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*Research Article*

# A Few Conditions for a $C^*$-Algebra to Be Commutative

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We present a few characterizations of the commutativity of $C^*$-algebras in terms of particular algebraic properties of power functions, the logarithmic and exponential functions, and the sine and cosine functions.

Let $G, H$ be subsets of groups which are closed under forming the Jordan triple product $aba$ of its elements. That means that for any $a, b$ from $G$ (resp., from $H$) we have $aba \in G$ (resp., $aba \in H$). A map $\phi : G \to H$ is called a Jordan triple map if it is a homomorphism relative to that operation, that is, if $\phi$ satisfies $\phi(aba) = \phi(a)\phi(b)\phi(a)$ for all $a, b \in G$. Such maps appear in many areas, in particular in ring theory and, as recent investigations have shown, also in correspondence with surjective isometries of noncommutative metric groups and related structures. Along this latter line of research, we have recently published several results concerning the descriptions of different sorts of surjective isometries of the sets of positive definite matrices and unitary matrices [1, 2]. In the paper [3] we have considered similar but much more general transformations that can be termed as maps preserving generalized distance measures in the context of operator algebras, namely, in that of factor von Neumann algebras. On the way to obtain structural results for those preserver transformations we have faced the following problem. In the case of the positive definite cone of a full matrix algebra or that of a nontrivial von Neumann factor we have shown that if for some nonzero exponent $c$ the power function $A \mapsto A^c$ is essentially a Jordan triple map, meaning that if $(ABA)^c$ and $A^c B^c A^c$ differ only by a scalar multiplier for any pair $A, B$ of positive definite elements, then $c$ must be trivial; that is, $c \in \{-1, 0, 1\}$. For details see the proofs of Theorem 1 in [2] and Theorem 5 in [3]. We have used a similar observation concerning power functions on unitary groups (the exponent being an integer there); see the proofs of Theorem 1 in [1] and Theorem 8 in [3].

In this short note we are going to prove a stronger result for general unital $C^*$-algebras. Namely, we show that if the power function corresponding to a nontrivial exponent is a Jordan triple map up to multiplication by central elements, then the underlying $C^*$-algebra is necessarily commutative. We also present several results of similar spirits concerning algebraic properties of the logarithmic and exponential functions as well as the sine and cosine functions. For example, we will show that it happens only in commutative algebras that the logarithm of the geometric mean of any two positive definite elements differs from the arithmetic means of their logarithms only by central elements.

We recall that in the literature one can find several conditions characterizing commutativity of $C^*$-algebras most of which are related to the order structure; see, for example, [4–8]. A characterization of algebraic character can be found in [9]. In this line we also mention that in the recent paper [10] we have investigated the standard $K$-loop structure of the positive definite cone of a $C^*$-algebra and obtained some results showing that the commutativity, associativity (even in certain weak senses), or distributivity of that structure are each equivalent to the commutativity of the underlying $C^*$-algebra.

We now turn to the results of the paper and first fix the notation. Let $\mathscr{A}$ be a unital $C^*$-algebra and $\mathscr{A}_s$ stand for the space of its self-adjoint elements. The cone of positive elements of $\mathscr{A}$, that is, those which are self-adjoint and have nonnegative spectrum, is denoted by $\mathscr{A}_+$, and $\mathscr{A}_+^{-1}$ stands for the set of all positive definite elements (in other words, positive invertibles), what we call the positive definite cone of $\mathscr{A}$. The geometric mean $A\#B$ (in Kubo-Ando sense) of the elements $A, B \in \mathscr{A}_+^{-1}$ is defined by

$$A\#B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}. \tag{1}$$

The unitary group in $\mathscr{A}$ is denoted by $\mathscr{A}_u$, and, finally, $\mathscr{Z}_{\mathscr{A}}$ stands for the center of $\mathscr{A}$.

Our main result reads as follows.

**Theorem 1.** *Let $\mathscr{A}$ be a unital $C^*$-algebra. The following assertions are equivalent.*

(i) *The algebra $\mathscr{A}$ is commutative.*

(ii) *There is a real number $c \notin \{-1, 0, 1\}$ with the property that for any pair $A, B \in \mathscr{A}_+^{-1}$ of positive invertibles there exists a central element $\lambda \in \mathscr{Z}_{\mathscr{A}}$ such that*

$$(ABA)^c = \lambda A^c B^c A^c. \tag{2}$$

(iii) *One has*

$$\log(ABA) - (2\log A + \log B) \in \mathscr{Z}_{\mathscr{A}}, \quad A, B \in \mathscr{A}_+^{-1}. \tag{3}$$

(iv) *One has*

$$\log(A\#B) - \frac{(\log A + \log B)}{2} \in \mathscr{Z}_{\mathscr{A}}, \quad A, B \in \mathscr{A}_+^{-1}. \tag{4}$$

(v) *For any pair $H, K \in \mathscr{A}_s$ of self-adjoint elements there exists a central element $\lambda \in \mathscr{Z}_{\mathscr{A}}$ such that*

$$\exp(2H + K) = \lambda \exp(H)\exp(K)\exp(H). \tag{5}$$

(vi) *For any pair $H, K \in \mathscr{A}_s$ of self-adjoint elements there exists a central element $\lambda \in \mathscr{Z}_{\mathscr{A}}$ such that*

$$\exp\left(\frac{H+K}{2}\right) = \lambda \exp(H)\#\exp(K). \tag{6}$$

*Proof.* The implication (i) ⇒ (ii) is trivial.

Suppose (ii) holds. We first show that the number $c$ can be assumed to be positive and less than 1. In fact, if $c$ is negative, then taking inverse in (2) we obtain a related property for the positive number $-c$. So, we can assume that $c$ is positive. If $c > 1$, then plug $A^{1/c}, B^{1/c}$ in the place of $A, B$ in (2) and then take $c$th root. Hence, we deduce an equality similar to (2) which holds for the number $1/c$. Therefore, we may and do assume that the constant $c$ is positive and less than 1.

Plugging $A^c, B^c$ in the place of $A, B$ in (2) again we obtain a similar equality with $c^2$ in the place of $c$. Repeating the argument we next deduce a similar equality for $c^3$ and so forth.

Now fix two elements $A, B \in \mathscr{A}_+^{-1}$. For the complex variable $z$ define $f(z) = (ABA)^z A^{-z}B^{-z}A^{-z}$ (here we mean $A^z = \exp(z\log A)$). Plainly, $f$ is a holomorphic (entire) function with values in $\mathscr{A}$ such that its values at $c, c^2, c^3, \ldots$ belong to the center $\mathscr{Z}_{\mathscr{A}}$ of $\mathscr{A}$. Since the sequence $(c^n)$ has a limit point in the domain of $f$, by the uniqueness theorem of holomorphic functions, it is easy to deduce that the whole range of $f$ lies in $\mathscr{Z}_{\mathscr{A}}$. Indeed, for any $T \in \mathscr{A}$ we have that

$$(ABA)^z A^{-z}B^{-z}A^{-z}T = T(ABA)^z A^{-z}B^{-z}A^{-z} \tag{7}$$

holds for all $z = c^n$, $n \in \mathbb{N}$. On both sides of this equality we have holomorphic (entire) functions which coincide on a set that has a limit point in $\mathbb{C}$. It follows that they necessarily coincide everywhere which means that the range of $f$ is indeed in $\mathscr{Z}_{\mathscr{A}}$. We have that

$$(ABA)^z = f(z)A^zB^zA^z, \quad z \in \mathbb{C}. \tag{8}$$

Clearly, $f(0) = I$. Differentiating both sides of the above equality at $z = 0$ we obtain

$$\log(ABA) = f'(0) + (2\log A + \log B). \tag{9}$$

This means that for every $A, B \in \mathscr{A}_+^{-1}$ we have

$$\log(ABA) - (2\log A + \log B) \in \mathscr{Z}_{\mathscr{A}}, \tag{10}$$

which proves (iii).

Suppose that (iii) holds. Clearly, by Gelfand-Naimark theorem we may assume that $\mathscr{A}$ is a $C^*$-subalgebra of the algebra $B(H)$ of all bounded linear operators acting on a Hilbert space $H$. Denote by $\mathscr{B}$ the strong closure of $\mathscr{A}$ in $B(H)$ which is a von Neumann algebra. Using Kaplansky density theorem it follows that every element of $\mathscr{B}_+^{-1}$ is the strong limit of a bounded net in $\mathscr{A}_+^{-1}$ which is bounded away from zero. Since the multiplication is strongly continuous on bounded sets of operators, bounded continuous complex valued functions of a real variable are strongly continuous, and the strong limit of a net of central elements is again central, we obtain that

$$\log(ABA) - (2\log A + \log B) \in \mathscr{Z}_{\mathscr{B}} \tag{11}$$

holds for all $A, B \in \mathscr{B}_+^{-1}$.

Now select projections $P, Q$ in $\mathscr{B}$. Let

$$A = I + tP, \qquad B = I + tQ, \tag{12}$$

where $t$ is a real number with $|t| < 1$. Easy computation shows that

$$\begin{aligned} ABA &= (I + tP)(I + tQ)(I + tP) \\ &= I + t(2P + Q) \\ &\quad + t^2(P + PQ + QP) + t^3(PQP). \end{aligned} \tag{13}$$

Recall that in an arbitrary unital Banach algebra, for any element $a$ with $\|a\| < 1$ we have

$$\log(1 + a) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}a^n}{n}. \tag{14}$$

This shows that for a suitable positive number $\epsilon(< 1)$, the elements $\log(ABA)$, $\log A$, and $\log B$ of $\mathscr{B}$ can be expressed by power series of $t$ ($|t| < \epsilon$) with coefficients in $\mathscr{B}$. Simple calculation shows that the coefficient of $t^3$ in the power series of the center valued function

$$g : t \longmapsto \log((I+tP)(I+tQ)(I+tP)) - (2\log(I+tP) + \log(I+tQ)) \tag{15}$$

is equal to

$$\begin{aligned}&\left(PQP - \frac{1}{2}((2P+Q)(P+PQ+QP) + (P+PQ+QP)(2P+Q)) + \frac{1}{3}(2P+Q)^3\right) - \left(\frac{1}{3}(2P+Q)\right)\\ &= \frac{1}{3}(PQP) - \frac{1}{3}(QPQ).\end{aligned} \tag{16}$$

This coefficient could also be obtained by calculating the third derivative of $g$ at $t = 0$. Since $g$ is center valued, so are its derivatives. Therefore, we obtain that the element $PQP - QPQ$ is central, which, in particular, means that it commutes with $P$. Clearly, this implies that $QPQ$ also commutes with $P$; that is, we have $PQPQ = QPQP$. We compute

$$\begin{aligned}(PQP)^3 &= PQPQPQP = PQ(PQPQ)P\\ &= PQ(QPQP)P = PQPQP = (PQP)^2.\end{aligned} \tag{17}$$

It follows from the equality $(PQP)^3 = (PQP)^2$ that the spectrum of the positive element $PQP$ is a subset of $\{0,1\}$ which implies that $PQP$ is a projection. But, by Lemma 2 in [10], this happens exactly when $P$ and $Q$ commute (referring to a comment by a referee, the handling editor suggested to include a proof of that observation and proposed the following nice argument for which the author is grateful; in paper [10] a different but also simple proof is presented. Consider the self-adjoint element $X = i(PQ - QP)$. By direct computation we have $X^3 = 0$, and thus $X = 0$ which gives $PQ = QP$). Consequently, we obtain that the projections in $\mathscr{B}$ all commute which immediately implies that the von Neumann algebra $\mathscr{B}$ is commutative. This gives us the commutativity of the smaller algebra $\mathscr{A}$, and hence we obtain (i).

We prove that (iv) implies (iii). We recall the well-known fact that the geometric mean of elements $A, B \in \mathscr{A}_+^{-1}$ can be characterized as the unique solution $X \in \mathscr{A}_+^{-1}$ of the Riccati equation $XA^{-1}X = B$. For any given $A, B \in \mathscr{A}_+^{-1}$ let $C = ABA$. Then we have $A = B^{-1}\#C$ and we compute by (iv)

$$\log A - \frac{(\log B^{-1} + \log(ABA))}{2} \in \mathscr{Z}_{\mathscr{A}}. \tag{18}$$

It is apparent that this implies

$$\log(ABA) - (2\log A + \log B) \in \mathscr{Z}_{\mathscr{A}}, \tag{19}$$

which means that (iii) holds true. Taking into account that (iii) is equivalent to the commutativity of $\mathscr{A}$, the implication (iii) $\Rightarrow$ (iv) is clear.

Finally, the assertions (iii) and (v) as well as (iv) and (vi) are obviously equivalent; thus the proof of the theorem is complete. □

*Remark 2.* In relation with the equivalence of (iii) and (iv) to (i) we make the following simple observation. Let $\mathscr{A}$ be a unital $C^*$-algebra and $(G,+)$ a commutative group in which every element is uniquely 2-divisible. We assert that if there is an injective transformation $\phi : \mathscr{A}_+^{-1} \to G$ such that either

$$\phi\left(A^{1/2}BA^{1/2}\right) = \phi(A) + \phi(B), \quad A, B \in \mathscr{A}_+^{-1} \tag{20}$$

or

$$\phi(A\#B) = \frac{(\phi(A) + \phi(B))}{2}, \quad A, B \in \mathscr{A}_+^{-1}, \tag{21}$$

then $\mathscr{A}$ is necessarily commutative. In fact, by the commutativity of $G$, in the first case we immediately have $A^{1/2}BA^{1/2} = B^{1/2}AB^{1/2}$ which is equivalent to $AB = BA$ as it has been shown, for example, in Proposition 1 in [10]. Concerning the second case, considering the map $\phi(\cdot) - \phi(I)$ we can clearly assume that $\phi(I) = 0$. Then applying the argument we have followed in the proof of the implication (iv) $\Rightarrow$ (iii) above, we obtain

$$\phi(A) = \frac{\left(\phi\left(B^{-1}\right) + \phi(ABA)\right)}{2}. \tag{22}$$

For any $D \in \mathscr{A}_+^{-1}$ we have

$$0 = \phi(I) = \phi\left(D\#D^{-1}\right) = \frac{\left(\phi(D) + \phi\left(D^{-1}\right)\right)}{2} \tag{23}$$

yielding $\phi(D^{-1}) = -\phi(D)$. Hence from (22) we deduce

$$2\phi(A) + \phi(B) = \phi(ABA). \tag{24}$$

Moreover, for any $D \in \mathscr{A}_+^{-1}$ we have

$$\phi\left(D^{1/2}\right) = \phi(D\#I) = \frac{(\phi(D) + 0)}{2}. \tag{25}$$

Hence, from (24) we infer

$$\phi(A) + \phi(B) = \phi\left(A^{1/2}BA^{1/2}\right) \tag{26}$$

and this, as we have seen above, already implies $AB = BA$, $A, B \in \mathscr{A}_+^{-1}$. Consequently, in both cases we deduce that $\mathscr{A}$ is necessarily commutative.

In the next corollary we present a characterization of the commutativity of $C^*$-algebras concerning the unitary group which is similar to (ii) in Theorem 1.

**Corollary 3.** *Let $\mathscr{A}$ be a unital $C^*$-algebra and assume that $m$ is an integer, $m \notin \{-1, 0, 1\}$, with the property that for any pair $U, V \in \mathscr{A}_u$ of unitaries there is a central element $\lambda \in \mathscr{Z}_{\mathscr{A}}$ such that*

$$(VWV)^m = \lambda V^m W^m V^m. \tag{27}$$

*Then $\mathscr{A}$ is commutative.*

*Proof.* Taking inverses in equality (27) if necessary, we can obviously assume that the integer $m$ is positive. For any $V, W \in \mathscr{A}_u$ we know that $(VWV)^m V^{-m} W^{-m} V^{-m}$ is central. Inserting one-parameter groups of unitaries in the places of $V$ and $W$ we deduce that for any given pair $H, J \in \mathscr{A}_s$ the element

$$\left(e^{itH} e^{isJ} e^{itH}\right)^m e^{-imtH} e^{-imsJ} e^{-imtH} \tag{28}$$

is central for all $t, s \in \mathbb{R}$. Fixing the real variable $s$ and putting the complex variable $z$ in the place of $it$ we obtain that the holomorphic (entire) function

$$z \longmapsto \left(e^{zH} e^{isJ} e^{zH}\right)^m e^{-mzH} e^{-imsJ} e^{-mzH} \tag{29}$$

has central values along the $y$-axis. Just as in the proof of Theorem 1 we deduce that the same holds for every complex number $z$ too. Now fixing $z$ and inserting the complex variable $w$ in the place of $is$, the same reasoning yields that

$$\left(e^{zH} e^{wJ} e^{zH}\right)^m e^{-mzH} e^{-mwJ} e^{-mzH} \tag{30}$$

is central for all $z, w$ in $\mathbb{C}$. In particular, we obtain that for any $t, s \in \mathbb{R}$ the element

$$\left(e^{tH} e^{sJ} e^{tH}\right)^m e^{-mtH} e^{-msJ} e^{-mtH} \tag{31}$$

is central. Since every element of $\mathscr{A}_+^{-1}$ is the exponential of an element of $\mathscr{A}_s$, we deduce that $(ABA)^m A^{-m} B^{-m} A^{-m}$ is central for all $A, B \in \mathscr{A}_+^{-1}$. By Theorem 1 this implies that $\mathscr{A}$ is commutative and the proof is complete. □

Concerning our final proposition we recall that in [9] Jeang and Ko presented the following result. Assume $\mathscr{A}$ is a $C^*$-algebra and there are nonconstant continuous functions $f$ and $g$ defined on intervals $I, J$ in the real line such that for any pair $X, Y \in \mathscr{A}_s$ of self-adjoint elements with $\sigma(X) \subset I$, $\sigma(Y) \subset J$ ($\sigma(\cdot)$ stands for the spectrum) we have $f(X)g(Y) = g(Y)f(X)$. Then it follows that $\mathscr{A}$ is necessarily commutative. We now give a simple extension of that result.

**Proposition 4.** *Assume $\mathscr{A}$ is a unital $C^*$-algebra and there are nonconstant complex-valued continuous functions $f, g$ defined on some intervals $I, J$ in the real line such that for any pair $X, Y \in \mathscr{A}_s$ of self-adjoint elements with $\sigma(X) \subset I$, $\sigma(Y) \subset J$ one has $f(X)g(Y) - g(Y)f(X) \in \mathscr{Z}_{\mathscr{A}}$. Then $\mathscr{A}$ is commutative.*

*Proof.* We can trivially assume that $I = J = [0, 1]$. Next, identify $\mathscr{A}$ with a $C^*$-algebra of operators acting on a Hilbert space and consider its strong closure $\mathscr{B}$ which is a von Neumann algebra. Similarly to the proof of the implication (iii) ⇒ (i) in Theorem 1, referring to Kaplansky's density theorem, we easily see that the relation $f(X)g(Y) - g(Y)f(X) \in \mathscr{Z}_{\mathscr{B}}$ holds for all pairs $X, Y$ of self-adjoint elements of $\mathscr{B}$ with $\sigma(X), \sigma(Y) \subset [0, 1]$. Considering operators $X, Y$ of the form $X = tI + sP$, $Y = t'I + s'P'$, where $P, P' \in \mathscr{B}$ are arbitrary projections and $t, t', s, s'$ are properly chosen real numbers, we deduce that $PP' - P'P \in \mathscr{Z}_{\mathscr{B}}$. It follows that $PP' - P'P$ commutes with $P$. It is then easy to verify that $P, P'$ commute. Therefore, all projections in $\mathscr{B}$ commute which implies that $\mathscr{B}$ is commutative, and hence we obtain that $\mathscr{A}$ is commutative too. □

In accordance with Example 5 in [9], as an immediate application of the above result, we mention that a unital $C^*$-algebra $\mathscr{A}$ is commutative if and only if any of the following relations

$$\begin{aligned}
&\sin(X+Y) - (\sin X \cos Y + \cos X \sin Y) \in \mathscr{Z}_{\mathscr{A}},\\
&\sin(X-Y) - (\sin X \cos Y - \cos X \sin Y) \in \mathscr{Z}_{\mathscr{A}},\\
&\cos(X+Y) - (\cos X \cos Y - \sin X \sin Y) \in \mathscr{Z}_{\mathscr{A}},\\
&\cos(X-Y) - (\cos X \cos Y + \sin X \sin Y) \in \mathscr{Z}_{\mathscr{A}}
\end{aligned} \tag{32}$$

hold true for all pairs $X, Y$ of self-adjoint elements in $\mathscr{A}$.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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