Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 697974, 10 pages
http://dx.doi.org/10.1155/2014/697974

Research Article

Variational Integrals of a Class of Nonhomogeneous $\mathscr{A}$-Harmonic Equations

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Received 20 September 2013; Accepted 10 January 2014; Published 27 February 2014

Academic Editor: Shusen Ding

We introduce a class of variational integrals whose Euler equations are nonhomogeneous $\mathscr{A}$-harmonic equations. We investigate the relationship between the minimization problem and the Euler equation and give a simple proof of the existence of some nonhomogeneous $\mathscr{A}$-harmonic equations by applying direct methods of the calculus of variations. Besides, we establish some interesting results on variational integrals.

1. Introduction

In this paper, we study the variational integral of the form

$$I_{(F_1,F_2)}(u,E) = \int_E (F_1(x,\nabla u(x)) + F_2(x,u(x)))\,dx, \quad (1)$$

whose Euler equations are nonhomogeneous $\mathscr{A}$-harmonic equations

$$-\operatorname{div}\mathscr{A}(x\nabla u) + \mathscr{B}(x,u) = 0, \quad (2)$$

where $F_1 : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, $F_2 : \mathbb{R}^n\times\mathbb{R} \to \mathbb{R}$, $\mathscr{A} : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, and $\mathscr{B} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are operators satisfying some assumptions. There are many literatures on (2) and a large of useful results have been established; see [1–3] and their references. We investigate the relationship between the minimization of $I_{(F_1,F_2)}(u,E)$ and solutions of the Euler equation. Based on that, we give a simple proof of the existence of some nonhomogeneous $\mathscr{A}$-harmonic equations by applying direct methods of the calculus of variations. Besides, we establish some interesting results on variational integrals. The results of this paper make the theory on (2) easier to comprehend.

We recall the weighted Sobolev spaces $H^{1,p}(\Omega;\mu)$ which are adopted in [4].

Let $\mathbb{R}^n$ be the real Euclidean space with the dimension n, $n \ge 2$. Throughout this paper, Ω will denote an open subset of $\mathbb{R}^n$ and $1 < p < \infty$. Let w be a locally integrable, nonnegative function in $\mathbb{R}^n$. A Radon measure μ is canonically associated with the weight w,

$$\mu(E) = \int_E w(x)\,dx. \quad (3)$$

Thus $d\mu(x) = w(x)dx$, where dx is the n-dimensional Lebesgue measure. In this paper, unless otherwise stated, we always assume that μ is a p-admissible measure and $d\mu(x) = w(x)dx$; see [4].

Let $L^p(\Omega;\mu) = \{\varphi : \Omega \to \mathbb{R} : \int_\Omega |\varphi|^p d\mu < \infty\}$ and $L^p(\Omega;\mu;\mathbb{R}^n) = \{\varphi : \Omega \to \mathbb{R}^n : \int_\Omega |\varphi|^p d\mu < \infty\}$. Denote the norm of $L^p(\Omega;\mu)$ and $L^p(\Omega;\mu;\mathbb{R}^n)$ by $\|\cdot\|_p$,

$$\|\phi\|_p = \left(\int_\Omega |\phi|^p d\mu\right)^{1/p}, \quad (4)$$

where $\phi \in L^p(\Omega;\mu)$ (or $L^p(\Omega;\mu;\mathbb{R}^n)$).

For $\varphi \in C^\infty(\Omega)$, let

$$\|\varphi\|_{1,p} = \left(\int_\Omega |\varphi|^p d\mu\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p d\mu\right)^{1/p}, \quad (5)$$

where $\nabla\varphi = (\partial_1\varphi,\ldots,\partial_n\varphi)$ is the gradient of φ. The Sobolev space $H^{1,p}(\Omega;\mu)$ is defined to be the completion of the set $\{\varphi \in C^\infty(\Omega) : \|\varphi\|_{1,p} < \infty\}$ with respect to the norm $\|\cdot\|_{1,p}$. In other words, $u \in H^{1,p}(\Omega;\mu)$ if and only if $u \in L^p(\Omega;\mu)$

and there is a function $v \in L^p(\Omega;\mu;\mathbb{R}^n)$ and a sequence $\varphi_i \in C^\infty(\Omega)$, such that

$$\int_\Omega |\varphi_i - u|^p d\mu \longrightarrow 0, \qquad \int_\Omega |\nabla\varphi_i - v|^p d\mu \longrightarrow 0, \qquad i \longrightarrow \infty. \tag{6}$$

We call v the gradient of u in $H^{1,p}(\Omega;\mu)$ and write $v = \nabla u$.

The space $H_0^{1,p}(\Omega;\mu)$ is the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega;\mu)$. Obviously, $H^{1,p}(\Omega;\mu)$ and $H_0^{1,p}(\Omega;\mu)$ are Banach space with respect to the norm $\|\cdot\|_{1,p}$. Moreover, $\|\cdot\|_{1,p}$ is uniformly convex and the Sobolev space $H^{1,p}(\Omega;\mu)$ and $H_0^{1,p}(\Omega;\mu)$ are reflexive; see [5] for details.

The corresponding local Sobolev space $H_{\mathrm{loc}}^{1,p}(\Omega;\mu)$ is defined in the obvious manner: a function u is in $H_{\mathrm{loc}}^{1,p}(\Omega;\mu)$ if and only if u is in $H_{\mathrm{loc}}^{1,p}(\Omega';\mu)$ each open set $\Omega' \Subset \Omega$.

2. Variational Integrals

Suppose that E is a measurable set and that $u \in H_{\mathrm{loc}}^{1,p}(\Omega;\mu)$ for an open neighborhood Ω of E. Then, we have the following variational integral:

$$I_{(F_1,F_2)}(u,E) = \int_E (F_1(x,\nabla u(x)) + F_2(x,u(x)))\,dx, \tag{7}$$

where $F_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a variational kernel satisfying the following assumptions for some constants $0 < \gamma_1 \le \delta_1 < \infty$:

$$\text{the mapping } x \longmapsto F_1(x,\xi) \text{ is measurable } \quad \forall \xi \in \mathbb{R}^n; \tag{8}$$

for a.e. $x \in \mathbb{R}^n$;

$$\gamma_1 w(x)|\xi|^p \le F_1(x,\xi) \le \delta_1 w(x)|\xi|^p, \quad \xi \in \mathbb{R}^n, \tag{9}$$

$$\text{the mapping } \xi \longmapsto F_1(x,\xi) \text{ is strictly convex and differentiable,} \tag{10}$$

$$F_1(x,\lambda\xi) = |\lambda|^p F_1(x,\xi), \quad \lambda \in \mathbb{R},\ \xi \in \mathbb{R}^n, \tag{11}$$

and $F_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is also a variational kernel satisfying the following assumptions for some constants $0 < \gamma_2 \le \delta_2 < \infty$:

$$\text{the mapping } x \longmapsto F_2(x,t) \text{ is measurable } \quad \forall t \in \mathbb{R}; \tag{12}$$

for a.e. $x \in \mathbb{R}^n$;

$$\gamma_2 w(x)|t|^p \le F_2(x,t) \le \delta_2 w(x)|t|^p, \quad t \in \mathbb{R}; \tag{13}$$

$$\text{the mapping } t \longmapsto F_2(x,t) \text{ is convex and differentiable.} \tag{14}$$

Remark 1. Note that a convex function is differential if and only if it is continuously differentiable; see [6]. Thus, by assumptions (10) and (14), mappings $\xi \mapsto F_1(x,\xi)$ and $t \mapsto F_2(x,t)$ are continuously differentiable for a.e. x. Denote by $\nabla_\xi F_1(x,\cdot)$ the usual gradient of F_1 with respect to the second variable and by $\partial_t F_2(x,\cdot)$ the usual derivative of F_1 with respect to the second variable. Obviously, $\nabla_\xi F_1(x,\cdot)$ and $\partial_t F_2(x,\cdot)$ exist for a.e. $x \in \mathbb{R}^n$.

The value $I_{(F_1,F_2)}(u,E)$ lies in the interval $[0,\infty]$ and by assumptions (9) and (13), $I_{(F_1,F_2)}(u,E) < \infty$ if and only if $u \in L^p(E;\mu)$ and $\nabla u \in L^p(E;\mu)$; that is, $u \in H^{1,p}(E;\mu)$.

The convexity assumptions (10) and (14) can imply the following useful inequalities.

Lemma 2. *For a.e. $x \in \mathbb{R}^n$,*

$$F_1(x,\xi_1) - F_1(x,\xi_2) > \nabla_\xi F_1(x,\xi_2)\cdot(\xi_1 - \xi_2), \tag{15}$$

$$F_2(x,t_1) - F_2(x,t_2) \ge \partial_t F_2(x,t_2)(t_1 - t_2), \tag{16}$$

whenever $\xi_1,\xi_2 \in \mathbb{R}^n$, $\xi_1 \ne \xi_2$, and $t_1,t_2 \in \mathbb{R}$.

Proof. The proof is based on assumptions (10) and (14) and the definition of directional derivative. Here, we only show the proof of the first inequality (15) and the other is similar.

Fix $x \in \mathbb{R}^n$ such that the mapping $\xi \mapsto F_1(x,\xi)$ is strictly convex and differentiable. Then, for $0 < s < 1$,

$$\begin{aligned} F_1(x,\xi_2 + s(\xi_1 - \xi_2)) &= F_1(x,(1-s)\xi_2 + s\xi_1) \\ &< (1-s)F_1(x,\xi_2) + sF_1(x,\xi_1). \end{aligned} \tag{17}$$

Setting $\xi = \xi_1 - \xi_2$, we can get

$$F_1(x,\xi_2 + s\xi) - F_1(x,\xi_2) < s(F_1(x,\xi_2+\xi) - F_1(x,\xi_2)). \tag{18}$$

Dividing by s and subtracting $\nabla_\xi F_1(x,\xi_2)\cdot\xi$ from both sides, we obtain that

$$\begin{aligned} &\frac{F_1(x,\xi_2 + s\xi) - F_1(x,\xi_2)}{s} - \nabla_\xi F_1(x,\xi_2)\cdot\xi \\ &\quad < F_1(x,\xi_2+\xi) - F_1(x,\xi_2) - \nabla_\xi F_1(x,\xi_2)\cdot\xi. \end{aligned} \tag{19}$$

By the definition of directional derivative, we have that

$$\lim_{s\to 0^+} \frac{F_1(x,\xi_2 + s\xi) - F_1(x,\xi_2)}{s} = \nabla_\xi F_1(x,\xi_2)\cdot\xi. \tag{20}$$

Then, we can get that $F_1(x,\xi_1) - F_1(x,\xi_2) \ge \nabla_\xi F_1(x,\xi_2)\cdot(\xi_1 - \xi_2)$.

Suppose that there exist $\xi_1,\xi_2 \in \mathbb{R}^n$, $\xi_1 \ne \xi_2$, such that $F_1(x,\xi_1) - F_1(x,\xi_2) = \nabla_\xi F_1(x,\xi_2)\cdot(\xi_1 - \xi_2)$, let $\xi = (1/2)(\xi_1 + \xi_2)$, and then we can obtain that

$$\begin{aligned} F_1(x,\xi) &= F_1\left(x, \frac{1}{2}(\xi_1+\xi_2)\right) \\ &< \frac{1}{2}(F_1(x,\xi_1) + F_1(x,\xi_2)) \\ &= F_1(x,\xi_2) + \frac{1}{2}\nabla_\xi F_1(x,\xi_2)\cdot(\xi_1 - \xi_2). \end{aligned} \tag{21}$$

On the other hand, since $\xi \neq \xi_2$, we have that

$$\begin{aligned} F_1(x,\xi) &\geq F_1(x,\xi_2) + \nabla_\xi F_1(x,\xi_2)\cdot(\xi-\xi_2) \\ &= F_1(x,\xi_2) + \frac{1}{2}\nabla_\xi F_1(x,\xi_2)\cdot(\xi_1-\xi_2). \end{aligned} \tag{22}$$

Then, (22) contradicts (21) and the lemma follows. □

3. Nonhomogeneous $\mathscr{A}$-Harmonic Equations and the Obstacles Problem

The following nonlinear elliptic equation:

$$-\operatorname{div}\mathscr{A}(x,\nabla u) + \mathscr{B}(x,u) = 0 \tag{23}$$

is called the nonhomogeneous $\mathscr{A}$-harmonic equation, where $\mathscr{A}:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is an operator satisfying the following assumptions for some constants $0<\alpha\leq\beta<\infty$:

$$\begin{aligned} &\text{the mapping } x \longmapsto \mathscr{A}(x,\xi) \text{ is measurable} \quad \forall\xi\in\mathbb{R}^n, \\ &\text{the mapping } \xi \longmapsto \mathscr{A}(x,\xi) \text{ is continuous for a.e. } x\in\mathbb{R}^n; \end{aligned} \tag{24}$$

for all $\xi\in\mathbb{R}^n$ and almost all $x\in\mathbb{R}^n$,

$$\mathscr{A}(x,\xi)\cdot\xi \geq \alpha w(x)|\xi|^p, \tag{25}$$

$$|\mathscr{A}(x,\xi)| \leq \beta w(x)|\xi|^{p-1}, \tag{26}$$

$$(\mathscr{A}(x,\xi_1)-\mathscr{A}(x,\xi_2))\cdot(\xi_1-\xi_2) > 0, \tag{27}$$

whenever $\xi_1,\xi_2\in\mathbb{R}^n$, $\xi_1\neq\xi_2$, and

$$\mathscr{A}(x,\lambda\xi) = \lambda|\lambda|^{p-2}\mathscr{A}(x,\xi), \tag{28}$$

whenever $\lambda\in\mathbb{R}$, $\lambda\neq 0$, and $\mathscr{B}:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is also an operator satisfying the following similar assumptions for some constants $0<\gamma\leq\delta<\infty$:

$$\begin{aligned} &\text{the mapping } x \longmapsto \mathscr{B}(x,t) \text{ is measurable} \quad \forall t\in\mathbb{R}, \\ &\text{the mapping } t \longmapsto \mathscr{A}(x,t) \text{ is continuous} \\ &\qquad\qquad \text{for a.e. } x\in\mathbb{R}^n; \end{aligned} \tag{29}$$

for all $t\in\mathbb{R}$ and almost all $x\in\mathbb{R}^n$,

$$\mathscr{B}(x,t)\,t \geq \gamma w(x)|t|^p, \tag{30}$$

$$|\mathscr{B}(x,t)| \leq \delta w(x)|t|^{p-1}, \tag{31}$$

$$(\mathscr{B}(x,t_1)-\mathscr{B}(x,t_2))(t_1-t_2) \geq 0, \tag{32}$$

whenever $t_1,t_2\in\mathbb{R}$.

Definition 3. A function $u\in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ is a (weak) solution of (2) in Ω if

$$\int_\Omega (\mathscr{A}(x,\nabla u)\cdot\nabla\varphi + \mathscr{B}(x,u)\varphi)\,dx = 0 \tag{33}$$

for all $\varphi\in C_0^\infty(\Omega)$. A function $u\in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ is a supersolution of (2) in Ω if

$$\int_\Omega (\mathscr{A}(x,\nabla u)\cdot\nabla\varphi + \mathscr{B}(x,u)\varphi)\,dx \geq 0, \tag{34}$$

whenever $\varphi\in C_0^\infty(\Omega)$ is nonnegative. A function $u\in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ is a subsolution of (2) in Ω if

$$\int_\Omega (\mathscr{A}(x,\nabla u)\cdot\nabla\varphi + \mathscr{B}(x,u)\varphi)\,dx \leq 0, \tag{35}$$

whenever $\varphi\in C_0^\infty(\Omega)$ is nonnegative.

Next is the obstacles problem associated with nonhomogeneous $\mathscr{A}$-harmonic equations (2).

Suppose that Ω is bounded in $\mathbb{R}^n$, $\psi:\Omega\to[-\infty,\infty]$ is a function, and $\vartheta\in H^{1,p}(\Omega;\mu)$. Let

$$\begin{aligned} \mathscr{K}_{\psi,\vartheta} = \mathscr{K}_{\psi,\vartheta}(\Omega) = \{&v\in H^{1,p}(\Omega;\mu) : v\geq\psi \\ &\text{a.e. in } \Omega, v-\vartheta\in H_0^{1,p}(\Omega;\mu)\}. \end{aligned} \tag{36}$$

If $\psi=\vartheta$, write $\mathscr{K}_{\psi,\psi}(\Omega)=\mathscr{K}_\psi(\Omega)$.

The problem is to find a function u in $\mathscr{K}_{\psi,\vartheta}$ such that

$$\int_\Omega (\mathscr{A}(x,\nabla u)\cdot(\nabla v-\nabla u) + \mathscr{B}(x,u)(v-u))\,dx \geq 0, \tag{37}$$

whenever $v\in\mathscr{K}_{\psi,\vartheta}$. We call the function ψ an obstacle.

Definition 4. If a function $u\in\mathscr{K}_{\psi,\vartheta}(\Omega)$ satisfies (37) for all $v\in\mathscr{K}_{\psi,\vartheta}(\Omega)$, we say that u is a solution to the obstacle problem with obstacle ψ and boundary values ϑ or a solution to the obstacle problem in $\mathscr{K}_{\psi,\vartheta}(\Omega)$.

If u is a solution to the obstacle problem in $\mathscr{K}_{\psi,u}(\Omega)$, we say that u is a solution to the obstacle problem with obstacle ψ.

Proposition 5. *(1) A solution u to the obstacle problem is always a supersolution to* (2) *in Ω.*

(2) If u is a supersolution to (2) *in Ω, u is a solution to the obstacle problem in $\mathscr{K}_{u,u}(D)$ for each open sets $D\Subset\Omega$. Moreover, if Ω is bounded and $u\in H^{1,p}(\Omega;\mu)$, u is a solution to the obstacle problem in $\mathscr{K}_{u,u}(\Omega)$.*

(3) A solution u to the obstacle problem in $\mathscr{K}_{-\infty,u}(\Omega)$ is a solution to (2) *in Ω.*

(4) If u is a solution to (2) *in Ω, u is a solution to the obstacle problem in $\mathscr{K}_{-\infty,u}(D)$ for each open set $D\Subset\Omega$. Moreover, if Ω is bounded $u\in H^{1,p}(\Omega;\mu)$, u is a solution to the obstacle problem in $\mathscr{K}_{-\infty,u}(\Omega)$.*

Proof. By the definition of supersolution and solution to (2) and the definition of solution to the obstacle problem, it is easy to prove that Proposition 5 is true. Here, we only give a proof of (1).

Suppose u is a solution to the obstacle problem in $\mathscr{K}_{\psi,\vartheta}(\Omega)$, and u is obviously in $H^{1,p}(\Omega;\mu)$. For all nonnegative $\varphi\in C_0^\infty(\Omega)$, $u+\varphi\in H^{1,p}(\Omega;\mu)$, $u+\varphi\geq u\geq\vartheta$ a.e. in Ω, and

$$u+\varphi-\vartheta = (u-\vartheta)+\varphi \in H_0^{1,p}(\Omega;\mu). \tag{38}$$

Then, $u + \varphi \in \mathscr{K}_{\psi,\vartheta}(\Omega)$. By (37), we can get

$$\int_{\Omega} (\mathscr{A}(x,\nabla u)\cdot\nabla\varphi + \mathscr{B}(x,u)\varphi)\,dx = \int_{\Omega} (\mathscr{A}(x,\nabla u)\cdot(\nabla u + \nabla\varphi - \nabla u) + \mathscr{B}(x,u)(u+\varphi-u))\,dx \ge 0. \tag{39}$$

Therefore, u is a supersolution to (2) in Ω. □

Theorem 6 (see [3]). *Suppose u is a solution to the obstacle problem in $\mathscr{K}_{\psi,\vartheta}(\Omega)$. If $v \in H^{1,p}(\Omega;\mu)$ is a supersolution to* (2) *in Ω, such that* $\min\{u,v\} \in \mathscr{K}_{\psi,\vartheta}(\Omega)$*, then $v \ge u$ a.e. in Ω.*

4. Relationship between the Minimization Problem and the Euler Equation

In this section, we establish that the variational integral (7) gives rise to an equation of the type (2) as its Euler equation, where the mappings $\mathscr{A}(x,\xi) = \nabla_\xi F_1(x,\xi)$ and $\mathscr{B}(x,t) = \partial_t F_2(x,t)$ satisfy the structural assumptions (24)–(32).

Theorem 7. *Suppose that F_1 and F_2 are two variational kernels satisfying* (8)–(14) *and let $\mathscr{A}(x,\xi) = \nabla_\xi F_1(x,\xi)$ and $\mathscr{B}(x,t) = \partial_t F_2(x,t)$. Then, $\mathscr{A}$ and $\mathscr{B}$ satisfy assumptions* (24)–(32) *with $\alpha = \gamma_1$, $\beta = 2^p\delta_1$, $\gamma = \gamma_2$, and $\delta = 2^p\delta_2$.*

Proof. For points x for which (9), (10), (11), (13), and (14) do not hold, we are free to define $\mathscr{A}(x,\xi)$ and $\mathscr{B}(x,t)$ arbitrarily. Fix $x \in \mathbb{R}^n$ such that F_1 satisfies (9)–(11) and F_2 satisfies (13) and (14).

(i) By the definition of partial derivative, the kth coordinate of $\mathscr{A}(x,\xi)$ equals

$$\lim_{i\to\infty} i\left(F_1\left(x,\xi+\frac{e_k}{i}\right) - F_1(x,\xi)\right). \tag{40}$$

Then, the mapping $x \mapsto \mathscr{A}(x,\xi)$ is measurable. Moreover, by (10), $\xi \mapsto F_1(x,\xi)$ is continuously differentiable. Then, $\xi \mapsto \mathscr{A}(x,\xi)$ is continuous and $\mathscr{A}$ satisfies (24).

(ii) If $\xi \ne 0$, then $\xi' = 0 \ne \xi$. By (15), we have that

$$F_1(x,\xi') - F_1(x,\xi) > \nabla_\xi F_1(x,\xi)\cdot(\xi'-\xi) = \mathscr{A}(x,\xi)\cdot(\xi'-\xi). \tag{41}$$

Since $\xi' = 0$ and $F_1(x,\xi') = F_1(x,0) = 0$, we can obtain that $-F_1(x,\xi) > \mathscr{A}(x,\xi)\cdot(-\xi)$. Then,

$$\mathscr{A}(x,\xi)\cdot\xi > F_1(x,\xi) \ge \gamma_1 w(x)|\xi|^p. \tag{42}$$

If $\xi = 0$, $\mathscr{A}(x,\xi)\cdot\xi = 0 = \gamma_1 w(x)|\xi|^p$. Therefore, $\mathscr{A}$ satisfies (25).

(iii) If $\xi \ne 0$, then $\mathscr{A}(x,\xi) \ne 0$ by (25). Write

$$v = \frac{\mathscr{A}(x,\xi)}{|\mathscr{A}(x,\xi)|} \tag{43}$$

and $\xi_1 = \xi + |\xi|v$. Then, $\xi_1 = \xi + |\xi|v \ne \xi$ and $|\xi_1| \le |\xi| + |\xi||v| = 2|\xi|$. Applying (15) with ξ_1 and ξ, we can obtain that

$$\begin{aligned}|\mathscr{A}(x,\xi)||\xi| &= \mathscr{A}(x,\xi)\cdot\frac{|\xi|\mathscr{A}(x,\xi)}{|\mathscr{A}(x,\xi)|} \\ &= \mathscr{A}(x,\xi)\cdot(|\xi|v) = \mathscr{A}(x,\xi)\cdot(\xi_1-\xi) \\ &< F_1(x,\xi_1) - F_1(x,\xi) \\ &\le F_1(x,\xi_1) \le \delta_1 w(x)|\xi_1|^p \le 2^p\delta_1 w(x)|\xi|^p.\end{aligned} \tag{44}$$

Since $\xi \ne 0$, we have that $|\mathscr{A}(x,\xi)| < 2^p\delta_1 w(x)|\xi|^{p-1}$.

If $\xi = 0$, we just need to verify that $\mathscr{A}(x,0) = 0$. If not, for each $k \in \mathbb{N}$, write

$$v_k = \frac{\mathscr{A}(x,0)}{k|\mathscr{A}(x,0)|}, \tag{45}$$

and $|v_k| = 1/k \ne 0$. By (15), $F_1(x,v_k) = F_1(x,v_k) - F_1(x,0) > \mathscr{A}(x,0)\cdot v_k$. Therefore,

$$\begin{aligned}\frac{1}{k}|\mathscr{A}(x,0)| &= \mathscr{A}(x,0)\cdot\frac{\mathscr{A}(x,0)}{k|\mathscr{A}(x,0)|} \\ &= \mathscr{A}(x,0)\cdot v_k < F_1(x,v_k) \le \delta_1 w(x)|v_k|^p \\ &= \delta_1 w(x)k^{-p}.\end{aligned} \tag{46}$$

Thus, $|\mathscr{A}(x,0)| \le \delta_1 w(x)k^{1-p}$ for $k \in \mathbb{N}$. The right hand side goes to zero as $k \to \infty$ and $\mathscr{A}(x,0) = 0$. Then, $\mathscr{A}$ satisfies (26).

(iv) For $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \ne \xi_2$, by (15), we have that

$$F_1(x,\xi_1) - F_1(x,\xi_2) > \nabla_\xi F_1(x,\xi_2)\cdot(\xi_1-\xi_2) = \mathscr{A}(x,\xi_2)\cdot(\xi_1-\xi_2), \tag{47}$$

$$F_1(x,\xi_2) - F_1(x,\xi_1) > \nabla_\xi F_1(x,\xi_1)\cdot(\xi_2-\xi_1) = \mathscr{A}(x,\xi_1)\cdot(\xi_2-\xi_1). \tag{48}$$

Combining (47) with (48), we obtain that

$$\begin{aligned}0 &> \mathscr{A}(x,\xi_2)\cdot(\xi_1-\xi_2) + \mathscr{A}(x,\xi_1)\cdot(\xi_2-\xi_1) \\ &= -(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2))\cdot(\xi_1-\xi_2).\end{aligned} \tag{49}$$

Then, $(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2))\cdot(\xi_1-\xi_2) > 0$ and $\mathscr{A}$ satisfies (27).

(v) For $\lambda \in \mathbb{R}$, $\lambda \ne 0$, $F_1(x,\lambda\xi_1) = |\lambda|^p F_1(x,\xi_1)$. Taking partial derivative from both sides with respect to ξ yields

$$\lambda\mathscr{A}(x,\lambda\xi) = \lambda\nabla_\xi F(x,\lambda\xi) = |\lambda|^p\nabla_\xi F(x,\xi) = |\lambda|^p\mathscr{A}(x,\xi). \tag{50}$$

Then, $\mathscr{A}(x,\lambda\xi) = \lambda|\lambda|^{p-2}\mathscr{A}(x,\xi)$.

By the similar argument, we can obtain that $\mathscr{B}(x,t)$ satisfies assumptions (29)–(32). □

The next theorem shows that minimizers of the variational integral $I_{(F_1,F_2)}(u,\Omega)$ are solutions to the corresponding Euler equation and vice versa.

Theorem 8. *Suppose that $K \subset H^{1,p}(\Omega;\mu)$ is a convex set and $u \in K$. Then,*

$$I_{(F_1,F_2)}(u,\Omega) = \min\left\{I_{(F_1,F_2)}(v,\Omega) : v \in K\right\} \tag{51}$$

if and only if

$$\int_\Omega \left(\nabla_\xi F_1(x,\nabla u)\cdot(\nabla v - \nabla u) + \partial_t F_2(x,u)(v-u)\right)dx \ge 0 \tag{52}$$

for all $v \in K$.

Proof. (i) By Lemma 2, we have that

$$\begin{aligned}\nabla_\xi F_1(x,\nabla u)\cdot(\nabla v - \nabla u) &\le F_1(x,\nabla v) - F_1(x,\nabla u)\\ \partial_t F_2(x,u)(v-u) &\le F_2(x,v) - F_2(x,u)\end{aligned} \tag{53}$$

for all $v \in K \subset H^{1,p}(\Omega;\mu)$. Then,

$$\begin{aligned}0 &\le \int_\Omega \left(\nabla_\xi F_1(x,\nabla u)\cdot(\nabla v - \nabla u) + \partial_t F_2(x,u)(v-u)\right)dx\\ &\le \int_\Omega \left(F_1(x,\nabla v) - F_1(x,\nabla u) + F_2(x,v) - F_2(x,u)\right)dx\\ &\le \int_\Omega \left(F_1(x,\nabla v) + F_2(x,v)\right)dx\\ &\quad - \int_\Omega \left(F_1(x,\nabla u) + F_2(x,u)\right)dx\\ &= I_{(F_1,F_2)}(v,\Omega) - I_{(F_1,F_2)}(u,\Omega).\end{aligned} \tag{54}$$

That is, $I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(v,\Omega)$. Therefore, $I_{(F_1,F_2)}(u,\Omega) = \min\{I_{(F_1,F_2)}(v,\Omega) : v \in K\}$.

(ii) Fix $v \in K$ and let $\varphi = v-u$. Then, since K is convex and by (51), we have that

$$u + \varepsilon\varphi = (1-\varepsilon)u + \varepsilon v \in K \tag{55}$$

for all $0 < \varepsilon < 1$ and $I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(u+\varepsilon\varphi,\Omega)$. Therefore,

$$\int_\Omega \left(\frac{F_1(x,\nabla u + \varepsilon\nabla\varphi) - F_1(x,\nabla u)}{\varepsilon} + \frac{F_2(x,u+\varepsilon\varphi) - F_2(x,u)}{\varepsilon}\right)dx \ge 0. \tag{56}$$

By assumptions (10) and (14) and the definition of directional derivative,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{F_1(x,\nabla u + \varepsilon\nabla\varphi) - F_1(x,\nabla u)}{\varepsilon} &= \nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi,\\ \lim_{\varepsilon\to 0}\frac{F_2(x,u+\varepsilon\varphi) - F_2(x,u)}{\varepsilon} &= \partial_t F_2(x,u)\varphi\end{aligned} \tag{57}$$

for a.e. $x \in \mathbb{R}^n$. By the mean value theorem, there exists a real number $t_0 \in (0,1)$, such that

$$\begin{aligned}&F_1(x,\nabla u + \varepsilon\nabla\varphi) + F_2(x,u+\varepsilon\varphi) - F_1(x,\nabla u) - F_2(x,u)\\ &\quad= \nabla_\xi F_1(x,\nabla u + t_0\varepsilon\nabla\varphi)\cdot\varepsilon\nabla\varphi + \partial_t F_2(x,u+t_0\varepsilon\varphi)\varepsilon\varphi.\end{aligned} \tag{58}$$

Then, we have

$$\begin{aligned}&\frac{F_1(x,\nabla u + \varepsilon\nabla\varphi) - F_1(x,\nabla u)}{\varepsilon} + \frac{F_2(x,u+\varepsilon\varphi) - F_2(x,u)}{\varepsilon}\\ &\quad= \nabla_\xi F_1(x,\nabla u + t_0\varepsilon\nabla\varphi)\cdot\nabla\varphi + \partial_t F_2(x,u+t_0\varepsilon\varphi)\varphi.\end{aligned} \tag{59}$$

By Theorem 7, the following inequalities hold:

$$\begin{aligned}&\left|\nabla_\xi F_1(x,\nabla u + t_0\varepsilon\nabla\varphi)\cdot\nabla\varphi\right|\\ &\quad\le \left|\nabla_\xi F_1(x,\nabla u + t_0\varepsilon\nabla\varphi)\right||\nabla\varphi|\\ &\quad\le 2^p\delta_1 w(x)\left|\nabla u + t_0\varepsilon\nabla\varphi\right|^{p-1}|\nabla\varphi|\\ &\quad\le 2^{2p-1}\delta_1 w(x)\left(|\nabla u|^{p-1} + |t_0\varepsilon\nabla\varphi|^{p-1}\right)|\nabla\varphi|\\ &\quad\le 2^{2p-1}\delta_1 w(x)\left(|\nabla u|^{p-1}|\nabla\varphi| + |\nabla\varphi|^p\right),\\ &\left|\partial_t F_2(x,u+t_0\varepsilon\varphi)\varphi\right| \le \left|\partial_t F_2(x,u+t_0\varepsilon\varphi)\right||\varphi|\\ &\quad\le 2^p\delta_2 w(x)\left|u+t_0\varepsilon\varphi\right|^{p-1}|\varphi|\\ &\quad\le 2^{2p-1}\delta_2 w(x)\left(|u|^{p-1}|\varphi| + |\varphi|^p\right),\\ &\left|\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi\right| \le 2^p\delta_1 w(x)|\nabla u|^{p-1}|\nabla\varphi|,\\ &\left|\partial_t F_2(x,u)\varphi\right| \le 2^p\delta_2 w(x)|u|^{p-1}|\varphi|.\end{aligned} \tag{60}$$

Write $g(x) = 2^{2p-1}\delta_1 w(x)(|\nabla u|^{p-1}|\nabla\varphi| + |\nabla\varphi|^p) + 2^{2p-1}\delta_2 w(x)(|u|^{p-1}|\varphi| + |\varphi|^p)$ and by $u,\varphi \in H^{1,p}(\Omega;\mu)$, we have that

$$\begin{aligned}\int_\Omega g(x)\,dx &\le 2^{2p-1}\delta_1\int_\Omega w(x)\left(|\nabla u|^{p-1}|\nabla\varphi| + |\nabla\varphi|^p\right)dx\\ &\quad+ 2^{2p-1}\delta_2\int_\Omega w(x)\left(|u|^{p-1}|\varphi| + |\varphi|^p\right)dx\\ &= 2^{2p-1}\delta_1\int_\Omega |\nabla u|^{p-1}|\nabla\varphi|\,d\mu + 2^{2p-1}\delta_1\\ &\quad\times\int_\Omega |\nabla\varphi|^p d\mu + 2^{2p-1}\delta_2\\ &\quad\times\int_\Omega |u|^{p-1}|\varphi|\,d\mu + 2^{2p-1}\delta_2\int_\Omega |\varphi|^p d\mu\end{aligned}$$

$$= 2^{2p-1}\delta_1\left(\int_\Omega |\nabla u|^p d\mu\right)^{1-1/p}\left(\int_\Omega |\nabla\varphi|^p d\mu\right)^{1/p} + 2^{2p-1}\delta_1\int_\Omega |\nabla\varphi|^p d\mu + 2^{2p-1}\delta_2\left(\int_\Omega |u|^p d\mu\right)^{1-1/p}\left(\int_\Omega |\varphi|^p d\mu\right)^{1/p} + 2^{2p-1}\delta_2\int_\Omega |\varphi|^p d\mu < \infty; \tag{61}$$

that is, $g \in L^1(\Omega; dx)$. Then, we can get the following conditions:

$$\left|\frac{F_1(x,\nabla u+\varepsilon\nabla\varphi)-F_1(x,\nabla u)}{\varepsilon}+\frac{F_2(x,u+\varepsilon\varphi)-F_2(x,u)}{\varepsilon}\right| \le g(x),$$
$$\left|\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi+\partial_t F_2(x,u)\varphi\right| \le g(x),$$
$$\frac{F_1(x,\nabla u+\varepsilon\nabla\varphi)-F_1(x,\nabla u)}{\varepsilon}+\frac{F_2(x,u+\varepsilon\varphi)-F_2(x,u)}{\varepsilon} \longrightarrow \nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi+\partial_t F_2(x,u)\varphi \tag{62}$$

as $\varepsilon \to 0$. By the Lebesgue's dominated convergence theorem, we obtain that

$$\int_\Omega\left(\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi+\partial_t F_2(x,u)\varphi\right)dx = \lim_{\varepsilon\to 0}\int_\Omega\left(\frac{F_1(x,\nabla u+\varepsilon\nabla\varphi)-F_1(x,\nabla u)}{\varepsilon}+\frac{F_2(x,u+\varepsilon\varphi)-F_2(x,u)}{\varepsilon}\right)dx \ge 0. \tag{63}$$

The theorem is proved. □

5. (F_1,F_2)-Extremals and (F_1,F_2)-Superextremals with Obstacles

Definition 9. A function $u \in H^{1,p}(\Omega;\mu)$ is called an (F_1,F_2)-extremal in Ω with boundary values $\vartheta \in H^{1,p}(\Omega;\mu)$ if $u-\vartheta \in H_0^{1,p}(\Omega;\mu)$ and

$$I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(v,\Omega) \tag{64}$$

whenever $v-\vartheta \in H_0^{1,p}(\Omega;\mu)$. A function $u \in H_{\mathrm{loc}}^{1,p}(\Omega;\mu)$ is called a (free) (F_1,F_2)-extremal in Ω if u is an (F_1,F_2)-extremal with boundary values u in each open set $D \Subset \Omega$.

It is immediate that an (F_1,F_2)-extremal with boundary values is a free (F_1,F_2)-extremal.

Theorem 10. *Suppose that $u \in H^{1,p}(\Omega;\mu)$ is a free (F_1,F_2)-extremal in Ω. Then,*

$$I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(v,\Omega) \tag{65}$$

whenever $v-u \in H_0^{1,p}(\Omega;\mu)$.

Proof. For $\varphi \in C_0^\infty(\Omega)$, φ has compact support. Choose an open set $D \Subset \Omega$ such that $\operatorname{spt}\varphi \subset D$. Then, $\varphi \in C_0^\infty(D)$ and u is an (F_1,F_2)-extremal with boundary values u in D. Since $(u+\varphi)-u \in H_0^{1,p}(D;\mu)$, we have that

$$I_{(F_1,F_2)}(u,D) \le I_{(F_1,F_2)}(u+\varphi,D). \tag{66}$$

Since φ vanishes outside D, we can obtain that

$$\begin{aligned} I_{(F_1,F_2)}(u,\Omega) &= \int_\Omega (F_1(x,\nabla u)+F_2(x,u))\,dx \\ &= \int_D (F_1(x,\nabla u)+F_2(x,u))\,dx + \int_{\Omega\setminus D}(F_1(x,\nabla u)+F_2(x,u))\,dx \\ &= I_{(F_1,F_2)}(u,D) + \int_{\Omega\setminus D}(F_1(x,\nabla u)+F_2(x,u))\,dx \\ &\le I_{(F_1,F_2)}(u+\varphi,D) + \int_{\Omega\setminus D}(F_1(x,\nabla u)+F_2(x,u))\,dx \\ &= \int_D (F_1(x,\nabla u+\nabla\varphi)+F_2(x,u+\varphi))\,dx + \int_{\Omega\setminus D}(F_1(x,\nabla u)+F_2(x,u))\,dx \\ &= \int_\Omega (F_1(x,\nabla u+\nabla\varphi)+F_2(x,u+\varphi))\,dx \\ &= I_{(F_1,F_2)}(u+\varphi,\Omega), \end{aligned} \tag{67}$$

whenever $\varphi \in C_0^\infty(\Omega)$.

Fix v with $v-u \in H_0^{1,p}(\Omega;\mu)$ and let $\varphi_j \in C_0^\infty(\Omega)$ be a sequence with φ_j converging to $v-u$ in $H^{1,p}(\Omega;\mu)$. By Lemma 2, we can get that

$$\begin{aligned} F_1(x,\nabla u+\nabla\varphi_j) &\le F_1(x,\nabla v) + \nabla_\xi F_1(x,\nabla u+\nabla\varphi_j)\cdot(\nabla u+\nabla\varphi_j-\nabla v), \\ F_2(x,u+\varphi_j) &\le F_2(x,v) + \partial_t F_2(x,u+\varphi_j)(u+\varphi_j-v). \end{aligned} \tag{68}$$

By the inequality (67), we obtain that

$$\begin{aligned}
I_{(F_1,F_2)}(u,\Omega) &\le I_{(F_1,F_2)}\left(u+\varphi_j,\Omega\right) \\
&= \int_\Omega \left(F_1\left(x,\nabla u+\nabla\varphi\right)+F_2\left(x,u+\varphi\right)\right)dx \\
&\le \int_\Omega \left(F_1\left(x,\nabla v\right)+F_2\left(x,v\right)\right)dx \\
&\quad + \int_\Omega \nabla_\xi F_1\left(x,\nabla u+\nabla\varphi_j\right) \\
&\qquad \cdot \left(\nabla u+\nabla\varphi_j-\nabla v\right)dx \\
&\quad + \int_\Omega \partial_t F_2\left(x,u+\varphi_j\right)\left(u+\varphi_j-v\right)dx \\
&\le I_{(F_1,F_2)}(v,\Omega) + \int_\Omega \left|\nabla_\xi F_1\left(x,\nabla u+\nabla\varphi_j\right)\right| \\
&\qquad \times \left|\nabla u+\nabla\varphi_j-\nabla v\right|dx \\
&\quad + \int_\Omega \left|\partial_t F_2\left(x,u+\varphi_j\right)\right|\left|u+\varphi_j-v\right|dx \\
&\le I_{(F_1,F_2)}(v,\Omega) + \int_\Omega 2^p\delta_1 w(x)\left|\nabla u+\nabla\varphi_j\right|^{p-1} \\
&\qquad \times \left|\nabla u+\nabla\varphi_j-\nabla v\right|dx \\
&\quad + \int_\Omega 2^p\delta_2 w(x)\left|u+\varphi_j\right|^{p-1}\left|u+\varphi_j-v\right|dx \\
&\le I_{(F_1,F_2)}(v,\Omega) \\
&\quad + 2^p\delta_1\left(\int_\Omega \left|\nabla u+\nabla\varphi_j\right|^p d\mu\right)^{1-1/p} \\
&\quad \times \left(\int_\Omega \left|\nabla u+\nabla\varphi_j-\nabla v\right|^p d\mu\right)^{1/p} \\
&\quad + 2^p\delta_2\left(\int_\Omega \left|u+\varphi_j\right|^p d\mu\right)^{1-1/p} \\
&\quad \times \left(\int_\Omega \left|u+\varphi_j-v\right|^p d\mu\right)^{1/p} \\
&\le I_{(F_1,F_2)}(v,\Omega) + 2^p\left(\delta_1+\delta_2\right) \\
&\quad \times \left\|u+\varphi_j\right\|_{1,p}^{p-1}\left\|u+\varphi_j-v\right\|_{1,p}.
\end{aligned} \tag{69}$$

Since φ_j converges to $v-u$ in $H^{1,p}(\Omega;\mu)$, letting j converge to ∞ in inequality (69), it follows that

$$I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(v,\Omega). \tag{70}$$

The theorem follows. □

Theorem 11. *A function $u \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ is an (free) (F_1,F_2)-extremal in Ω if and only if*

$$-\operatorname{div}\nabla_\xi F_1(x,\nabla u)+\partial_t F_2(x,u)=0 \tag{71}$$

in Ω, that is,

$$\int_\Omega \left(\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi+\partial_t F_2(x,u)\varphi\right)dx=0 \tag{72}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Proof. Write

$$K_D=\left\{v\in H^{1,p}(D;\mu): u-v\in H_0^{1,p}(D;\mu)\right\} \tag{73}$$

for each open set $D \Subset \Omega$.

(i) Fix $\varphi \in C_0^\infty(\Omega)$ and let $D \Subset \Omega$ be an open set such that $\operatorname{spt}\varphi \subset D$. Then, $u-(u+\varphi) \in H_0^{1,p}(D;\mu)$ and $u-(u-\varphi) \in H_0^{1,p}(D;\mu)$. Thus, $u \in K_D$, $u+\varphi \in K_D$, $u-\varphi \in K_D$, and $K_D \subset H^{1,p}(D;\mu)$ is a convex set. Since u is an (free) (F_1,F_2)-extremal in Ω and $D \Subset \Omega$, we have that

$$I_{(F_1,F_2)}(u,D)=\min\left\{I_{(F_1,F_2)}(v,D): v\in K_D\right\}. \tag{74}$$

By Theorem 8, it follows that

$$\begin{aligned}
&\int_D \left(\nabla_\xi F_1(x,\nabla u)\cdot\left(\nabla u+\nabla\varphi-\nabla u\right)\right. \\
&\qquad \left.+\partial_t F_2(x,u)\left(u+\varphi-u\right)\right)dx \ge 0, \\
&\int_D \left(\nabla_\xi F_1(x,\nabla u)\cdot\left(\nabla u-\nabla\varphi-\nabla u\right)\right. \\
&\qquad \left.+\partial_t F_2(x,u)\left(u-\varphi-u\right)\right)dx \ge 0.
\end{aligned} \tag{75}$$

Then,

$$\int_D \left(\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi+\partial_t F_2(x,u)\varphi\right)dx=0. \tag{76}$$

Since φ and $\nabla\varphi$ vanish outside D, it follows that

$$\int_\Omega \left(\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi+\partial_t F_2(x,u)\varphi\right)dx=0. \tag{77}$$

(ii) Fix the open set $D \Subset \Omega$ and $v \in K_D$. Then, $u \in H^{1,p}(D;\mu)$ and $u-v \in H_0^{1,p}(D;\mu)$. Choose a sequence $\varphi_j \in C_0^\infty(D)$ with φ_j converging to $u-v$ in $H^{1,p}(D;\mu)$. By Theorem 7, we obtain that

$$\begin{aligned}
&\left|\int_\Omega \nabla_\xi F_1(x,\nabla u)\cdot(\nabla v-\nabla u)\,dx+\int_\Omega \nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi_j\,dx\right| \\
&\quad \le \int_\Omega \left|\nabla_\xi F_1(x,\nabla u)\right|\left|\nabla v-\nabla u+\nabla\varphi_j\right|dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_\Omega 2^p \delta_1 w(x) |\nabla u|^{p-1} \left|\nabla v - \nabla u + \nabla \varphi_j\right| dx \\
&\le 2^p \delta_1 \left(\int_\Omega |\nabla u|^p d\mu\right)^{1-1/p} \\
&\quad \times \left(\int_\Omega \left|\nabla v - \nabla u + \nabla \varphi_j\right|^p d\mu\right)^{1/p} \longrightarrow 0, \\
&\left|\int_\Omega \partial_t F_2(x,u)(v-u)\,dx + \int_\Omega \partial_t F_2(x,u)\varphi_j\,dx\right| \\
&\le \int_\Omega \left|\partial_t F_2(x,u)\right| \left|v - u + \varphi_j\right| dx \\
&\le \int_\Omega 2^p \delta_2 w(x) |u|^{p-1} \left|v - u + \varphi_j\right| dx \\
&\le 2^p \delta_2 \left(\int_\Omega |u|^p d\mu\right)^{1-1/p} \\
&\quad \times \left(\int_\Omega \left|v - u + \varphi_j\right|^p d\mu\right)^{1/p} \longrightarrow 0
\end{aligned}
\tag{78}
$$

as $j \to \infty$. Therefore, it follows that

$$
\begin{aligned}
&\int_D \left(\nabla_\xi F_1(x,\nabla u)\cdot(\nabla v - \nabla u) + \partial_t F_2(x,u)(v-u)\right) dx \\
&= -\lim_{j\to\infty} \int_D \left(\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi_j + \partial_t F_2(x,u)\varphi_j\right) dx \\
&= -\lim_{j\to\infty} \int_\Omega \left(\nabla_\xi F_1(x,\nabla u)\cdot\nabla\varphi_j + \partial_t F_2(x,u)\varphi_j\right) dx = 0.
\end{aligned}
\tag{79}
$$

By Theorem 8, we have that $I_{(F_1,F_2)}(u,D) \le I_{(F_1,F_2)}(v,D)$. Then, u is a free (F_1,F_2)-extremal in Ω. □

Based on the proof of Theorem 11, we easily infer the following corollary.

Corollary 12. *Suppose that a sequence u_j converges to u in $H^{1,p}(\Omega;\mu)$, and then*

$$
I_{(F_1,F_2)}(u,\Omega) = \lim_{j\to\infty} I_{(F_1,F_2)}\left(u_j,\Omega\right). \tag{80}
$$

Next, we formulate the obstacle problem in terms of variational integrals. This makes the essence of the problem clearer.

Definition 13. Suppose that Ω is bounded. Let $\psi : \Omega \to [-\infty,\infty]$ be an arbitrary function and call it an obstacle. For $\vartheta \in H^{1,p}(\Omega;\mu)$, write

$$
\begin{aligned}
K_{\psi,\vartheta}(\Omega) = \{&v \in H^{1,p}(\Omega;\mu) : v - \vartheta \in H_0^{1,p}(\Omega;\mu), \\
&v \ge \psi \text{ a.e. in } \Omega\}.
\end{aligned}
\tag{81}
$$

A function $u \in K_{\psi,\vartheta}(\Omega)$ is called an (F_1,F_2)-superextremal with obstacle ψ and boundary values ϑ if

$$
I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(v,\Omega) \tag{82}
$$

for all $v \in K_{\psi,\vartheta}(\Omega)$.

A function $u \in H^{1,p}_{\mathrm{loc}}(\Omega;\mu)$ is called a (free) (F_1,F_2)-superextremal in Ω if u is an (F_1,F_2)-superextremal with obstacle and boundary values u in each open set $D \Subset \Omega$.

Remark 14. (1) The (F_1,F_2)-superextremal u with obstacle ψ and boundary values ϑ minimizes the variational integral $I_{(F_1,F_2)}(v,\Omega)$ among all functions v which, roughly speaking, coincide with ϑ on the boundary $\partial\Omega$ and lie above the obstacle ψ. Naturally, this problem makes sense only if $K_{\psi,\vartheta}(\Omega)$ is nonempty. Moreover, we always assume that the notation $K_{\psi,\vartheta}(\Omega)$ that includes the assumptions Ω is bounded in this paper.

(2) (F_1,F_2)-extremal can be interpreted as (F_1,F_2)-superextremal with ψ identically $-\infty$.

Theorem 15. *Suppose that $\psi : \Omega \to [-\infty,\infty]$ and $\vartheta \in H^{1,p}(\Omega;\mu)$. Then, a function $u \in K_{\psi,\vartheta}(\Omega)$ is an (F_1,F_2)-superextremal with obstacle ψ and boundary values ϑ if and only if u is a solution to the obstacle problem in $K_{\psi,\vartheta}(\Omega)$ with $\mathscr{A} = \nabla_\xi F_1$ and $\mathscr{B} = \partial_t F_2$.*

Proof. It is easy to see that $K_{\psi,\vartheta}(\Omega)$ is a convex subset of $H^{1,p}(\Omega;\mu)$ and $u \in H^{1,p}(\Omega;\mu)$. Then, u is a (F_1,F_2)-superextremal with obstacle ψ and boundary values ϑ if and only if

$$
I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(v,\Omega) \tag{83}
$$

for all $v \in K_{\psi,\vartheta}(\Omega)$.

By Theorem 8, we can get that $I_{(F_1,F_2)}(u,\Omega) \le I_{(F_1,F_2)}(v,\Omega)$ for all $v \in K_{\psi,\vartheta}(\Omega)$ if and only if

$$
\begin{aligned}
&\int_\Omega (\mathscr{A}(x,\nabla u)\cdot(\nabla v - \nabla u) + \mathscr{B}(x,u)(v-u))\,dx \\
&= \int_\Omega \left(\nabla_\xi F_1(x,\nabla u)\cdot(\nabla v - \nabla u) + \partial_t F_2(x,u)(v-u)\right) dx \\
&\ge 0
\end{aligned}
\tag{84}
$$

for all $v \in K_{\psi,\vartheta}(\Omega)$.

Thus, the theorem follows by the definition of the solution to the obstacle problem. □

6. Existence of (F_1,F_2)-Superextremals

In this section, we establish the existence of (F_1,F_2)-superextremals by the direct methods of the calculus of variations.

First, we show a lemma, which is a direct corollary of Mazur lemma.

Lemma 16. *If X is a normed space with the norm $\|\cdot\|$ and x_j converges weakly in X to x, then there exists a sequence $\tilde{x}_j$ of convex combinations of x_j,*

$$
\tilde{x}_j = \sum_{k=j}^{l} \lambda_{j,k} x_k, \quad \lambda_{j,k} \ge 0, \quad \sum_{k=j}^{l} \lambda_{j,k} = 1 \tag{85}
$$

such that $\tilde{x}_j$ converges to x in the norm topology of X.

Proof. Fix $j = 1, 2, \ldots$ and it is easily to see that the subsequence x_k, $k \geq j$, of x_j converges weakly in X to x. By the Mazur lemma, we can get that y_k converges to x in the norm topology of X, where

$$y_k = \sum_{s=j}^{k} \lambda_{k,s} u_s, \quad \lambda_{k,s} \geq 0, \qquad \sum_{s=j}^{k} \lambda_{k,s} = 1. \tag{86}$$

Then, there exists a number $k_j \in \mathbb{N}$, such that

$$\| y_k - x \| \leq \frac{1}{j} \tag{87}$$

for all $k \geq k_j$. Let $\tilde{x}_j = y_{k_j}$ and the lemma follows. □

Theorem 17. *Suppose that $K \subset H^{1,p}(\Omega; \mu)$ is a nonempty convex closed set. Then there is $u \in K$ such that*

$$I_{(F_1,F_2)}(u, \Omega) = \min \left\{ I_{(F_1,F_2)}(v, \Omega) : v \in K \right\}. \tag{88}$$

Proof. Let $u_j \in K$ be a minimizing sequence, that is,

$$I_{(F_1,F_2)}\left(u_j, \Omega\right) \longrightarrow I_0 = \min \left\{ I_{(F_1,F_2)}(v, \Omega) : v \in K \right\} \tag{89}$$

as $j \to \infty$. Since $K \neq \emptyset$, $0 \leq I_0 < \infty$, and we can assume that

$$I_{(F_1,F_2)}\left(u_j, \Omega\right) \leq I_0 + 1 \tag{90}$$

for all j. By assumptions (9) and (13), we have that

$$\begin{aligned} &\gamma_1 \int_\Omega \left|\nabla u_j\right|^p d\mu + \gamma_2 \int_\Omega \left|u_j\right|^p d\mu \\ &\quad \leq I_{(F_1,F_2)}\left(u_j, \Omega\right) \\ &\quad \leq I_0 + 1 < \infty. \end{aligned} \tag{91}$$

Therefore, u_j is a bounded sequence in $H^{1,p}(\Omega; \mu)$. Thus a subsequence which we still denote by u_j converges weakly in $H^{1,p}(\Omega; \mu)$ to a function $u \in H^{1,p}(\Omega; \mu)$. By Lemma 16, there exists a sequence $\tilde{u}_j$ of convex combinations of u_j,

$$\tilde{u}_j = \sum_{k=j}^{l} \lambda_{j,k} u_k, \quad \lambda_{j,k} \geq 0, \qquad \sum_{k=j}^{l} \lambda_{j,k} = 1 \tag{92}$$

such that $\tilde{u}_j$ converges to u in $H^{1,p}(\Omega; \mu)$. Since K is closed and convex, $u \in H^{1,p}(\Omega; \mu)$. By Corollary 12, we have that

$$I_0 \leq I_{(F_1,F_2)}(u, \Omega) = \varliminf_{j \to \infty} I_{(F_1,F_2)}\left(\tilde{u}_j, \Omega\right). \tag{93}$$

For each $\varepsilon > 0$, since $I_{(F_1,F_2)}(u_j, \Omega) \to I_0$ as $j \to \infty$, there exists a number $j_\varepsilon \in \mathbb{N}$ such that

$$\max \left\{ I_0 - \varepsilon, 0 \right\} \leq I_{(F_1,F_2)}\left(u_j, \Omega\right) < I_0 + \varepsilon \tag{94}$$

for all $j \geq j_\varepsilon$. By assumptions (10) and (14), we obtain that

$$\begin{aligned} I_{(F_1,F_2)}\left(\tilde{u}_j, \Omega\right) &= \int_\Omega \left(F_1\left(x, \nabla \tilde{u}_j\right) + F_2\left(x, \tilde{u}_j\right)\right) dx \\ &= \int_\Omega \left(F_1\left(x, \sum_{k=j}^{l} \lambda_{j,k} \nabla u_k \right) \right. \\ &\qquad \left. + F_2\left(x, \sum_{k=j}^{l} \lambda_{j,k} u_k \right) \right) dx \\ &\leq \int_\Omega \sum_{k=j}^{l} \lambda_{j,k} \left(F_1\left(x, \nabla u_k\right) + F_2\left(x, u_k\right)\right) dx \\ &= \sum_{k=j}^{l} \lambda_{j,k} \int_\Omega \left(F_1\left(x, \nabla u_k\right) + F_2\left(x, u_k\right)\right) dx \\ &= \sum_{k=j}^{l} \lambda_{j,k} I_{(F_1,F_2)}\left(u_k, \Omega\right) \leq \sum_{k=j}^{l} \lambda_{j,k} \left(I_0 + \varepsilon\right) \\ &= I_0 + \varepsilon \end{aligned} \tag{95}$$

whenever $j \geq j_\varepsilon$. By (93) and (95), it follows that

$$I_0 \leq I_{(F_1,F_2)}(u, \Omega) \leq I_0 + \varepsilon. \tag{96}$$

Then, $I_0 = I_{(F_1,F_2)}(u, \Omega)$ and u is the desired minimizer. □

Theorem 18. *Suppose that Ω is bounded, that $\psi : \Omega \to [-\infty, \infty]$, and that $\vartheta \in H^{1,p}(\Omega; \mu)$. If*

$$\begin{aligned} &K_{\psi,\vartheta}(\Omega) \\ &\quad = \left\{ v \in H^{1,p}(\Omega) : v \geq \psi \ \text{a.e. in } \Omega, \ v - \vartheta \in H_0^{1,p}(\Omega) \right\} \neq \emptyset, \end{aligned} \tag{97}$$

there exists a unique (F_1, F_2)-superextremal with obstacle ψ and boundary values ϑ.

Proof. Since the set $K_{\psi,\vartheta}(\Omega)$ is nonempty convex subset of $H^{1,p}(\Omega; \mu)$, the existence follows from Theorem 17 if we can show that $K_{\psi,\vartheta}(\Omega)$ is closed in $H^{1,p}(\Omega; \mu)$. To accomplish this, let u_j be a sequence such that u_j converges to a function u in $H^{1,p}(\Omega; \mu)$. Since $u_j - \vartheta \in H_0^{1,p}(\Omega; \mu)$, $u - \vartheta \in H_0^{1,p}(\Omega; \mu)$. Since u_j converges to u in $H^{1,p}(\Omega; \mu)$, there is subsequence of u_j that converges a.e. to u. Therefore, $u \geq \psi$ a.e. Ω. Then, $u \in K_{\psi,\vartheta}(\Omega)$ and the existence part is thereby established.

For the uniqueness, suppose that $u_1, u_2 \in K_{\psi,\vartheta}(\Omega)$ are two distinct minimizers. Since $u_1 - \vartheta, u_1 - \vartheta \in H_0^{1,p}(\Omega; \mu)$, the set $\{\nabla u_1 \neq \nabla u_2\}$ has positive measure. By the strict convexity (10) of F_1, we can get that

$$F_1(x, \nabla v(x)) < \frac{1}{2}\left(F_1\left(x, \nabla u_1(x)\right) + F_1\left(x, \nabla u_2(x)\right)\right) \tag{98}$$

for each $x \in \{\nabla u_1 \neq \nabla u_2\}$ and

$$\int_{\{\nabla u_1 \neq \nabla u_2\}} F_1(x, \nabla v)\,dx < \frac{1}{2}\left(\int_{\{\nabla u_1 \neq \nabla u_2\}} F_1(x, \nabla u_1)\,dx + \int_{\{\nabla u_1 \neq \nabla u_2\}} F_1(x, \nabla u_2)\,dx\right). \tag{99}$$

Then,

$$\int_\Omega F_1(x, \nabla v)\,dx < \frac{1}{2}\left(\int_\Omega F_1(x, \nabla u_1)\,dx + \int_\Omega F_1(x, \nabla u_2)\,dx\right). \tag{100}$$

By the convexity (14) of F_2, we can obtain that

$$\int_\Omega F_2(x, v)\,dx \leq \frac{1}{2}\left(\int_\Omega F_2(x, u_1)\,dx + \int_\Omega F_2(x, u_2)\,dx\right). \tag{101}$$

Thus,

$$\begin{aligned} I_{(F_1,F_2)}(v, \Omega) &= \int_\Omega \left(F_1(x, \nabla v) + F_2(x, v)\right)dx \\ &< \frac{1}{2}\left(\int_\Omega F_1(x, \nabla u_1)\,dx + \int_\Omega F_1(x, \nabla u_2)\,dx\right) \\ &\quad + \frac{1}{2}\left(\int_\Omega F_2(x, u_1)\,dx + \int_\Omega F_2(x, u_2)\,dx\right) \\ &= \frac{1}{2}\left(I_{(F_1,F_2)}(u_1, \Omega) + I_{(F_1,F_2)}(u_2, \Omega)\right) \\ &= \min\left\{I_{(F_1,F_2)}(u, \Omega) : u \in K_{\psi,\vartheta}(\Omega)\right\}. \end{aligned} \tag{102}$$

This contradiction completes the proof. □

Similar to the proof of Theorem 18, we can obtain the existence of (F_1, F_2)-extremals.

Theorem 19. *Suppose that $\vartheta \in H^{1,p}(\Omega; \mu)$, then there exists a unique (F_1, F_2)-extremals u in Ω with $u - \vartheta \in H^{1,p}(\Omega; \mu)$.*

Remark 20. In Theorem 19, the open set Ω can be unbounded.

Conflict of Interests

The authors declare that they have no competing interests.

Author's Contributions

Guanfeng Li carried out the theorem proofs and drafted the paper. Yong Wang and Gejun Bao participated in the design of the study and revising the paper. All authors read and approved the final paper.

Acknowledgments

This work was supported by the National Youth Science Foundation (Grant no. 61104124) and Program for Innovation Research of Science in Harbin Institute of Technology.

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