Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 698354, 14 pages
http://dx.doi.org/10.1155/2014/698354

Research Article

Two-Level Brezzi-Pitkäranta Stabilized Finite Element Methods for the Incompressible Flows

Rong An and Xian Wang

College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, China

Correspondence should be addressed to Rong An; anrong702@gmail.com

Received 28 November 2013; Accepted 14 April 2014; Published 8 May 2014

Academic Editor: Grzegorz Nowak

We present a new stabilized finite element method for incompressible flows based on Brezzi-Pitkäranta stabilized method. The stability and error estimates of finite element solutions are derived for classical one-level method. Combining the techniques of two-level discretizations, we propose two-level Stokes/Oseen/Newton iteration methods corresponding to three different linearization methods and show the stability and error estimates of these three methods. We also propose a new Newton correction scheme based on the above two-level iteration methods. Finally, some numerical experiments are given to support the theoretical results and to check the efficiency of these two-level iteration methods.

1. Introduction

In this paper, we consider steady Navier-Stokes equations with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\mu\Delta u + (u\cdot\nabla)\,u - \nabla p &= f, \quad \text{in } \Omega, \\ \operatorname{div} u &= 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with boundary $\partial\Omega$. $\mu > 0$ represents the viscous coefficient. $u = (u_1(x), u_2(x))$ denotes the velocity vector, $p = p(x)$ the pressure, and $f = (f_1(x), f_2(x))$ the prescribed body force vector. The solenoidal condition $\operatorname{div} u = 0$ means that the flows are incompressible.

In computational fluid dynamics, it is very important in searching the appropriate mixed finite element approximation to solve the numerical solutions of the problem (1) quickly and efficiently. Roughly speaking, the selected finite element spaces are required to satisfy the inf-sup condition, such as the finite element space constructed by the P_2-P_1 pair. However, from the computational cost point of view, the P_1-P_1 pair is of practical importance in scientific computation with the lower computational cost. Therefore, much attention has been attracted by the $P_1 - P_1$ pair for simulating the incompressible flow. But, in this case, the inf-sup condition is not satisfied. A usual technique is to introduce the stabilized term in the finite element variational equation such that the inf-sup condition is enforced. There exist many stabilized methods, such as Brezzi-Pitkäranta stabilized method [1], locally stabilized method [2, 3], pressure stabilized method [4], stream upwind Petrov-Galerkin method [5], Douglas-Wang absolutely stabilized method [6], and pressure projection stabilized method [7, 8] and the references cited therein. Most of these stabilized methods necessarily introduce the stabilized parameters. Moreover, some of these methods are conditionally stable; that is, the stabilized parameters must satisfy some stable condition. Therefore, the development of stabilized methods free from stabilized parameters has become increasingly important.

In this paper, we combine the Brezzi-Pitkäranta stabilized method, which is unconditionally stable [9], with techniques of two-level discretizations to solve the numerical solution of the problem (1) under the uniqueness condition. Two-level discretization method has become a powerful tool in solving nonlinear partial differential equations. The basic idea is to capture "large eddies" by computing the initial approximation on the coarse mesh and then to obtain the fine approximation by solving a linearized problem corresponding to nonlinear partial differential equations on the fine mesh. More details can be referred to in the works of Xu [10, 11]. There exists a large amount of references about two-level finite element method for Navier-Stokes equations. For details, please see the works of An and Qiu [12], Ervin et al. [13], Franca and Nesliturk [14], de Frutos et al. [15, 16], Girault and Lions [17], Goswami and Damázio [18], He [19], He and Li [20], He and Wang [21], He et al. [22], Huang et al. [23], Layton [24], Layton and Tobiska [25], Li [26], Li and An [27, 28], Liu and Hou [29], and Zhu and Chen [30] and the references cited therein.

Based on the Brezzi-Pitkäranta stabilized finite element method, in this paper, we solve the nonlinear Navier-Stokes equations on the coarse mesh with mesh size H in Step I and then solve a linear system according to Stokes/Oseen/Newton iterative method on the fine mesh with mesh size h in Step II. Denote by (u^h, p^h) the finite element approximation solution on the fine mesh. If we suppose $(u, p) \in (H^2(\Omega)^2, H^1(\Omega))$, then the error estimate derived is

$$\left\|u - u^h\right\|_V + \left\|p - p^h\right\| \le c\left(h + H^2\right), \tag{2}$$

where $c > 0$ is independent of h and H and the norms $\|\cdot\|_V$ and $\|\cdot\|$ are defined in the next section. It is obvious that if we choose $H = O(h^{1/2})$, then two-level method discussed in this paper provides the same convergence order as the classical one-level method. Finally, we propose a Newton correction scheme on the fine mesh. The numerical solution (u^h, p^h) in Step II is as the iterative initial value. Then the finite element approximation solution (u_*^h, p_*^h) is solved in terms of Newton iterative scheme on the fine mesh in Step III. The error estimate derived for this Newton correction scheme is

$$\left\|u - u_*^h\right\|_V + \left\|p - p_*^h\right\| \le c\left(h + H^4\right). \tag{3}$$

Thus, if $H = O(h^{1/4})$, then this new two-level method also is of the same convergence order as the classical one-level method.

This paper is organized as follows. In Section 2, we introduce some function spaces and some classical results about Navier-Stokes equations. In Section 3, the Brezzi-Pitkäranta stabilized finite element approximation will be applied and the error estimates about the velocity in H^1-norm and L^2-norm and the pressure in L^2-norm are derived. In Section 4, the two-level discretization finite element methods are proposed and the error estimates (2) and (3) are shown. In the final section, the numerical experiments are displaced to support the theoretical results.

2. Navier-Stokes Equations

In what follows, we employ the standard notation $H^l(\Omega)$ (or $H^l(\Omega)^2$), $l \ge 0$, for the Sobolev spaces of all functions having square integrable derivatives up to order l in Ω. Denote the standard Sobolev norm by $\|\cdot\|_l$. If $l = 0$, we write $L^2(\Omega)$ (or $L^2(\Omega)^2$) and $\|\cdot\|$ instead of $H^0(\Omega)$ (or $H^0(\Omega)^2$) and $\|\cdot\|_0$, respectively. The symbol c always denotes some positive constant which is independent of the mesh parameters h and H and can be a different constant even in the same formulation.

Introduce the following spaces usually used in this paper:

$$V = H_0^1(\Omega)^2, \qquad M = L_0^2(\Omega) = \left\{q \in L^2(\Omega);\ \int_\Omega q\,dx = 0\right\}. \tag{4}$$

The space V is equipped with the norm

$$\|v\|_V = \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}, \quad \forall v \in V. \tag{5}$$

It is well known that $\|v\|_V$ is equivalent to $\|v\|_1$ due to Poincaré inequality. Introduce the following bilinear and trilinear forms:

$$\begin{aligned} a(u, v) &= \mu \int_\Omega \nabla u \cdot \nabla v\, dx, \quad \forall u, v \in V, \\ d(v, q) &= \int_\Omega q \operatorname{div} v\, dx, \quad \forall v \in V,\ q \in M, \\ b(u, v, w) &= \int_\Omega (u \cdot \nabla) v \cdot w\, dx - \frac{1}{2}\int_\Omega \operatorname{div} u v \cdot w\, dx \\ &= \frac{1}{2}\int_\Omega (u \cdot \nabla) v \cdot w\, dx - \frac{1}{2}\int_\Omega (u \cdot \nabla) w \cdot v\, dx. \end{aligned} \tag{6}$$

It is easy to check that this trilinear form satisfies the following important properties [20, 31]:

$$b(u, v, w) = -b(u, w, v), \tag{7}$$

$$b(u, v, w) \le N\|u\|_V\|v\|_V\|w\|_V, \tag{8}$$

$$\begin{aligned} b(u, v, w) \le{}& \frac{N}{2}\|u\|^{1/2}\|u\|_V^{1/2} \\ &\times \left(\|v\|_V\|w\|^{1/2}\|w\|_V^{1/2} + \|w\|_V\|v\|^{1/2}\|v\|_V^{1/2}\right), \end{aligned} \tag{9}$$

for all $u, v, w \in V$ and

$$|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)| \le N\|u\|_V\|v\|_2\,\|w\|, \tag{10}$$

for all $u \in V$, $v \in H^2(\Omega)^2$, and $w \in L^2(\Omega)^2$, where $N > 0$ depends only on Ω.

Given $f \in L^2(\Omega)^2$, under the above notations, the variational formulation of the problem (1) reads as follows: find $(u, p) \in (V, M)$ such that for all $(v, q) \in (V, M)$

$$\begin{aligned} a(u, v) + b(u, u, v) - d(v, p) &= (f, v), \\ d(u, q) &= 0. \end{aligned} \tag{11}$$

Define a generalized bilinear form on $(V, M) \times (V, M)$ by

$$\mathscr{B}(u, p; v, q) = a(u, v) - d(v, p) + d(u, q); \tag{12}$$

then the problem (11) also takes the following form:

$$\mathscr{B}(u, p; v, q) + b(u, u, v) = (f, v), \quad \forall (v, q) \in (V, M). \tag{13}$$

The following existence, uniqueness, and regularity results concerning the solution (u, p) to the problem (13) are classical [32–34].

Theorem 1. *Assuming that μ and f satisfy the following uniqueness condition:*

$$2\mu^{-2}N\|f\| < 1, \tag{14}$$

then the problem (13) *exists a unique solution $(u, p) \in (V, M)$ satisfying*

$$\|u\|_V \leq \frac{1}{\mu}\|f\| < \frac{\mu}{2N}. \tag{15}$$

Furthermore, if $\partial\Omega$ is of class C^2, then the solution (u, p) to the problem (13) *satisfies the following regularity property:*

$$\|u\|_2 + \|p\|_1 \leq c\|f\|. \tag{16}$$

3. Stabilized Finite Element Approximation

Let $\mathcal{T}_h$ be a family of quasiuniform triangular partitions of Ω into triangles. The corresponding ordered triangles are denoted by $K_1, K_2, \ldots, K_n$. Let $h_i = \operatorname{diam}(K_i)$, $i = 1, \ldots, n$, and $h = \max\{h_1, h_2, \ldots, h_n\}$. For every $K \in \mathcal{T}_h$, let $P_r(K)$ denote the space of the polynomials on K of degree at most r. Consider the conforming finite element spaces V_h and M_h given by

$$\begin{aligned} V_h &= \left\{v_h \in C(\overline{\Omega})^2 \cap V, v_h|_K \in [P_1(K)]^2, \forall K \in \mathcal{T}_h\right\}, \\ M_h &= \left\{q_h \in C(\overline{\Omega}) \cap M, q_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\right\}. \end{aligned} \tag{17}$$

Then the Brezzi-Pitkäranta stabilized finite element approximation of (11) is as follows: find $u_h \in V_h$ and $p_h \in M_h$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{aligned} &a(u_h, v_h) + b(u_h, u_h, v_h) - d(v_h, p_h) = (f, v_h), \\ &d(u_h, q_h) + C_h(p_h, q_h) = 0, \end{aligned} \tag{18}$$

where the stabilized term is defined by

$$C_h(p_h, q_h) = \alpha\sum_{i=1}^{n} h_i^2 \int_{K_i} \nabla p_h \cdot \nabla q_h dx, \quad \forall p_h, q_h \in M_h \tag{19}$$

with some positive constant $\alpha > 0$. Define a mesh-dependent norm $[\cdot]_h$ on M_h by

$$[q_h]_h^2 = \frac{1}{\alpha}C_h(q_h, q_h), \quad \forall q_h \in M_h. \tag{20}$$

Then, it holds that $C_h(p_h, q_h) \leq \alpha[p_h]_h[q_h]_h$ for all $p_h, q_h \in M_h$ and

$$d(v, q_h) \leq \frac{c}{h}\|v\|\,[q_h]_h, \quad \forall v \in V,\ q_h \in M_h, \tag{21}$$

which has been shown by Latché and Vola [35]. Moreover, $C_h(p, q)$ also is defined for any couple of functions $p, q \in H^1(\Omega)$ and satisfies

$$[q]_h \leq ch\|q\|_1, \quad \forall q \in H^1(\Omega). \tag{22}$$

Introduce another generalized bilinear form $\mathcal{B}_h(\cdot,\cdot;\cdot,\cdot)$ on $(V_h, M_h) \times (V_h, M_h)$ defined by

$$\mathcal{B}_h(u_h, p_h; v_h, q_h) = \mathcal{B}(u_h, p_h; v_h, q_h) + C_h(p_h, q_h). \tag{23}$$

Then the discrete problem (18) can be rewritten as follows:

$$\mathcal{B}_h(u_h, p_h; v_h, q_h) + b(u_h, u_h, v_h) = (f, v_h). \tag{24}$$

Denote by $I_h : H^2(\Omega)^2 \cap V \longrightarrow V_h$ and $J_h : H^1(\Omega) \cap M \longrightarrow M_h$ the standard interpolation operators satisfying

$$\begin{aligned} \|v - I_h v\| + h\|v - I_h v\|_V &\leq ch^2\|v\|_2, \quad \forall v \in H^2(\Omega)^2 \cap V, \\ \|q - J_h q\| &\leq ch\|q\|_1, \quad \forall q \in H^1(\Omega) \cap M. \end{aligned} \tag{25}$$

Moreover, we suppose that the inverse inequalities hold:

$$\|\nabla q_h\|_{K_i} \leq ch_i^{-1}\|q_h\|_{K_i}, \qquad \|\nabla q_h\| \leq ch^{-1}\|q_h\|. \tag{26}$$

First, we recall the following stable theorem [9].

Theorem 2. *For any $\alpha > 0$, there exist two positive constants β_1 and β_2 independent of h such that $\mathcal{B}_h(\cdot,\cdot;\cdot,\cdot)$ on $(V_h, M_h) \times (V_h, M_h)$ satisfies the following continuous property:*

$$\mathcal{B}_h(w_h, r_h; v_h, q_h) \leq \beta_1(\|w_h\|_V + \|r_h\|)(\|v_h\|_V + \|q_h\|) \tag{27}$$

and the weakly coercive property:

$$\beta_2(\|w_h\|_V + \|r_h\|) \leq \sup_{(v_h, q_h) \in (V_h, M_h)} \frac{\mathcal{B}_h(w_h, r_h; v_h, q_h)}{\|v_h\|_V + \|q_h\|}. \tag{28}$$

A direct result of Theorem 2 is that the problem (24) exists a unique solution. In order to derive the error estimate between (u, p) and (u_h, p_h), we introduce the following Galerkin projection operator $(R_h, Q_h) : (V, M) \longrightarrow (V_h, M_h)$ defined by

$$\mathcal{B}_h(R_h w, Q_h r; w_h, r_h) = \mathcal{B}(w, r; w_h, r_h) \tag{29}$$

for each $(w, r) \in (V, M)$ and all $(w_h, r_h) \in (V_h, M_h)$. According to Theorem 2, it is easy to check that $(R_h w, Q_h r)$ is well defined. Moreover, there holds

$$\mathcal{B}_h(R_h w, Q_h r; w_h, r_h) = \mathcal{B}_h(w, r; w_h, r_h) - C_h(r, r_h). \tag{30}$$

About the Galerkin projection operator (R_h, Q_h), the following approximation property has been derived in [9].

Theorem 3. *For any $w \in H^2(\Omega)^2 \cap V$ and $r \in H^1(\Omega) \cap M$, there holds*

$$\begin{aligned} &\|w - R_h w\| + h\|w - R_h w\|_V + h\|r - Q_h r\| \\ &\quad + h[r - Q_h r]_h \leq ch^2(\|w\|_2 + \|r\|_1). \end{aligned} \tag{31}$$

Next, we begin to show the error estimate for the one-level finite element approximation solution (u_h, p_h).

Theorem 4. *Suppose that the uniqueness condition* (14) *holds. If* $(u,p) \in H^2(\Omega)^2 \cap V \times H^1(\Omega) \cap M$ *and* $(u_h, p_h) \in (V_h, M_h)$ *are the solutions of* (13) *and* (24)*, respectively, then, for any* $\alpha > 0$*, the following optimal error estimate holds:*

$$\|u-u_h\|_V + \|p-p_h\| + [p-p_h]_h \le ch. \tag{32}$$

Proof. First, we estimate $\|u_h\|_V$. Setting $v_h = u_h$ and $q_h = p_h$ in (24), using (7) and Young inequality, we obtain

$$\mu\|u_h\|_V^2 \le (f, u_h) \le \frac{\mu}{2}\|u_h\|_V^2 + \frac{1}{2\mu}\|f\|^2. \tag{33}$$

Then under the uniqueness condition (14), u_h satisfies

$$\|u_h\|_V \le \frac{1}{\mu}\|f\| < \frac{\mu}{2N}. \tag{34}$$

It follows from (30) that

$$\begin{aligned}
\mu\|u_h - R_h u\|_V^2 &+ \alpha[p_h - Q_h p]_h^2 \\
&= \mathscr{B}_h(u_h - R_h u, p_h - Q_h p; u_h - R_h u, p_h - Q_h p) \\
&= \mathscr{B}_h(u_h - u, p_h - p; u_h - R_h u, p_h - Q_h p) \\
&\quad + \mathscr{B}_h(u - R_h u, p - Q_h p; u_h - R_h u, p_h - Q_h p) \\
&= b(u, u, u_h - R_h u) - b(u_h, u_h, u_h - R_h u).
\end{aligned} \tag{35}$$

According to (7), (15), (34), and Young inequality, we get

$$\begin{aligned}
b(u, u, &u_h - R_h u) - b(u_h, u_h, u_h - R_h u) \\
&= b(u - u_h, u, u_h - R_h u) + b(u_h, u - u_h, u_h - R_h u) \\
&= b(u - R_h u, u, u_h - R_h u) + b(R_h u - u_h, u, u_h - R_h u) \\
&\quad + b(u_h, u - R_h u, u_h - R_h u) \\
&\le N(\|u\|_V + \|u_h\|_V)\|u - R_h u\|_V\|u_h - R_h u\|_V \\
&\quad + N\|u\|_V\|u_h - R_h u\|_V^2 \\
&\le \mu\|u - R_h u\|_V\|u_h - R_h u\|_V + \frac{\mu}{2}\|u_h - R_h u\|_V^2 \\
&\le \frac{\mu}{2}\|u_h - R_h u\|_V^2 + \frac{\mu}{4}\|u_h - R_h u\|_V^2 + \mu\|u - R_h u\|_V^2.
\end{aligned} \tag{36}$$

Thus, from (31) we obtain

$$\begin{aligned}
\|u - u_h\|_V &\le \|u - R_h u\|_V + \|u_h - R_h u\|_V \\
&\le 3\|u - R_h u\|_V \le ch.
\end{aligned} \tag{37}$$

Next, we estimate $\|p_h - Q_h p\|$. It follows from (15), (28), (34), and (37) that

$$\begin{aligned}
\beta_2\|p_h - Q_h p\| &\\
\le \sup_{(v_h,q_h)\in(V_h,M_h)} &\frac{\mathscr{B}_h(u_h - R_h u, p_h - Q_h p; v_h, q_h)}{\|v_h\|_V + \|q_h\|} \\
= \sup_{(v_h,q_h)\in(V_h,M_h)} &\Big(\big(\mathscr{B}_h(u_h - u, p_h - p; v_h, q_h) \\
&+ \mathscr{B}_h(u - R_h u, p - Q_h p; v_h, q_h)\big) \\
&\times(\|v_h\|_V + \|q_h\|)^{-1}\Big) \\
= \sup_{(v_h,q_h)\in(V_h,M_h)} &\frac{b(u, u, v_h) - b(u_h, u_h, v_h)}{\|v_h\|_V + \|q_h\|} \\
= \sup_{(v_h,q_h)\in(V_h,M_h)} &\frac{b(u - u_h, u, v_h) + b(u_h, u - u_h, v_h)}{\|v_h\|_V + \|q_h\|} \\
\le \mu\|u - u_h\|_V. &
\end{aligned} \tag{38}$$

Moreover,

$$\begin{aligned}
[p - p_h]_h &\le [p - Q_h p]_h + [p_h - Q_h p]_h \\
&\le [p - Q_h p]_h + c\|u - R_h u\|_V \le ch.
\end{aligned} \tag{39}$$

□

Next, we give the L^2 error estimate $\|u - u_h\|$ by Aubin-Nitsche technique. This error analysis is based on the regularity assumption that the following linearized problem (40) is $(H^2(\Omega)^2, H^1(\Omega))$ regular. Given $z \in L^2(\Omega)^2$, find $(w, \pi) \in (V, M)$ such that for all $(v, q) \in (V, M)$

$$\begin{aligned}
a(w, v) + b(u, v, w) + b(v, u, w) - d(v, \pi) &= (z, v), \\
d(w, q) &= 0.
\end{aligned} \tag{40}$$

According to (7) and (15), it is easy to verify that the problem (40) exists a unique solution $(w, \pi) \in (V, M)$. The assumption that (40) is $(H^2(\Omega)^2, H^1(\Omega))$ regular means that (w, π) also belongs to $(H^2(\Omega)^2, H^1(\Omega))$ and the following estimate holds:

$$\|w\|_2 + \|\pi\|_1 \le c\|z\|. \tag{41}$$

Under the above assumption, we prove the following theorem.

Theorem 5. *Suppose that the uniqueness condition* (14) *holds. If* $(u,p) \in (H^2(\Omega)^2 \cap V, H^1(\Omega) \cap M)$ *and* $(u_h, p_h) \in (V_h, M_h)$ *are the solutions of* (13) *and* (24)*, respectively, then, for any* $\alpha > 0$*, the following optimal* L^2 *error estimate holds:*

$$\|u - u_h\| \le ch^2. \tag{42}$$

Proof. Setting $z = v = u - u_h$ in the first equation of (40), it yields

$$\begin{aligned}
\|u - u_h\|^2 &= a(w, u - u_h) + b(u, u - u_h, w) \\
&\quad + b(u - u_h, u, w) - d(u - u_h, \pi).
\end{aligned} \tag{43}$$

Subtracting (11) from (18) yields

$$\begin{aligned} &a(u-u_h,v_h)+b(u,u,v_h)-b(u_h,u_h,v_h)\\ &\quad -d(v_h,p-p_h)=0,\quad \forall v_h\in V_h,\\ &d(u-u_h,q_h)-C_h(p_h,q_h)=0,\quad \forall q_h\in M_h. \end{aligned} \tag{44}$$

Taking $v_h = R_h w$ and $q_h = Q_h\pi$ in (44) and combining them with (43), we obtain

$$\begin{aligned} \|u-u_h\|^2 &= a(w-R_hw,u-u_h)+b(u,u-u_h,w)\\ &\quad + b(u-u_h,u,w)+b(u_h,u_h,R_hw)\\ &\quad - b(u,u,R_hw)+d(R_hw-w,p-p_h)\\ &\quad - d(u-u_h,\pi-Q_h\pi)-C_h(p_h,Q_h\pi)\\ &= I_1+\cdots+I_4. \end{aligned} \tag{45}$$

Using (31), (32), and (41), I_1 is estimated by

$$\begin{aligned} I_1 &= a(w-R_hw,u-u_h)\le \mu\|u-u_h\|_V\|w-R_hw\|_V\\ &\le ch^2(\|w\|_2+\|\pi\|_1)\le ch^2\|u-u_h\|. \end{aligned} \tag{46}$$

Similarly, I_3 is estimated by

$$\begin{aligned} I_3 &= d(R_hw-w,p-p_h)-d(u-u_h,\pi-Q_h\pi)\\ &\le \|R_hw-w\|_V\|p-p_h\|+\|u-u_h\|_V\|\pi-Q_h\pi\|\\ &\le ch^2(\|w\|_2+\|\pi\|_1)\le ch^2\|u-u_h\|. \end{aligned} \tag{47}$$

About I_2, we rewrite it as

$$\begin{aligned} I_2 &= b(u,u-u_h,w)+b(u-u_h,u,w)\\ &\quad + b(u_h,u_h,R_hw)-b(u,u,R_hw)\\ &= b(u-u_h,u-u_h,w)+b(u-u_h,u,w-R_hw)\\ &\quad + b(u,u-u_h,w-R_hw)\\ &\quad + b(u-u_h,u-u_h,R_hw-w). \end{aligned} \tag{48}$$

Then it follows from (8), (15), (31), (32), and (41) that

$$\begin{aligned} I_2 &\le N\|u-u_h\|_V^2(\|w\|_2+\|w-R_hw\|_V)\\ &\quad + N\|u\|_V\|u-u_h\|_V\|w-R_hw\|_V\\ &\le ch^2\|w\|_2\le ch^2\|u-u_h\|. \end{aligned} \tag{49}$$

Finally, using (22), (31), (32), and (41) we estimate I_4 by

$$\begin{aligned} I_4 &= -C_h(p_h,Q_h\pi)=C_h(p-p_h,Q_h\pi-\pi)+C_h(p-p_h,\pi)\\ &\quad + C_h(p,\pi-Q_h\pi)-C_h(p,\pi)\le [p-p_h]_h[Q_h\pi-\pi]_h\\ &\quad + [p-p_h]_h[\pi]_h+[p]_h[\pi-Q_h\pi]_h+[p]_h[\pi]_h\\ &\le ch^2(\|w\|_2+\|\pi\|_1)\le ch^2\|u-u_h\|. \end{aligned} \tag{50}$$

Combining these estimates for I_1 to I_4 with (45), we complete the proof of (42). □

4. Two-Level Brezzi-Pitkäranta Stabilized Methods

In this section, the two-level Brezzi-Pitkäranta stabilized finite element methods for (13) are proposed in terms of Oseen/Stokes/Newton iteration method. From now on, H and h with $h < H < 1$ are two real positive parameters. The coarse mesh triangulation $\mathcal{T}_H$ is made as like in Section 3. And a fine mesh triangulation $\mathcal{T}_h$ is generated by a mesh refinement process to $\mathcal{T}_H$. The conforming finite element space pairs (V_h, M_h) and $(V_H, M_H) \subset (V_h, M_h)$ corresponding to the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, are constructed as like in Section 3. With the above notations, we propose the following two-level Brezzi-Pitkäranta stabilized finite element methods in the next subsections.

4.1. Two-Level Oseen Iteration Method

Step I. We solve (24) on the coarse mesh; that is, find $(u_H, p_H) \in (V_H, M_H)$ such that for all $(v_H, q_H) \in (V_H, M_H)$

$$\mathcal{B}_H(u_H,p_H;v_H,q_H)+b(u_H,u_H,v_H)=(f,v_H). \tag{51}$$

Step II. We solve a discrete Oseen problem according to Oseen iteration on the fine mesh; that is, find $(u^h, p^h) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\mathcal{B}_h(u^h,p^h;v_h,q_h)+b(u_H,u^h,v_h)=(f,v_h). \tag{52}$$

First, we discuss the existence and uniqueness of the solution to the problem (52) under the uniqueness condition (14). In view of Theorem 2, the problem (51) exists a unique solution $(u_H, p_H) \in (V_H, M_H)$ with

$$\|u_H\|_V\le \frac{1}{\mu}\|f\|<\frac{\mu}{2N}. \tag{53}$$

Moreover, it follows from Theorems 4 and 5 that

$$\|u-u_H\|+H\|u-u_H\|_V+H\|p-p_H\|\le cH^2. \tag{54}$$

On the other hand, setting $v_h = u^h$ and $q_h = p^h$ in (52), it yields

$$\mathcal{B}_h(u^h,p^h;u^h,p^h)+b(u_H,u^h,u^h)=\mu\|u^h\|_V^2+[p^h]_h^2. \tag{55}$$

Then it is easy to show that the problem (52) also exists a unique solution $(u^h, p^h) \in (V_h, M_h)$ such that

$$\|u^h\|_V\le \frac{1}{\mu}\|f\|<\frac{\mu}{2N}. \tag{56}$$

Next, we give the error estimate for the two-level Oseen iteration method.

Theorem 6. *Suppose that the uniqueness condition* (14) *holds. If* $(u, p) \in (H^2(\Omega)^2 \cap V, H^1(\Omega) \cap M)$ *and* $(u^h, p^h) \in (V_h, M_h)$ *are the solutions of* (13) *and* (52), *respectively, then there holds*

$$\left\|u - u^h\right\|_V + \left\|p - p^h\right\| \leq c\left(h + H^2\right). \quad (57)$$

Proof. In terms of the definition $\mathscr{B}_h(\cdot,\cdot;\cdot,\cdot)$ and (30), we get

$$\begin{aligned}
&\mu\left\|u^h - R_h u\right\|_V^2 + \left[p^h - Q_h p\right]_h^2 \\
&\quad = \mathscr{B}_h\left(u^h - R_h u, p^h - Q_h p; u^h - R_h u, p^h - Q_h p\right) \\
&\quad = \mathscr{B}_h\left(u^h - u, p^h - p; u^h - R_h u, p^h - Q_h p\right) \\
&\qquad + \mathscr{B}_h\left(u - R_h u, p - Q_h p; u^h - R_h u, p^h - Q_h p\right) \\
&\quad = b\left(u, u, u^h - R_h u\right) - b\left(u_H, u^h, u^h - R_h u\right).
\end{aligned} \quad (58)$$

We rewrite $b(u, u, u^h - R_h u) - b(u_H, u^h, u^h - R_h u)$ as

$$\begin{aligned}
&b\left(u, u, u^h - R_h u\right) - b\left(u_H, u^h, u^h - R_h u\right) \\
&\quad = b\left(u - u_H, u, u^h - R_h u\right) - b\left(u_H, u^h - u, u^h - R_h u\right) \\
&\quad = b\left(u - u_H, u, u^h - R_h u\right) - b\left(u_H, R_h u - u, u^h - R_h u\right).
\end{aligned} \quad (59)$$

Then using (8), (10), and (53), we obtain

$$\begin{aligned}
&b\left(u, u, u^h - R_h u\right) - b\left(u_H, u^h, u^h - R_h u\right) \\
&\quad \leq N\|u\|_2\left\|u^h - R_h u\right\|_V \|u - u_H\| \\
&\qquad + N\|u_H\|_V\|R_h u - u\|_V\left\|u^h - R_h u\right\|_V \\
&\quad \leq \frac{\mu}{2}\left\|u^h - R_h u\right\|_V^2 + \frac{N^2}{\mu}\|u\|_2^2\|u - u_H\|^2 \\
&\qquad + \frac{\mu}{4}\|R_h u - u\|_V^2.
\end{aligned} \quad (60)$$

Thus, there holds

$$\left\|u^h - R_h u\right\|_V \leq c\left(\|u - u_H\| + \|R_h u - u\|_V\right) \leq c\left(h + H^2\right), \quad (61)$$

where we use (31) and (54). A direct consequence of the above estimate is

$$\left\|u - u^h\right\|_V \leq c\left(h + H^2\right). \quad (62)$$

From (28), (30), (54), and (62), we have

$$\begin{aligned}
&\beta_2\left\|p^h - Q_h p\right\| \\
&\quad \leq \sup_{(v_h,q_h)\in(V_h,M_h)} \frac{\mathscr{B}_h\left(u^h - R_h u, p^h - Q_h p; v_h, q_h\right)}{\|v_h\|_V + \|q_h\|} \\
&\quad = \sup_{(v_h,q_h)\in(V_h,M_h)} \left(\left(\mathscr{B}_h\left(u^h - u, p^h - p; v_h, q_h\right)\right.\right. \\
&\qquad \left. + \mathscr{B}_h\left(u - R_h u, p - Q_h p; v_h, q_h\right)\right) \\
&\qquad \left. \times\left(\|v_h\|_V + \|q_h\|\right)^{-1}\right) \\
&\quad = \sup_{(v_h,q_h)\in(V_h,M_h)} \frac{b\left(u, u, v_h\right) - b\left(u_H, u^h, v_h\right)}{\|v_h\|_V + \|q_h\|} \\
&\quad = \sup_{(v_h,q_h)\in(V_h,M_h)} \frac{b\left(u - u_H, u, v_h\right) + b\left(u_H, u - u^h, v_h\right)}{\|v_h\|_V + \|q_h\|} \\
&\quad \leq c\left(\|u - u_H\| + \left\|u - u^h\right\|_V\right) \leq c\left(h + H^2\right),
\end{aligned} \quad (63)$$

which together with (31) yields

$$\left\|p - p^h\right\| \leq c\left(h + H^2\right). \quad (64)$$

□

4.2. Two-Level Stokes Iteration Method

Step I. We solve (24) on the coarse mesh; that is, find $(u_H, p_H) \in (V_H, M_H)$ such that for all $(v_H, q_H) \in (V_H, M_H)$

$$\mathscr{B}_H\left(u_H, p_H; v_H, q_H\right) + b\left(u_H, u_H, v_H\right) = \left(f, v_H\right). \quad (65)$$

Step II. We solve a discrete Stokes problem according to Stokes iteration on the fine mesh; that is, find $(u^h, p^h) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\mathscr{B}_h\left(u^h, p^h; v_h, q_h\right) + b\left(u_H, u_H, v_h\right) = \left(f, v_h\right). \quad (66)$$

In this subsection, we assume that the following uniqueness conditions hold:

$$3\mu^{-2}N\|f\| < 1. \quad (67)$$

Proceeding the argument as in Section 4.1, the problem (65) exists a unique solution $(u_H, p_H) \in (V_H, M_H)$ and u_H satisfies $\|u_H\|_V \leq (1/\mu)\|f\| < \mu/3N$. According to the definition of $\mathscr{B}_h(\cdot,\cdot;\cdot,\cdot)$, the discrete Stokes problem (66) also exists a unique solution $(u^h, p^h) \in (V_h, M_h)$. Moreover, u^h satisfies

$$\begin{aligned}\left\|u^h\right\|_V &\le \frac{1}{\mu}\|f\| + \frac{N}{\mu}\|u_H\|_V^2 \\ &\le \frac{1}{\mu}\|f\| + \frac{1}{3}\|u_H\|_V \\ &\le \frac{4}{3\mu}\|f\| \le \frac{4\mu}{9N} < \frac{\mu}{2N}.\end{aligned} \tag{68}$$

Then the error estimate for two-level Stokes iteration method is derived in the following theorem.

Theorem 7. *Suppose that the uniqueness condition* (67) *holds. If* $(u, p) \in (H^2(\Omega)^2 \cap V, H^1(\Omega) \cap M)$ *and* $(u^h, p^h) \in (V_h, M_h)$ *are the solutions of* (13) *and* (66)*, respectively, then one has*

$$\left\|u - u^h\right\|_V + \left\|p - p^h\right\| \le c\left(h + H^2\right). \tag{69}$$

Proof. Subtracting (13) from (66), we get

$$\begin{aligned}&\mathscr{B}_h\left(u^h - u, p^h - p; v_h, q_h\right) \\ &\quad = b\left(u, u, v_h\right) - b\left(u_H, u_H, v_h\right) - C_h\left(p, q_h\right).\end{aligned} \tag{70}$$

Then, from (10), (28), (30), and (54), we have

$$\begin{aligned}&\beta_2\left(\left\|u^h - R_h u\right\|_V + \left\|p^h - Q_h p\right\|\right) \\ &\le \sup_{(v_h,q_h)\in(V_h,M_h)} \frac{\mathscr{B}_h\left(u^h - R_h u, p^h - Q_h p; v_h, q_h\right)}{\left\|v_h\right\|_V + \left\|q_h\right\|} \\ &= \sup_{(v_h,q_h)\in(V_h,M_h)} \Big(\Big(\mathscr{B}_h\left(u^h - u, p^h - p; v_h, q_h\right) \\ &\qquad + \mathscr{B}_h\left(u - R_h u, p - Q_h p; v_h, q_h\right)\Big) \\ &\qquad \times \left(\left\|v_h\right\|_V + \left\|q_h\right\|\right)^{-1}\Big) \\ &= \sup_{(v_h,q_h)\in(V_h,M_h)} \frac{b\left(u, u, v_h\right) - b\left(u_H, u_H, v_h\right)}{\left\|v_h\right\|_V + \left\|q_h\right\|} \\ &= \sup_{(v_h,q_h)\in(V_h,M_h)} \Big(\Big(b\left(u - u_H, u, v_h\right) + b\left(u, u - u_H, v_h\right) \\ &\qquad - b\left(u - u_H, u - u_H, v_h\right)\Big) \\ &\qquad \times \left(\left\|v_h\right\|_V + \left\|q_h\right\|\right)^{-1}\Big) \\ &\le 2N\|u\|_2\left\|u - u_H\right\| + N\left\|u - u_H\right\|_V^2 \le cH^2,\end{aligned} \tag{71}$$

which together with (31) completes the proof of (69). □

4.3. Two-Level Newton Iteration Method

Step I. We solve (24) on the coarse mesh; that is, find $(u_H, p_H) \in (V_H, M_H)$ such that for all $(v_H, q_H) \in (V_H, M_H)$

$$\mathscr{B}_H\left(u_H, p_H; v_H, q_H\right) + b\left(u_H, u_H, v_H\right) = \left(f, v_H\right). \tag{72}$$

Step II. We solve discrete linearized Navier-Stokes equations according to Newton iteration on the fine mesh; that is, find $(u^h, p^h) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{aligned}&\mathscr{B}_h\left(u^h, p^h; v_h, q_h\right) + b\left(u_H, u^h, v_h\right) + b\left(u^h, u_H, v_h\right) \\ &\quad = \left(f, v_h\right) + b\left(u_H, u_H, v_h\right).\end{aligned} \tag{73}$$

As in Section 4.2, we modify the uniqueness condition as

$$4\mu^{-2}N\|f\| < 1. \tag{74}$$

In this case, the solution u_H of the problem (72) satisfies $\|u_H\|_V \le (1/\mu)\|f\| < \mu/4N$. Setting $v_h = u^h$ and $q_h = p^h$ in (73), we have

$$\begin{aligned}&\mathscr{B}_h\left(u^h, p^h; u^h, p^h\right) + b\left(u^h, u_H, u^h\right) + b\left(u_H, u^h, u^h\right) \\ &\quad \ge \mu\left\|u^h\right\|_V^2 + [p_h]_h^2 - N\|u_H\|_V\left\|u^h\right\|_V^2 \\ &\quad \ge \frac{3\mu}{4}\left\|u^h\right\|_V^2 + [p_h]_h^2.\end{aligned} \tag{75}$$

Moreover, we can estimate u^h by

$$\begin{aligned}\left\|u^h\right\|_V &\le \frac{4}{3\mu}\|f\| + \frac{4N}{3\mu}\|u_H\|_V^2 \\ &\le \frac{4}{3\mu}\|f\| + \frac{1}{3}\|u_H\|_V \\ &\le \frac{5}{3\mu}\|f\| \le \frac{5\mu}{12N} < \frac{\mu}{2N}.\end{aligned} \tag{76}$$

The error estimate for two-level Newton iteration method is derived in the following theorem.

Theorem 8. *Suppose that the uniqueness condition* (74) *holds. If* $(u, p) \in (H^2(\Omega)^2 \cap V, H^1(\Omega) \cap M)$ *and* $(u^h, p^h) \in (V_h, M_h)$ *are the solutions of* (13) *and* (73)*, respectively, then one has*

$$\left\|u - u^h\right\|_V + \left\|p - p^h\right\| \le c\left(h + H^2\right). \tag{77}$$

Proof. Proceeding as in the proof of Theorem 7, we have

$$\begin{aligned}&\mu\left\|u^h - R_h u\right\|_V^2 + \left[p^h - Q_h p\right]_h^2 \\ &\quad = \mathscr{B}_h\left(u^h - R_h u, p^h - Q_h p; u^h - R_h u, p^h - Q_h p\right) \\ &\quad = \mathscr{B}_h\left(u^h - u, p^h - p; u^h - R_h u, p^h - Q_h p\right) \\ &\qquad + \mathscr{B}_h\left(u - R_h u, p - Q_h p; u^h - R_h u, p^h - Q_h p\right) \\ &\quad = b\left(u, u, u^h - R_h u\right) - b\left(u_H, u^h, u^h - R_h u\right) \\ &\qquad - b\left(u^h, u_H, u^h - R_h u\right) + b\left(u_H, u_H, u^h - R_h u\right).\end{aligned} \tag{78}$$

We rewrite the right-hand side of the above identity as follows:

$$\begin{aligned}
&b\left(u,u,u^h-R_hu\right)-b\left(u_H,u^h,u^h-R_hu\right)\\
&\quad-b\left(u^h,u_H,u^h-R_hu\right)+b\left(u_H,u_H,u^h-R_hu\right)\\
&=b\left(u-u^h,u,u^h-R_hu\right)+b\left(u^h,u-u^h,u^h-R_hu\right)\\
&\quad+b\left(u^h-u_H,u^h-u_H,u^h-R_hu\right)\\
&=b\left(u-R_hu,u,u^h-R_hu\right)+b\left(R_hu-u^h,u,u^h-R_hu\right)\\
&\quad+b\left(u^h,u-R_hu,u^h-R_hu\right)\\
&\quad+b\left(u^h-u_H,R_hu-u_H,u^h-R_hu\right)=J_1+\cdots+J_4.
\end{aligned}\tag{79}$$

Using (8) and (15), we have

$$\begin{aligned}
J_1&=b\left(u-R_hu,u,u^h-R_hu\right)\\
&\le N\|u\|_V\|u-R_hu\|_V\left\|u^h-R_hu\right\|_V\\
&\le\frac{\mu}{2}\|u-R_hu\|_V\left\|u^h-R_hu\right\|_V\\
&\le\frac{\mu}{8}\left\|u^h-R_hu\right\|_V^2+8\mu\|u-R_hu\|_V^2.
\end{aligned}\tag{80}$$

Similarly, J_2 and J_3 can be estimated, respectively, by

$$\begin{aligned}
J_2&=b\left(R_hu-u^h,u,u^h-R_hu\right)\\
&=N\|u\|_V\left\|u^h-R_hu\right\|_V^2\le\frac{\mu}{2}\left\|u^h-R_hu\right\|_V^2,\\
J_3&=b\left(u^h,u-R_hu,u^h-R_hu\right)\\
&\le N\left\|u^h\right\|_V\|u-R_hu\|_V\left\|u^h-R_hu\right\|_V\\
&\le\frac{\mu}{2}\|u-R_hu\|_V\left\|u^h-R_hu\right\|_V\\
&\le\frac{\mu}{8}\left\|u^h-R_hu\right\|_V^2+8\mu\|u-R_hu\|_V^2.
\end{aligned}\tag{81}$$

Finally, we estimate J_4 by

$$\begin{aligned}
J_4&=b\left(u^h-u_H,R_hu-u_H,u^h-R_hu\right)\\
&=b\left(u^h-R_hu,R_hu-u_H,u^h-R_hu\right)\\
&\quad+b\left(R_hu-u_H,R_hu-u_H,u^h-R_hu\right)\\
&\le N\|R_hu-u_H\|_V\left\|u^h-R_hu\right\|_V^2\\
&\quad+N\|R_hu-u_H\|_V^2\left\|u^h-R_hu\right\|_V\\
&\le cH\left\|u^h-R_hu\right\|_V^2+\frac{\mu}{8}\left\|u^h-R_hu\right\|_V^2+c\|R_hu-u_H\|_V^4.
\end{aligned}\tag{82}$$

Combining these estimates for J_1 to J_4 with (79), for sufficiently small H, we get

$$\begin{aligned}
\left\|u^h-R_hu\right\|_V&\le c\left(\|u-R_hu\|_V+\|R_hu-u_H\|_V^2\right)\\
&\le c\left(\|u-R_hu\|_V+\|R_hu-u\|_V^2+\|u-u_H\|_V^2\right)\\
&\le c\left(h+H^2\right),
\end{aligned}\tag{83}$$

which implies that

$$\left\|u-u^h\right\|_V\le c\left(h+H^2\right).\tag{84}$$

From (28) and (30), we have

$$\begin{aligned}
&\beta_2\left\|p^h-Q_hp\right\|\\
&\le\sup_{(v_h,q_h)\in(V_h,M_h)}\frac{\mathscr{B}_h\left(u^h-R_hu,p^h-Q_hp;v_h,q_h\right)}{\|v_h\|_V+\|q_h\|}\\
&=\sup_{(v_h,q_h)\in(V_h,M_h)}\Big(\Big(\mathscr{B}_h\left(u^h-u,p^h-p;v_h,q_h\right)\\
&\qquad+\mathscr{B}_h\left(u-R_hu,p-Q_hp;v_h,q_h\right)\Big)\\
&\qquad\times\left(\|v_h\|_V+\|q_h\|\right)^{-1}\Big)\\
&=\sup_{(v_h,q_h)\in(V_h,M_h)}\Big(\Big(b\left(u,u,v_h\right)-b\left(u_H,u^h,v_h\right)\\
&\qquad-b\left(u^h,u_H,v_h\right)+b\left(u_H,u_H,v_h\right)\Big)\\
&\qquad\times\left(\|v_h\|_V+\|q_h\|\right)^{-1}\Big).
\end{aligned}\tag{85}$$

Since

$$\begin{aligned}
&b\left(u,u,v_h\right)\\
&\quad-b\left(u_H,u^h,v_h\right)-b\left(u^h,u_H,v_h\right)+b\left(u_H,u_H,v_h\right)\\
&=b\left(u-u^h,u,v_h\right)+b\left(u,u-R_hu,v_h\right)\\
&\quad-b\left(u-u^h,u-R_hu,v_h\right)+b\left(u^h-u_H,R_hu-u_H,v_h\right)\\
&\quad-b\left(u_H,u^h-R_hu,v_h\right)\\
&\le N\|u\|_V\left(\left\|u-u^h\right\|_V+\|u-R_hu\|_V\right)\|v_h\|_V\\
&\quad+N\left\|u-u^h\right\|_V\|u-R_hu\|_V\|v_h\|_V\\
&\quad+N\|u_H\|_V\left\|u^h-R_hu\right\|_V\|v_h\|_V\\
&\quad+N\left(\left\|u-u^h\right\|_V+\|u-u_H\|_V\right)\\
&\quad\times\left(\|u-R_hu\|_V+\|u-u_H\|_V\right)\|v_h\|_V,
\end{aligned}\tag{86}$$

then using (31), (54), (83), and (84), we obtain

$$\begin{aligned}\left\|p^h - Q_h p\right\| &\\ &\le N\|u\|_V \left(\left\|u-u^h\right\|_V + \|u - R_h u\|_V\right)\\ &\quad + N\left\|u-u^h\right\|_V \|u - R_h u\|_V + N\|u_H\|_V \left\|u^h - R_h u\right\|_V\\ &\quad + N\left(\left\|u-u^h\right\|_V + \|u-u_H\|_V\right)\\ &\quad \times \left(\|u - R_h u\|_V + \|u - u_H\|_V\right) \le c\left(h + H^2\right),\end{aligned} \tag{87}$$

which together with (31) yields

$$\left\|p - p^h\right\| \le c\left(h + H^2\right). \tag{88}$$

□

4.4. Newton Correction Scheme. As a result of Theorems 6–8, if we choose $H = O(h^{1/2})$, then two-level Stokes/Oseen/Newton iteration methods in the above subsections provide the same convergence order as the usual one-level finite element method (24). In this subsection, we propose a new Newton correction scheme. The error estimate for this scheme implies that if $H = O(h^{1/4})$, then this correction scheme also provides the same convergence order as the usual one-level finite element method (24).

Step I. Solve $(u_H, p_H) \in (V_H, M_H)$ on the coarse mesh by the problem (51).

Step II. Solve $(u^h, p^h) \in (V_h, M_h)$ on the fine mesh by the problem (52) or (66) or (73).

Step III. Solve a Newton correction solution $(u^h_\star, p^h_\star)$ on the fine mesh; that is, find $(u^h_\star, p^h_\star) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{aligned}&\mathscr{B}_h\left(u^h_\star, p^h_\star; v_h, q_h\right) + b\left(u^h, u^h_\star, v_h\right) + b\left(u^h_\star, u^h, v_h\right)\\ &\quad = \left(f, v_h\right) + b\left(u^h, u^h, v_h\right).\end{aligned} \tag{89}$$

First, we discuss the existence and uniqueness of the solution $(u^h_\star, p^h_\star)$ to the problem (89). In terms of (56), (83), and (76), the solution u^h to the problem (52) or (66) or (73) satisfies $\|u^h\|_V \le \mu/2N$. Then taking $v_h = u^h_\star$ and $q_h = p^h_\star$ in (89), we get

$$\begin{aligned}&\mathscr{B}_h\left(u^h_\star, p^h_\star; u^h_\star, p^h_\star\right) + b\left(u^h, u^h_\star, u^h_\star\right) + b\left(u^h_\star, u^h, u^h_\star\right)\\ &\quad \ge \mu\left\|u^h_\star\right\|_V^2 + \left[p^h_\star\right]_h^2 - N\left\|u^h\right\|_V\left\|u^h_\star\right\|_V^2\\ &\quad \ge \frac{\mu}{2}\left\|u^h_\star\right\|_V^2 + \left[p^h_\star\right]_h^2.\end{aligned} \tag{90}$$

Thus, we conclude that the problem (89) exists a unique solution $(u^h_\star, p^h_\star) \in (V_h, M_h)$. Moreover, it is easy to check that $u^h_\star$ satisfies

$$\left\|u^h_\star\right\|_V \le \frac{2}{\mu}\|f\| + \frac{\mu}{2N} \le \frac{3\mu}{2N}. \tag{91}$$

Theorem 9. *Suppose that the uniqueness condition* (14) *or* (67) *or* (74) *holds. If* $(u_h, p_h) \in (V_h, M_h)$ *and* $(u^h_\star, p^h_\star) \in (V_h, M_h)$ *are the solutions of* (24) *and* (89), *respectively, then one has*

$$\left\|u_h - u^h_\star\right\|_V + \left\|p_h - p^h_\star\right\| \le c\left\|u_h - u^h\right\|_V^2, \tag{92}$$

where u^h *is the solution to the problem* (52) *or* (66) *or* (73).

Proof. Subtracting (24) from (89), we get

$$\begin{aligned}&\mathscr{B}_h\left(u_h - u^h_\star, p_h - p^h_\star; v_h, q_h\right)\\ &\quad = b\left(u^h, u^h_\star, v_h\right) + b\left(u^h_\star, u^h, v_h\right) - b\left(u_h, u_h, v_h\right)\\ &\qquad - b\left(u^h, u^h, v_h\right) = b\left(u^h_\star - u_h, u^h, v_h\right)\\ &\qquad + b\left(u^h, u^h_\star - u_h, v_h\right) - b\left(u_h - u^h, u_h - u^h, v_h\right).\end{aligned} \tag{93}$$

Setting $v_h = u_h - u^h_\star$ and $q_h = p_h - p^h_\star$ in (93) and using (56), it yields

$$\begin{aligned}&\mu\left\|u_h - u^h_\star\right\|_V^2\\ &\quad \le b\left(u^h_\star - u_h, u^h, u_h - u^h_\star\right) - b\left(u_h - u^h, u_h - u^h, u_h - u^h_\star\right)\\ &\quad \le N\left\|u^h\right\|_V\left\|u_h - u^h_\star\right\|_V^2 + N\left\|u_h - u^h\right\|_V^2\left\|u_h - u^h_\star\right\|_V\\ &\quad \le \frac{\mu}{2}\left\|u_h - u^h_\star\right\|_V^2 + \frac{\mu}{4}\left\|u_h - u^h_\star\right\|_V^2 + \frac{N^2}{\mu}\left\|u_h - u^h\right\|_V^4.\end{aligned} \tag{94}$$

Thus, we obtain

$$\left\|u_h - u^h_\star\right\|_V \le \frac{2N}{\mu}\left\|u_h - u^h\right\|_V^2. \tag{95}$$

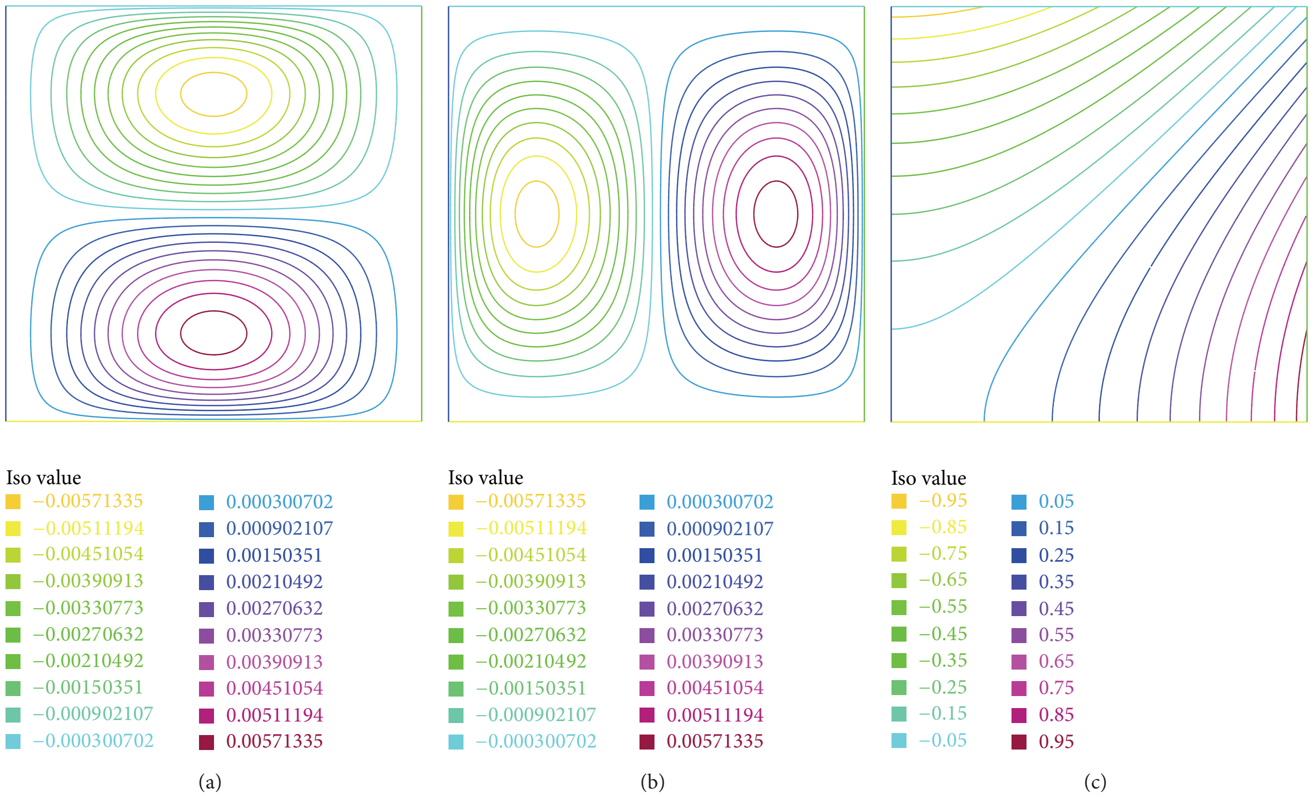


Figure 1: Contour plots of exact solution. From left to right: two components of velocity and pressure.

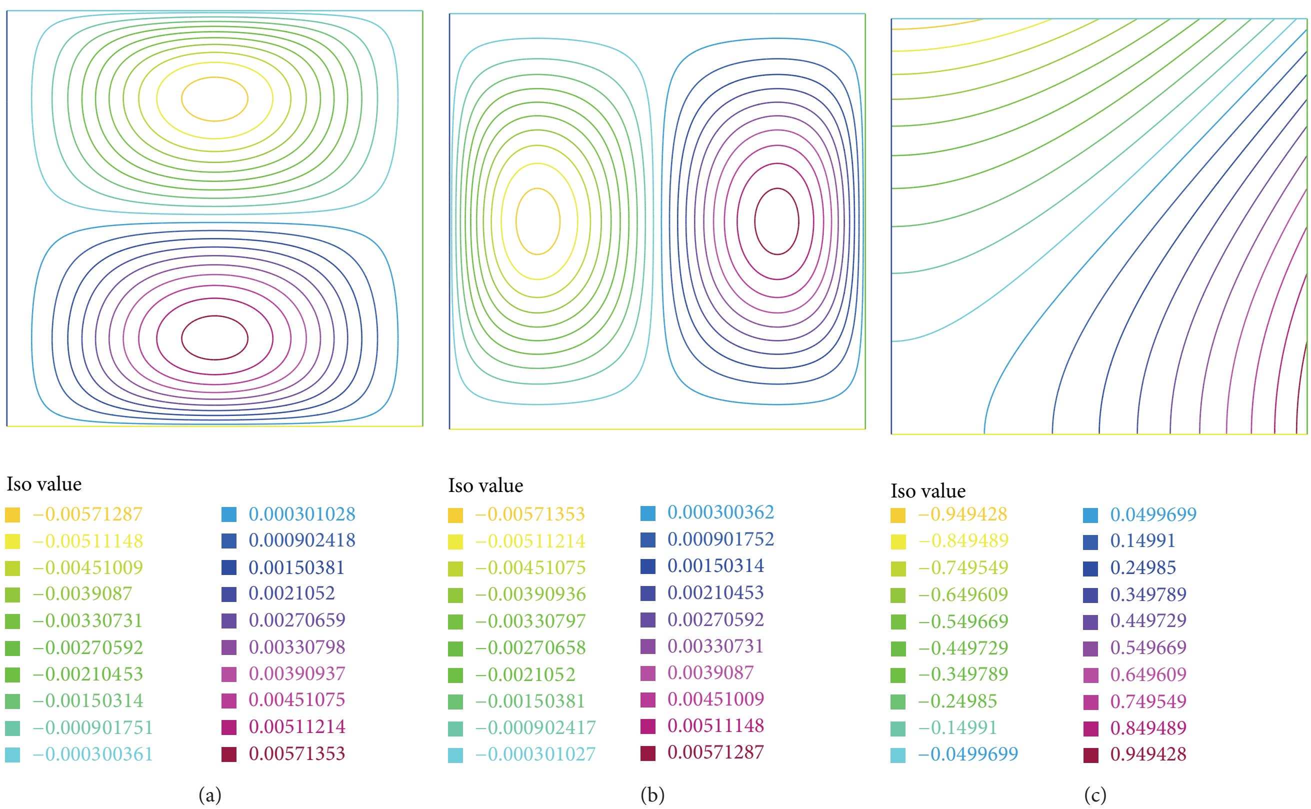
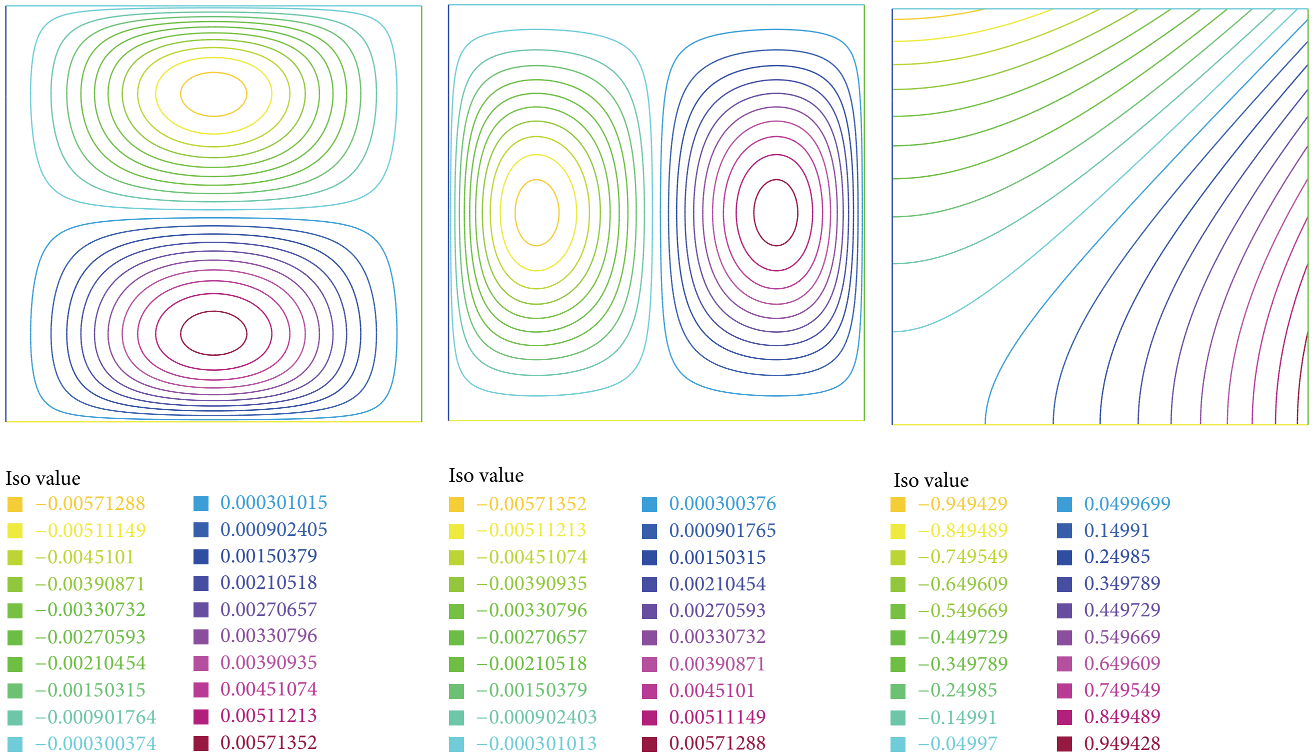


Figure 2: Contour plots of numerical solution by one-level stabilized method. From left to right: two components of velocity and pressure.



(a) (b) (c)

Figure 3: Contour plots of numerical solution by two-level Newton iteration method. From left to right: two components of velocity and pressure.

Table 1: Convergence of one-level method.

$1/h$	$\|u-u_h\|/\|u\|$	Rate	$\|u-u_h\|_V/\|u\|_V$	Rate	$\|p-p_h\|/\|p\|$	Rate	CPU (s)
4^2	$5.05728e-02$	/	$2.04342e-01$	/	$7.00342e-03$	/	0.271
6^2	$9.71970e-03$	2.0338	$8.42186e-02$	1.0930	$1.74965e-03$	1.7104	1.211
8^2	$3.03424e-03$	2.0234	$4.57602e-02$	1.0602	$6.91717e-04$	1.6129	4.526
10^2	$1.23515e-03$	2.0139	$2.87904e-02$	1.0383	$3.42162e-04$	1.5772	10.509
12^2	$5.94367e-04$	2.0059	$1.98039e-02$	1.0261	$1.93897e-04$	1.5576	22.588
14^2	$3.21053e-04$	1.9977	$1.44653e-02$	1.0189	$1.20442e-04$	1.5445	44.181
16^2	$1.88826e-04$	1.9875	$1.10329e-02$	1.0143	$7.99308e-05$	1.5352	80.286
18^2	OUT		OF		MEMORY		

Table 2: Convergence of two-level Oseen iteration method.

$1/H$	$1/h$	$\|u-u^h\|/\|u\|$	Rate	$\|u-u^h\|_V/\|u\|_V$	Rate	$\|p-p^h\|/\|p\|$	Rate	CPU (s)
4	4^2	$5.05858e-02$	/	$2.04349e-01$	/	$7.00341e-03$	/	0.179
6	6^2	$9.73019e-03$	2.0328	$8.42217e-02$	1.0930	$1.74965e-03$	1.7104	0.738
8	8^2	$3.04358e-03$	2.0199	$4.57618e-02$	1.0602	$6.91725e-04$	1.6129	2.105
10	10^2	$1.24408e-03$	2.0046	$2.87914e-02$	1.0383	$3.42169e-04$	1.5772	5.081
12	12^2	$6.03073e-04$	1.9858	$1.98046e-02$	1.0261	$1.93904e-04$	1.5575	10.546
14	14^2	$3.29574e-04$	1.9599	$1.44658e-02$	1.0189	$1.20448e-04$	1.5444	19.846
16	16^2	$1.97162e-04$	1.9238	$1.10332e-02$	1.0143	$7.99367e-05$	1.5352	36.305
18	18^2	$1.26739e-04$	1.8759	$8.69472e-03$	1.0112	$5.57676e-05$	1.5284	56.658

Table 3: Convergence of two-level Stokes iteration method.

$1/H$	$1/h$	$\|u-u^h\|/\|u\|$	Rate	$\|u-u^h\|_V/\|u\|_V$	Rate	$\|p-p^h\|/\|p\|$	Rate	CPU (s)
4	4^2	$5.05773e-02$	/	$2.04346e-01$	/	$7.00344e-03$	/	0.163
6	6^2	$9.72434e-03$	2.0333	$8.42205e-02$	1.0930	$1.74968e-03$	1.7103	0.606
8	8^2	$3.03897e-03$	2.0215	$4.57613e-02$	1.0602	$6.91743e-04$	1.6128	1.808
10	10^2	$1.23981e-03$	2.0089	$2.87911e-02$	1.0383	$3.42185e-04$	1.5772	4.451
12	12^2	$5.98991e-04$	1.9950	$1.98044e-02$	1.0261	$1.93918e-04$	1.5575	9.314
14	14^2	$3.25639e-04$	1.9768	$1.44657e-02$	1.0189	$1.20460e-04$	1.5443	17.481
16	16^2	$1.93359e-04$	1.9518	$1.10331e-02$	1.0143	$7.99470e-05$	1.5350	30.246
18	18^2	$1.23069e-04$	1.9179	$8.69461e-03$	1.0112	$5.57768e-05$	1.5283	49.980

Table 4: Convergence of two-level Newton iteration method.

$1/H$	$1/h$	$\|u-u^h\|/\|u\|$	Rate	$\|u-u^h\|_V/\|u\|_V$	Rate	$\|p-p^h\|/\|p\|$	Rate	CPU (s)
4	4^2	$5.05710e-02$	/	$2.04342e-01$	/	$7.00340e-03$	/	0.204
6	6^2	$9.71757e-03$	2.0340	$8.42186e-02$	1.0930	$1.74964e-03$	1.7104	0.848
8	8^2	$3.03203e-03$	2.0243	$4.57601e-02$	1.0602	$6.91711e-04$	1.6129	2.768
10	10^2	$1.23292e-03$	2.0163	$2.87903e-02$	1.0383	$3.42158e-04$	1.5772	6.316
12	12^2	$5.92128e-04$	2.0113	$1.98039e-02$	1.0261	$1.93895e-04$	1.5576	13.269
14	14^2	$3.18804e-04$	2.0083	$1.44653e-02$	1.0189	$1.20440e-04$	1.5445	24.717
16	16^2	$1.86566e-04$	2.0062	$1.10328e-02$	1.0143	$7.99295e-05$	1.5352	42.030
18	18^2	$1.16338e-04$	2.0049	$8.69440e-03$	1.0112	$5.57612e-05$	1.5285	69.469

It follows from (28), (93), and (95) that

$$\begin{aligned}
\beta_2 \left\| p_h - p_\star^h \right\| & \\
&\leq \sup_{(v_h,q_h)\in(V_h,M_h)} \frac{\mathscr{B}_h\left(u_h - u_\star^h, p_h - p_\star^h; v_h, q_h\right)}{\|v_h\|_V + \|q_h\|} \\
&= \sup_{(v_h,q_h)\in(V_h,M_h)} \Big(\Big(b\left(u_\star^h - u_h, u^h, v_h\right) + b\left(u^h, u_\star^h - u_h, v_h\right) \\
&\qquad - b\left(u_h - u^h, u_h - u^h, v_h\right)\Big) \\
&\qquad \times \left(\|v_h\|_V + \|q_h\|\right)^{-1}\Big) \\
&\leq 2N\left\|u^h\right\|_V \left\|u_h - u_\star^h\right\|_V + N\left\|u_h - u^h\right\|_V^2 \\
&\leq \mu\left\|u_h - u_\star^h\right\|_V + N\left\|u_h - u^h\right\|_V^2 \\
&\leq 3N\left\|u_h - u^h\right\|_V^2.
\end{aligned} \tag{96}$$

□

Combining Theorem 9 with Theorems 6–8 and Theorem 4, we obtain the following error estimate between the solutions (u, p) and $(u_\star^h, p_\star^h)$ to the problems (13) and (89), respectively.

Theorem 10. *Under the assumption in Theorem 9, if* $(u, p) \in (H^2(\Omega)^2 \cap V, H^1(\Omega) \cap M)$ *and* $(u_\star^h, p_\star^h) \in (V_h, M_h)$ *are the solutions of* (13) *and* (89)*, respectively, then one has*

$$\left\|u - u_\star^h\right\|_V + \left\|p - p_\star^h\right\| \leq c\left(h + H^4\right). \tag{97}$$

5. Numerical Experiments

In this section, we make some numerical experiments to support the theoretical results derived in Section 4. The body force f is appropriately selected such that the exact solution of the problem (1) is given by

$$\begin{gathered}
u(x,y) = \left(u_1(x,y), u_2(x,y)\right), \qquad p(x,y) = x^2 - y^2, \\
u_1(x,y) = x^2(x-1)^2 y(y-1)(2y-1), \\
u_2(x,y) = -x(x-1)(2x-1)y^2(y-1)^2
\end{gathered} \tag{98}$$

in the unit square $\Omega = (0,1) \times (0,1)$.

In all experiments, we choose the viscous coefficient $\mu = 0.1$ and stabilized parameter $\alpha = 0.01$ in (18). According to Theorems 6–8, we choose $H = h^{1/2}$; then two-level finite element approximation solution is of the following optimal error estimate:

$$\left\|u - u^h\right\|_V + \left\|p - p^h\right\| \leq ch. \tag{99}$$

Table 5: Convergence of two-level Newton correction scheme.

$1/H$	$1/h$	$\|u-u_*^h\|/\|u\|$	Rate	$\|u-u_*^h\|_V/\|u\|_V$	Rate	$\|p-p_*^h\|/\|p\|$	Rate	CPU (s)
2	4^2	$5.05706e-02$	/	$2.04341e-01$	/	$7.00308e-03$	/	0.263
2	6^2	$9.71752e-03$	2.0340	$8.42185e-02$	1.0930	$1.74964e-03$	1.7104	1.206
3	8^2	$3.03204e-03$	2.0243	$4.57601e-02$	1.0602	$6.91710e-04$	1.6129	3.729
3	10^2	$1.23293e-03$	2.0163	$2.87903e-02$	1.0383	$3.42158e-04$	1.5772	9.294
3	12^2	$5.92131e-04$	2.0113	$1.98039e-02$	1.0261	$1.93895e-04$	1.5576	19.497
4	14^2	$3.18806e-04$	2.0082	$1.44653e-02$	1.0189	$1.20440e-04$	1.5445	37.133
4	16^2	$1.86567e-04$	2.0062	$1.10328e-02$	1.0143	$7.99295e-05$	1.5352	65.678
4	18^2	OUT		OF		MEMORY		

Here we select eight fine mesh values $h = 1/4^2, 1/6^2, \ldots, 1/18^2$. Then the corresponding coarse mesh values are obtained. These fine mesh values also are used in the numerical experiment for one-level finite element method. The numerical results are displayed in Tables 1, 2, 3, and 4, from which the observations and conclusions are presented as follows.

(i) Based on Table 1, the numerical convergence orders reach the optimal orders which coincide with the theoretical results derived in Theorems 4 and 5, namely, $O(h)$ for the velocity in H^1-norm and the pressure in L^2-norm and $O(h^2)$ for the velocity in L^2-norm. We also observe that if $h = 1/18^2$, in this case, the standard one-level method can not work and does not obtain the predicted numerical results.

(ii) From Tables 2–4, we can see that if $H = h^{1/2}$, all three two-level Stokes/Oseen/Newton iteration methods can reach the optimal convergence orders of $O(h)$ for both velocity and pressure, in H^1-norm and L^2-norm, respectively, as proven in Theorems 6–8. Besides, we find that these methods can achieve the optimal convergence orders of $O(h^2)$ for velocity in the sense of L^2-norm as expected.

(iii) From the view of computational cost, we can obviously observe by comparing Table 1 and Tables 2–4 that these two-level iteration methods significantly save CPU time compared with the one-level method and, meanwhile, obtain nearly the same approximation results.

The numerical results for two-level Newton correction method also are displayed in Table 5. Based on Theorem 10, the optimal convergence order $O(h)$ for the velocity in H^1-norm and the pressure in L^2-norm can be reached as $H \approx h^{1/4}$, which has been reflected in Table 5. However, this Newton correction method only can save about 85% CPU time compared with the one-level method. The reason is that the Newton correction method needs two-step computation in the fine mesh.

Finally, we show the contour plots of the exact solution and the numerical solution to exhibit the approximation profiles. Figures 1 and 2 display the exact solution and the numerical solution by one-level stabilized method. Besides, as to the two-level method, here only the numerical solution by Newton iteration method is displayed in Figure 3. From these three groups of contour plots, we can observe the good coincidence with each other to illustrate the stability of the present stabilized methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This material is based upon work funded by the National Natural Science Foundation of China under Grant nos. 10901122 and 11001205 and by Zhejiang Provincial Natural Science Foundation of China under Grant no. LY12A01015.

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