligned} \tag{46}$$

for $t \neq t_k$.

Take $c_1 = 1/4$, $c_2 = 1/2$, $c = -9$, $q = 0.445$, $a_k \equiv 1 + \varepsilon$, $b_k \equiv (1 + 1/\varepsilon)\alpha^2$, $\delta = \ln[1 + \varepsilon + (1 + 1/\varepsilon)\alpha^2]/\varrho$, $\gamma = 1 + \varepsilon + (1 + 1/\varepsilon)\alpha^2$. It is easy to check that all conditions of Theorem 6 are satisfied under conditions (45), which means that (44) is globally mean square exponentially stable for any bounded impulsive input delays $\{d_k\}$.

*Remark.* It is noted that (44) without impulses is globally mean square exponentially stable and the impulses are destabilizing since $\alpha \geqslant 0$. Hence, the existing stability theorems in [30, 31] fail to work. This shows that our results have a wider adaptive range.

*Example 2.* Consider the following impulsive stochastic delayed neural network:

$$\begin{aligned}
\mathrm{d}x(t) &= \left[-x(t) + Af(x(t - \tau(t)))\right]\mathrm{d}t \\
&\qquad + Bx(t - \tau(t))\,\mathrm{d}w(t), \quad t \neq t_k, \\
x(t_k) &= 0.3x(t_k^-) \\
&\qquad + 0.2x\left((t_k - d_k)^-\right), \quad k \in \mathbb{N},
\end{aligned} \tag{47}$$

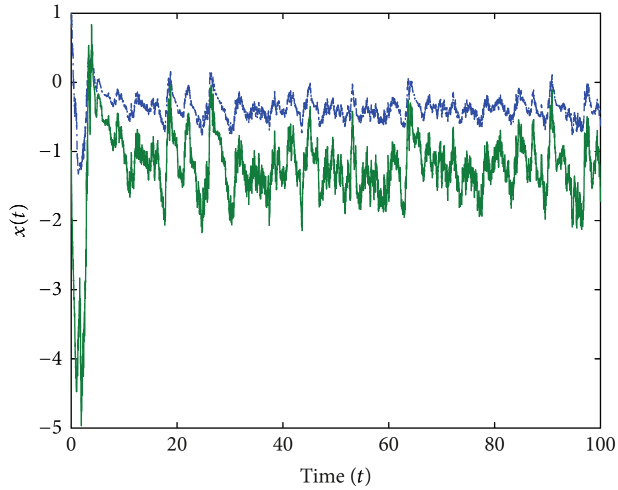


FIGURE 1: The solution of system (47) without impulses (single sample).

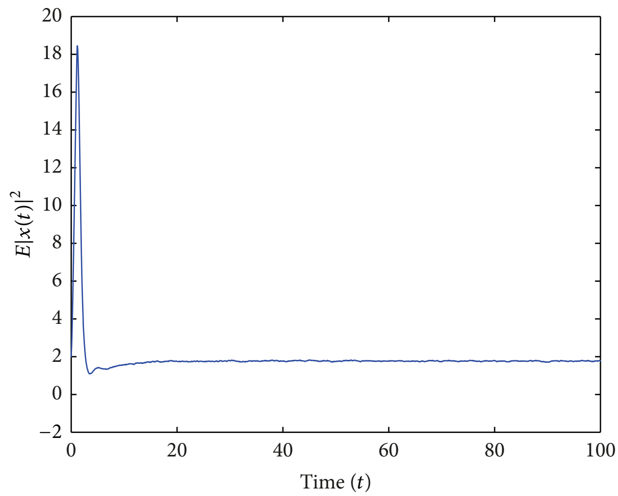


FIGURE 2: The mean square of the solution of system (47) without impulses (2000 samples).

where

$$A = \begin{bmatrix} -1.5 & 1 \\ -3 & 2.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \tag{48}$$

$f(x) = (f_1(x_1), f_2(x_2))^T$ with $f_1(s) = f_2(s) = (1/2)(|s+1| - |s-1|)$.

It is noted that (47) without impulse is not stable, and its simulation with delay $\tau(t) = 1$ and initial data $\xi(s) = [1, -1]^T$and $s \in [-1, 0]$ are shown in Figures 1 and 2.

In the following, applying Theorem 5, we will show that under impulsive control law, (47) is mean square exponentially stable if $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} \leqslant 0.0681$.

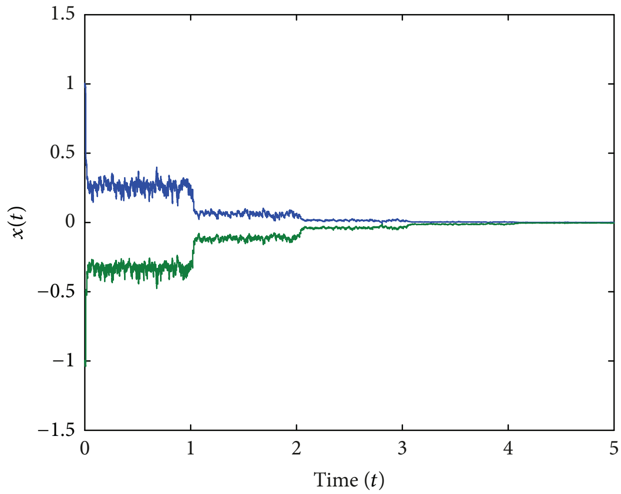


FIGURE 3: The solution of system (47) with impulses (single sample).

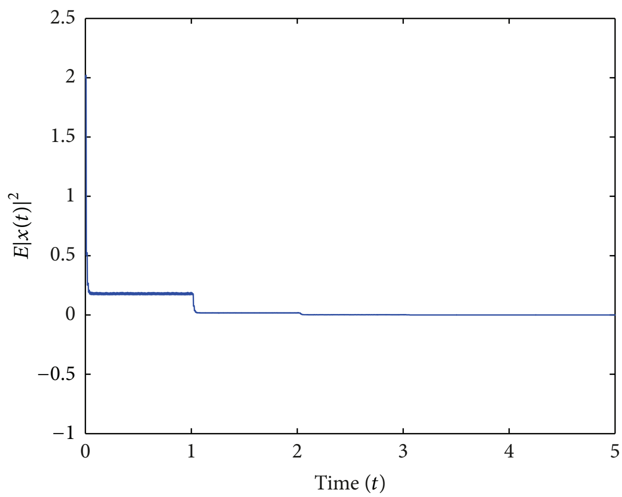


FIGURE 4: The mean square of the solution of system (47) with impulses (2000 samples).

Denote $I_k(t_k, X, Y) = 0.3X + 0.2Y$. Choose $V(t,x) = |x|^2$. Then condition (i) of Theorem 5 holds with $c_1 = c_2 = 1$,

$$\begin{aligned}
&\mathbb{E}V\left(t_k, I_k\left(t_k, X, Y\right)\right) \\
&\quad = \left[0.3X + 0.2Y\right]^T\left[0.3X + 0.2Y\right] \\
&\quad \leqslant 0.18\mathbb{E}|X|^2 + 0.08\mathbb{E}|Y|^2 \\
&\quad = 0.18\mathbb{E}V\left(t_k^-, X\right) + 0.08\mathbb{E}V\left(\left(t_k - d_k\right)^-, Y\right), \\
&\mathbb{E}\mathscr{L}V\left(t, \varphi\right) \\
&\quad = \mathbb{E}\left[2\varphi^T(0)\left(-\varphi(0) + Af\left(\varphi\left(-\tau(t)\right)\right)\right)\right] \\
&\qquad + \mathbb{E}\left[\varphi^T\left(-\tau(t)\right)B^T B\varphi\left(-\tau(t)\right)\right] \\
&\quad \leqslant \mathbb{E}\left[\left(-2 + \|A\|\right)\left|\varphi(0)\right|^2\right. \\
&\qquad \left. + \left(\|A\| + \|B\|^2\right)\left|\varphi\left(-\tau(t)\right)\right|^2\right] \\
&\quad \leqslant 2.2976\mathbb{E}\left|\varphi(0)\right|^2 + 4.5476\mathbb{E}\left|\varphi\left(-\tau(t)\right)\right|^2 \\
&\quad = 2.2976\mathbb{E}V\left(t, \varphi(0)\right) \\
&\qquad + 4.5476\mathbb{E}V\left(t - \tau(t), \varphi\left(-\tau(t)\right)\right),
\end{aligned} \tag{49}$$

for $t \neq t_k$.

Thus, the comparison system is

$$\begin{aligned}
\dot{u}(t) &= 2.2976u(t) + 4.5476u\left(t - \tau(t)\right), \quad t \neq t_k, \\
&\qquad\qquad t \geqslant t_0, \\
u\left(t_k\right) &= 0.18u\left(t_k^-\right) + 0.08u\left(\left(t_k - d_k\right)^-\right), \quad k \in \mathbb{N},
\end{aligned} \tag{50}$$

which according to case (iii) of Corollary 1 in [19] is globally exponentially stable for any bounded impulsive input delays $\{d_k\}$ if $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \ln(1/0.26)/(2.2976 + 4.5476/0.26) = 0.0681$. Hence, we conclude by Theorem 6 that system (47) is mean square exponentially stable if $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} \leqslant 0.0681$. With the same initial value, the simulations of the impulsive stochastic delay neural network (47) under the delayed impulsive control law $x(t_k) = 0.3x(t_k^-) + 0.2x((t_k - d_k)^-)$, $t_k - t_{k-1} = 0.06$, $d_k = 0.4$ are shown in Figures 3 and 4.

## 5. Conclusions

This paper has investigated the exponential stability of ISFDEs-DI based on the comparison approach and an impulsive delay differential inequality. Some criteria on the $p$th moment global exponential stability are established. The obtained results complement some recent works. Two examples have been given to illustrate the effectiveness and the advantages of the results obtained. One of the drawbacks of the proposed method is perhaps that our results require the condition $\delta + c + q\gamma < 0$ and thus cannot deal with the time delay system with $\Delta x(t_k) = B_k x((t_k - d_k)^-)$. There will be future work to establish a criterion for the above system.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (11226247, 11301004), the 211 Project of Anhui University (32030018/33010205), the Anhui Provincial Nature Science Foundation (1308085QA15/1308085MA01), the Natural Science Foundation of Jiangsu Province (BK20130239), the Research Fund for the Doctoral Program of Higher Education of China (20130094120015), the Key Foundation of Anhui Education Bureau (KJ2012A019), and the Research Fund for the Doctoral Program of Higher Education (20103401120002).

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