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Research Article

Oscillation Criteria for Functional Dynamic Equations with Nonlinearities Given by Riemann-Stieltjes Integral

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We present new oscillation criteria for the second order nonlinear dynamic equation $[r(t)\phi_\gamma(x^\Delta(t))]^\Delta + q_0(t)\phi_\gamma(x(g_0(t))) + \int_a^b q(t,s)\phi_{\alpha(s)}(x(g(t,s)))\Delta\zeta(s) = 0$ under mild assumptions. Our results generalize and improve some known results for oscillation of second order nonlinear dynamic equations. Several examples are worked out to illustrate the main results.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of the second order nonlinear functional dynamic equation with γ-Laplacian and nonlinearities given by Riemann-Stieltjes integral

$$\left[r(t)\phi_\gamma\left(x^\Delta(t)\right)\right]^\Delta + q_0(t)\phi_\gamma\left(x\left(g_0(t)\right)\right) + \int_a^b q(t,s)\phi_{\alpha(s)}\left(x\left(g(t,s)\right)\right)\Delta\zeta(s) = 0, \tag{1}$$

where the time scale $\mathbb{T}$ is unbounded above; $\phi_\gamma(u) := |u|^{\gamma-1}u$, $\gamma > 0$; $\alpha \in C[a,b)_{\widehat{\mathbb{T}}}$ with $-\infty < a < b \leq \infty$ is strictly increasing; $\widehat{\mathbb{T}}$ is a time scale; r is a positive rd-continuous function on $\mathbb{T}$; q_0 and q are nonnegative rd-continuous functions on $\mathbb{T}$ and $\mathbb{T}\times\widehat{\mathbb{T}}$ with $q_0,\ q \not\equiv 0$; the functions $g_0 : \mathbb{T} \to \mathbb{T}$ and $g : \mathbb{T}\times\widehat{\mathbb{T}} \to \mathbb{T}$ are rd-continuous functions such that $\lim_{t\to\infty} g_0(t) = \infty$ and $\lim_{t\to\infty} g(t,s) = \infty$ for $t \in \mathbb{T}$ and $s \in \widehat{\mathbb{T}}$.

Both of the following two cases:

$$\int_{t_0}^\infty r^{-1/\gamma}(t)\,\Delta t = \infty, \qquad \int_{t_0}^\infty r^{-1/\gamma}(t)\,\Delta t < \infty, \tag{2}$$

are considered. We define the time scale interval $[t_0,\infty)_\mathbb{T}$ by $[t_0,\infty)_\mathbb{T} := [t_0,\infty)\cap\mathbb{T}$. By a solution of (1) we mean a nontrivial real-valued function $x \in C^1_{\rm rd}[T_x,\infty)_\mathbb{T}$, $T_x \geq t_0$, which has the property that $r\phi_\gamma(x^\Delta) \in C^1_{\rm rd}[T_x,\infty)$ and x satisfies (1) on $[T_x,\infty)_\mathbb{T}$, where $C_{\rm rd}$ is the space of rd-continuous functions. The solutions vanishing identically in some neighborhood of infinity will be excluded from our consideration. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory.

Not only does the theory of the so-called "dynamic equations" unify theories of differential equations and difference equations, but also it extends these classical cases to cases "in between," for example, to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$ (which has important applications in quantum theory (see [1])) and can be applied in different types of time scales like $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$, and $\mathbb{T} = \{H_n\}$ the set of harmonic numbers. In this work knowledge and understanding of time scales and time scale notation is assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [2–4].

In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations; we refer the reader to [5–25] and the references cited therein. Recently, Erbe et al. [26] considered

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\sum_{i=0}^{n}q_i(t)\,\Phi_{\alpha_i}\left(x\left(g_i(t)\right)\right)=0 \tag{3}$$

on an arbitrary time scale $\mathbb{T}$, where γ is a quotient of odd positive integers and $\Phi_{\alpha_i}(u)=|u|^{\alpha_i}\operatorname{sgn}u$ with $\alpha_i>0$ and $\alpha_0=\gamma$, r is a positive rd-continuous function on $\mathbb{T}$, q_i, $i=0,1,2,\ldots,n$, are nonnegative rd-continuous functions on $\mathbb{T}$, and $g_i:\mathbb{T}\to\mathbb{T}$, $i=0,1,2,\ldots,n$, satisfy $\lim_{t\to\infty}g_i(t)=\infty$. In [26], some oscillation criteria have been established when $g_i(t)\equiv\tau(t)$, $i=1,2,\ldots,n$, $\tau(t)\le t$, and τ is nondecreasing and delta differentiable with $\tau o\sigma=\sigma o\tau$ on $[t_0,\infty)_{\mathbb{T}}$. In this paper, we will establish oscillation criteria for the more general equation (1) under mild assumptions on the time scale $\mathbb{T}$ and the time delay. Note that (1) not only contains a p-Laplacian term $\gamma>0$ and the advanced/delayed function g, but also allows an infinite number of nonlinear terms and even continuous nonlinearities determined by the function ζ.

2. Main Results

Throughout this paper, we denote

$$d_+(t):=\max\{0,d(t)\},\qquad d_-(t):=\max\{0,-d(t)\},$$
$$\lambda(u):=\int_u^{\infty}r^{-1/\gamma}(u)\,\Delta u,\qquad R(v,u):=\int_u^{v}r^{-1/\gamma}(s)\,\Delta s. \tag{4}$$

Lemma 1. *Assume that*

$$\int_{t_0}^{\infty}r^{-1/\gamma}(t)\,\Delta t=\infty, \tag{5}$$

or

$$\int_{t_0}^{\infty}r^{-1/\gamma}(t)\,\Delta t<\infty,$$
$$\int_{t_0}^{\infty}r^{-1/\gamma}(v)\left[\int_{t_0}^{v}Q_1(u)\Delta u\right]^{1/\gamma}\Delta v=\infty, \tag{6}$$

where

$$Q_1(w):=q_0(w)\,\lambda^{\gamma}(g_0(w))+\int_a^b q(w,s)\left[\lambda^{\alpha(s)}(g(w,s))\right]\Delta\zeta(s). \tag{7}$$

If (1) *has a positive solution x on $[t_0,\infty)_{\mathbb{T}}$, then there exists a $T\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that*

$$x^{\Delta}(t)>0,\qquad\left[r(t)\,\phi_\gamma\left(x^{\Delta}(t)\right)\right]^{\Delta}\le 0,\qquad t\in[T,\infty)_{\mathbb{T}}. \tag{8}$$

Proof. Pick $T\in[t_0,\infty)_{\mathbb{T}}$ sufficiently large such that $(t)>0$, $x(g_0(t))>0$, and $x(g(t,s))>0$ on $[T,\infty)_{\mathbb{T}}\times[a,b]_{\tilde{\mathbb{T}}}$. From (1), we have, for $t\in[T,\infty)_{\mathbb{T}}$,

$$\left[r(t)\,\phi_\gamma\left(x^{\Delta}(t)\right)\right]^{\Delta}=-q_0(t)\left[x\left(g_0(t)\right)\right]^{\gamma}-\int_a^b q(t,s)\left[x(g(t,s))\right]^{\alpha(s)}\Delta\zeta(s)\le 0. \tag{9}$$

Then $r\phi_\gamma(x^{\Delta})$ is nonincreasing on $[T,\infty)_{\mathbb{T}}$, and x^{Δ} is of definite sign eventually. We claim that x^{Δ} is eventually positive. If not, x^{Δ} is eventually negative; that is, there exists $T_1\ge T$ such that $x^{\Delta}(t)<0$ for $t\ge T_1$.

First, we assume (5) holds. Using the fact that $r\phi_\gamma(x^{\Delta})$ is nonincreasing, we obtain, for $t\in[T_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}x(t)&=x(T_1)+\int_{T_1}^{t}\phi_\gamma^{-1}\left[r(u)\,\phi_\gamma\left(x^{\Delta}(u)\right)\right]r^{-1/\gamma}(u)\,\Delta u\\&<x(T_1)+\phi_\gamma^{-1}\left[r(T_1)\,\phi_\gamma\left(x^{\Delta}(T_1)\right)\right]\int_{T_1}^{t}r^{-1/\gamma}(u)\,\Delta u.\end{aligned} \tag{10}$$

Hence, by (5), we have $\lim_{t\to\infty}x(t)=-\infty$, which contradicts the fact that x is a positive solution of (1).

Second, we assume that (6) holds. Using the fact that $r\phi_\gamma(x^{\Delta})$ is nonincreasing, we obtain, for $t\in[T_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}-x(t)&<\int_t^{\infty}\phi_\gamma^{-1}\left[r(u)\,\phi_\gamma\left(x^{\Delta}(u)\right)\right]r^{-1/\gamma}(u)\,\Delta u\\&\le\phi_\gamma^{-1}\left[r(t)\,\phi_\gamma\left(x^{\Delta}(t)\right)\right]\int_t^{\infty}r^{-1/\gamma}(u)\,\Delta u\\&\le\phi_\gamma^{-1}\left[r(T_1)\,\phi_\gamma\left(x^{\Delta}(T_1)\right)\right]\int_t^{\infty}r^{-1/\gamma}(u)\,\Delta u\\&=L_1\lambda(t),\end{aligned} \tag{11}$$

where $L_1:=\phi_{\gamma_1}^{-1}[r(T_1)\phi_\gamma(x^{\Delta}(T_1))]<0$. By choosing sufficiently large $T_2\in[T_1,\infty)_{\mathbb{T}}$ such that $g_0(t)\ge T_1$ and $g(t,s)\ge T_1$, for $t\ge T_2$ and $s\in[a,b]_{\tilde{\mathbb{T}}}$, we get, for $t\ge T_2$ and $s\in[a,b]_{\tilde{\mathbb{T}}}$,

$$\begin{aligned}[x(g_0(t))]^{\gamma}&>L\lambda^{\gamma}(g_0(t)),\\ [x(g(t,s))]^{\alpha(s)}&>L\lambda^{\alpha(s)}(g(t,s)),\end{aligned} \tag{12}$$

where $L:=\inf_{s\in[a,b]_{\tilde{\mathbb{T}}}}\{-L_1^{\gamma},-L_1^{\alpha(s)}\}>0$. From (1) and (12) we find that

$$\begin{aligned}\left[r(t)\,\phi_\gamma\left(x^{\Delta}(t)\right)\right]^{\Delta}&<-Lq_0(t)\,\lambda^{\gamma}(g_0(t))-L\int_a^b q(t,s)\left[\lambda^{\alpha(s)}(g(t,s))\right]\Delta\zeta(s)\\&=-LQ_1(t).\end{aligned} \tag{13}$$

Integrating this last inequality from T_2 to t, we see that

$$\begin{aligned}r(t)\,\phi_\gamma\left(x^{\Delta}(t)\right)&\le r(t)\,\phi_\gamma\left(x^{\Delta}(t)\right)-r(T_2)\,\phi_\gamma\left(x^{\Delta}(T_2)\right)\\&<-L\int_{T_2}^{t}Q_1(w)\,\Delta w,\end{aligned} \tag{14}$$

which implies

$$x^{\Delta}(t) < -r^{-1/\gamma}(t)\left[L\int_{T_2}^{t} Q_1(u)\Delta u\right]^{1/\gamma}. \quad (15)$$

Again, integrating this last inequality from T_2 to t, we get

$$x(t) - x(T_2) < -\int_{T_2}^{t} r^{-1/\gamma}(v)\left[L\int_{T_2}^{v} Q_1(u)\Delta u\right]^{1/\gamma}\Delta v. \quad (16)$$

From (6), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is a positive solution of (1). This completes the proof. □

Lemma 2. *Assume that there exists sufficiently large $T \ge t_0$ such that*

$$x(t) > 0, \qquad x^{\Delta}(t) > 0,$$
$$\left[r(t)\phi_\gamma\left(x^{\Delta}(t)\right)\right]^{\Delta} \le 0, \quad (17)$$
$$t \in [T,\infty)_{\mathbb{T}}.$$

Then

$$x(g_0(t)) \ge \varphi_1(t)\,x(t),$$
$$x(g(t,s)) \ge \varphi_2(t,s)\,x(t), \quad (18)$$
$$t \ge T_1 \ge T,$$

where

$$\varphi_1(t) := \begin{cases} 1, & g_0(t) \ge t, \\ \dfrac{R(g_0(t),T)}{R(t,T)}, & g_0(t) \le t, \end{cases} \quad (19)$$

$$\varphi_2(t,s) := \begin{cases} 1, & g(t,s) \ge t, \\ \dfrac{R(g(t,s),T)}{R(t,T)}, & g(t,s) \le t. \end{cases} \quad (20)$$

Proof. Since $r\phi_\gamma(x^{\Delta})$ is strictly decreasing on $[T,\infty)_{\mathbb{T}}$. If $\tau \ge t$, then $x(\tau) > x(t)$ by the fact that x is strictly increasing. Now we consider the case when $T \le \tau \le t$. We first have

$$\begin{aligned} x(t) - x(\tau) &= \int_{\tau}^{t} x^{\Delta}(s)\,\Delta s \\ &= \int_{\tau}^{t}\left[r(s)\phi_\gamma\left(x^{\Delta}(s)\right)\right]^{1/\gamma} r^{-1/\gamma}(s)\,\Delta s \\ &\le \left[r(\tau)\phi_\gamma\left(x^{\Delta}(\tau)\right)\right]^{1/\gamma}\int_{\tau}^{t} r^{-1/\gamma}(s)\,\Delta s \\ &= \left[r(\tau)\phi_\gamma\left(x^{\Delta}(\tau)\right)\right]^{1/\gamma} R(t,g(t,s)), \end{aligned} \quad (21)$$

which implies

$$\frac{x(t)}{x(\tau)} \le 1 + \frac{\left[r(\tau)\phi_\gamma\left(x^{\Delta}(\tau)\right)\right]^{1/\gamma}}{x(\tau)} R(t,g(t,s)). \quad (22)$$

On the other hand, we have

$$\begin{aligned} x(\tau) &> x(\tau) - x(T) \\ &= \int_{T}^{\tau}\left[r(s)\phi_\gamma\left(x^{\Delta}(s)\right)\right]^{1/\gamma} r^{-1/\gamma}(s)\,\Delta s \\ &\ge \left[r(\tau)\phi_\gamma\left(x^{\Delta}(\tau)\right)\right]^{1/\gamma}\int_{T}^{\tau} r^{-1/\gamma}(s)\,\Delta s \\ &= \left[r(\tau)\phi_\gamma\left(x^{\Delta}(\tau)\right)\right]^{1/\gamma} R(\tau,T). \end{aligned} \quad (23)$$

It implies that

$$\frac{\left[r(\tau)\phi_\gamma\left(x^{\Delta}(\tau)\right)\right]^{1/\gamma}}{x(\tau)} \le \frac{1}{R(\tau,T)}. \quad (24)$$

Therefore, (22) and (24) yield that

$$\frac{x(t)}{x(\tau)} \le 1 + \frac{R(t,\tau)}{R(\tau,T)} = \frac{R(t,T)}{R(\tau,T)}, \quad (25)$$

and hence

$$x(\tau) \ge \frac{R(\tau,T)}{R(t,T)}x(t), \quad t \ge T. \quad (26)$$

Let $T_1 \ge T$ so that $g_0(t) > T$ and $g(t,s) > T$ for $t \ge T_1$ and $s \in [a,b]_{\hat{\mathbb{T}}}$. Thus, we have that, for $t \ge T_1$,

$$x(g_0(t)) \ge \varphi_1(t)\,x(t), \qquad x(g(t,s)) \ge \varphi_2(t,s)\,x(t). \quad (27)$$

This completes the proof. □

We denote by $L_{\zeta}(a,b)_{\hat{\mathbb{T}}}$ the set of Riemann-Stieltjes integrable functions on $[a,b)_{\hat{\mathbb{T}}}$ with respect to ζ. Let $b \in [a,b)_{\hat{\mathbb{T}}}$ such that $\alpha(c) = \gamma$. We further assume that

$$\alpha, \alpha^{-1} \in L_{\zeta}(a,b)_{\hat{\mathbb{T}}} \quad (28)$$

such that

$$\int_{a}^{c}\Delta\zeta(s) > 0, \qquad \int_{c}^{b}\Delta\zeta(s) > 0. \quad (29)$$

We start with the following two lemmas cited from [25] which will play an important role in the proofs of our results.

Lemma 3. *Let*

$$\begin{aligned} m &:= \gamma\int_{\sigma(c)}^{b}\alpha^{-1}(s)\,\Delta\zeta(s)\left(\int_{\sigma(c)}^{b}\Delta\zeta(s)\right)^{-1}, \\ n &:= \gamma\int_{a}^{\sigma(c)}\alpha^{-1}(s)\,\Delta\zeta(s)\left(\int_{a}^{\sigma(c)}\Delta\zeta(s)\right)^{-1}. \end{aligned} \quad (30)$$

Then there exists $\eta \in L_{\zeta}(a,b)_{\hat{\mathbb{T}}}$ such that $\eta(s) > 0$ on $[a,b)_{\hat{\mathbb{T}}}$,

$$\int_{a}^{b}\alpha(s)\,\eta(s)\,\Delta\zeta(s) = \gamma, \qquad \int_{a}^{b}\eta(s)\,\Delta\zeta(s) = 1. \quad (31)$$

Lemma 4. *Let* $u \in C[a,b)_{\tilde{\mathbb{T}}}$ *and* $\eta \in L_\zeta(a,b)_{\tilde{\mathbb{T}}}$ *satisfying* $u > 0$, $\eta > 0$ *on* $[a,b)_{\tilde{\mathbb{T}}}$ *and* $\int_a^b \eta(s)\Delta\zeta(s) = 1$. *Then*

$$\int_a^b \eta(s)\, u(s)\, \Delta\zeta(s) \geq \exp\left(\int_a^b \eta(s) \ln[u(s)]\, \Delta\zeta(s)\right), \quad (32)$$

where we use the convention that $\ln 0 = -\infty$ *and* $e^{-\infty} = 0$.

Theorem 5. *Assume that one of conditions* (5) *and* (6) *holds. Furthermore, suppose that there exists a positive* Δ-*differentiable function* $\delta(t)$ *such that, for all sufficiently large* T,

$$\limsup_{t\to\infty} \int_T^t \left[\delta(u)\, Q_2(u) - \frac{r(u)\left(\left(\delta^\Delta(u)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(u)}\right]\Delta u = \infty, \quad (33)$$

where

$$Q_2(u) := q_0(u)\,\varphi_1^\gamma(u) + \exp\left(\int_a^b \eta(s) \ln\left[\frac{q(u,s)\,\varphi_2^{\alpha(s)}(u,s)}{\eta(s)}\right]\Delta\zeta(s)\right), \quad (34)$$

with φ_1 *and* φ_2 *being defined by* (19) *and* (20), *respectively. Then every solution of* (1) *is oscillatory.*

Proof. Assume (1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is $T \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that $x(t) > 0$ and $x(g(t,s)) > 0$ on $[T,\infty)_{\mathbb{T}} \times [a,b]_{\tilde{\mathbb{T}}}$. By Lemma 1, we have, for $t \in [T,\infty)_{\mathbb{T}}$,

$$x^\Delta(t) > 0, \qquad \left[r(t)\phi_\gamma\left(x^\Delta(t)\right)\right]^\Delta < 0, \quad t \geq T. \quad (35)$$

Define

$$w(t) = \delta(t)\, \frac{r(t)\,\phi_\gamma\left(x^\Delta(t)\right)}{\phi_\gamma(x(t))}. \quad (36)$$

By the product rule and the quotient rule, we have that

$$\begin{aligned} w^\Delta(t) &= \left[\frac{\delta(t)}{\phi_\gamma(x(t))}\right]^\Delta \left[r(t)\,\phi_\gamma\left(x^\Delta(t)\right)\right]^\sigma + \frac{\delta(t)}{\phi_\gamma(x(t))}\left[r(t)\,\phi_\gamma\left(x^\Delta(t)\right)\right]^\Delta \\ &= \left[\frac{\delta^\Delta(t)}{\phi_\gamma(x^\sigma(t))} - \frac{\delta(t)\left(x^\gamma(t)\right)^\Delta}{\phi_\gamma(x(t))\,\phi_\gamma(x^\sigma(t))}\right] \times \left[r(t)\,\phi_\gamma\left(x^\Delta(t)\right)\right]^\sigma + \frac{\delta(t)}{\phi_\gamma(x(t))}\left[r(t)\,\phi_\gamma\left(x^\Delta(t)\right)\right]^\Delta \\ &= \delta^\Delta(t)\left[\frac{r(t)\,\phi_\gamma\left(x^\Delta(t)\right)}{\phi_\gamma(x(t))}\right]^\sigma - \delta(t)\,\frac{\left(x^\gamma(t)\right)^\Delta}{x^\gamma(t)}\left[\frac{r(t)\,\phi_\gamma\left(x^\Delta(t)\right)}{\phi_\gamma(x(t))}\right]^\sigma + \delta(t)\,\frac{\left[r(t)\,\phi_\gamma\left(x^\Delta(t)\right)\right]^\Delta}{\phi_\gamma(x(t))}. \end{aligned} \quad (37)$$

From (1) and the definition of $w(t)$, we have

$$w^\Delta(t) = -\delta(t)\int_a^b q(t,s)\,\frac{\left[x(g(t,s))\right]^{\alpha(s)}}{x^\gamma(t)}\Delta\zeta(s) + \frac{\delta^\Delta(t)}{\delta^\sigma(t)}w^\sigma(t) - \frac{\delta(t)}{\delta^\sigma(t)}\frac{\left(x^\gamma(t)\right)^\Delta}{x^\gamma(t)}w^\sigma(t). \quad (38)$$

By the Pötzsche chain rule [3, Theorem 1.90], we obtain

$$\begin{aligned} \left(x^\gamma(t)\right)^\Delta &= \gamma\int_0^1 \left[x(t) + h\mu(t)\,x^\Delta(t)\right]^{\gamma-1} dh\, x^\Delta(t) \\ &= \gamma\int_0^1 \left[(1-h)\,x(t) + hx^\sigma(t)\right]^{\gamma-1} dh\, x^\Delta(t) \\ &\geq \begin{cases} \gamma(x(t))^{\gamma-1}x^\Delta(t), & \gamma \geq 1 \\ \gamma(x^\sigma(t))^{\gamma-1}x^\Delta(t), & 0 < \gamma \leq 1. \end{cases} \end{aligned} \quad (39)$$

If $0 < \gamma \leq 1$, we have that

$$w^\Delta(t) \leq -\delta(t)\left[\frac{x(g_0(t))}{x(t)}\right]^\gamma - \delta(t)\int_a^b q(t,s)\,\frac{\left[x(g(t,s))\right]^{\alpha(s)}}{x^\gamma(t)}\Delta\zeta(s) + \frac{\delta^\Delta(t)}{\delta^\sigma(t)}w^\sigma(t) - \frac{\gamma\delta(t)}{\delta^\sigma(t)}\frac{x^\Delta(t)}{x^\sigma(t)}\left(\frac{x^\sigma(t)}{x(t)}\right)^\gamma w^\sigma(t), \quad (40)$$

whereas if $\gamma \geq 1$, we have that

$$w^\Delta(t) \leq -\delta(t)\left[\frac{x(g_0(t))}{x(t)}\right]^\gamma - \delta(t)\int_a^b q(t,s)\,\frac{\left[x(g(t,s))\right]^{\alpha(s)}}{x^\gamma(t)}\Delta\zeta(s) + \frac{\delta^\Delta(t)}{\delta^\sigma(t)}w^\sigma(t) - \frac{\gamma\delta(t)}{\delta^\sigma(t)}\frac{x^\Delta(t)}{x^\sigma(t)}\frac{x^\sigma(t)}{x(t)}w^\sigma(t). \quad (41)$$

Using the fact that $x(t)$ is strictly increasing and $r(t)(x^\Delta(t))^\gamma$ is nonincreasing, we get that

$$x^\sigma(t) \geq x(t), \qquad x^\Delta(t) \geq \left(\frac{r^\sigma(t)}{r(t)}\right)^{1/\gamma}\left(x^\Delta(t)\right)^\sigma. \quad (42)$$

From (40), (41), and (42), we obtain

$$w^{\Delta}(t) \leq -\delta(t)\left[\frac{x(g_0(t))}{x(t)}\right]^{\gamma} - \delta(t)\int_a^b q(t,s)\frac{[x(g(t,s))]^{\alpha(s)}}{x^{\gamma}(t)}\Delta\zeta(s) + \frac{(\delta^{\Delta}(t))_+}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)}, \tag{43}$$

where $\lambda := (\gamma+1)/\gamma$. By (18) and the definition of $\check{q}(t,s)$, we have that, for $t \geq T_2$ and $s \in [a,b]_{\hat{\mathbb{T}}}$,

$$w^{\Delta}(t) \leq -\delta(t)q_1(t) - \delta(t)\int_a^b q_2(t,s)x^{\alpha(s)-\gamma}(t)\Delta\zeta(s) + \frac{(\delta^{\Delta}(t))_+}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)}, \tag{44}$$

where $q_1(t) := q_0(t)\varphi_1^{\gamma}(t)$ and $q_2(t,s) := q(t,s)\varphi^{\alpha(s)}(t,s)$. We let $\eta \in L_{\zeta}(a,b)_{\hat{\mathbb{T}}}$ be defined as in Lemma 3. Then η satisfies (31). This follows the fact that

$$\int_a^b \eta(s)[\alpha(s)-\gamma]\Delta\zeta = 0. \tag{45}$$

From Lemma 4 we get

$$\begin{aligned}\int_a^b q_2(t,s)[x(t)]^{\alpha(s)-\gamma}\Delta\zeta(s) &= \int_a^b \eta(s)\frac{q_2(t,s)}{\eta(s)}[x(t)]^{\alpha(s)-\gamma}\Delta\zeta(s)\\ &\geq \exp\left(\int_a^b \eta(s)\ln\left(\frac{q_2(t,s)}{\eta(s)}[x(t)]^{\alpha(s)-\gamma}\right)\Delta\zeta(s)\right)\\ &= \exp\left(\int_a^b \eta(s)\ln\left[\frac{q_2(t,s)}{\eta(s)}\right]\Delta\zeta(s) + \ln(x(t))\int_a^b \eta(s)[\alpha(s)-\gamma]\Delta\zeta(s)\right)\\ &= \exp\left(\int_a^b \eta(s)\ln\left[\frac{q_2(t,s)}{\eta(s)}\right]\Delta\zeta(s)\right).\end{aligned} \tag{46}$$

This together with (44) shows that, for $t \geq T_2$,

$$w^{\Delta}(t) \leq -\delta(t)Q_2(t) + \frac{(\delta^{\Delta}(t))_+}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)}. \tag{47}$$

Define $A \geq 0$ and $B \geq 0$ by

$$A^{\lambda} := \frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)}, \qquad B^{\lambda-1} := \frac{(r^{1/\lambda}(t))^{1/\lambda}(\delta^{\Delta}(t))_+}{\lambda\gamma^{1/\lambda}(\delta(t))^{1/\lambda}}. \tag{48}$$

Then, using the inequality [27]

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda}, \tag{49}$$

we get that

$$\frac{(\delta^{\Delta}(t))_+}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{1/\gamma}(t)} \leq \frac{r(t)((\delta^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}. \tag{50}$$

From this last inequality and (47) we get, for $t \geq T_2$,

$$w^{\Delta}(t) \leq -\delta(t)Q_2(t) + \frac{r(t)((\delta^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}. \tag{51}$$

Integrating both sides from T_2 to t, we get

$$\int_{T_2}^t \left[\delta(u)Q_2(u) - \frac{r(u)((\delta^{\Delta}(u))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(u)}\right]\Delta u \leq w(T_2) - w(t) \leq w(T_2), \tag{52}$$

which leads to a contradiction to (33). □

In the following examples, for $\hat{\mathbb{T}} = \mathbb{R}$, $n \in \mathbb{N}$, and $s \in [0, n+1)$, we assume that

$$\zeta(s) = \sum_{j=1}^{n}\chi(s-j) \quad \text{with } \chi(s) = \begin{cases}1, & s \geq 0\\ 0, & s < 0;\end{cases} \tag{53}$$

$\alpha \in C[0, n+1)$ such that $\alpha(j) = \alpha_j$, $j = 1, \ldots, n$,

$$\begin{aligned}&\alpha_j > \gamma, \quad j = 1, 2, \ldots, l,\\ &\alpha_j < \gamma, \quad j = l+1, l+2, \ldots, n;\end{aligned} \tag{54}$$

$q(t,j) = q_j(t)$ and $g(t,j) = g_j(t)$ for $j = 1, \ldots, n$.

Example 6. Consider the nonlinear dynamic equation

$$[t^{\gamma-1}\phi_{\gamma}(x^{\Delta}(t))]^{\Delta} + \frac{1}{t^{1/(\gamma+1)}}x^{\gamma}(g_0(t)) + \sum_{j=1}^{n}q_j(t)\phi_{\alpha_j}(x(g_j(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{55}$$

where g_j, $j = 0, 1, 2, \ldots, n$, are rd-continuous functions with $g_0(t) \geq t$ on $[t_0,\infty)_{\mathbb{T}}$, γ and α_j, $j = 1, 2, \ldots, n$, are positive constants, and q_j, $j = 1, 2, \ldots, n$, are nonnegative rd-continuous functions on $\mathbb{T}$. Here,

$$r(t) = t^{\gamma-1}, \quad q_0(t) = \frac{1}{t^{1/(\gamma+1)}}. \tag{56}$$

Choose an n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ with $0 < \eta_j < 1$ satisfying (31). By Example 5.60 in [4], condition (5) holds since

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)\Delta t = \int_{t_0}^{\infty}\frac{\Delta t}{t^{1-1/\gamma}} = \infty. \tag{57}$$

Also, by choosing $\delta(t) \equiv 1$, we have

$$\limsup_{t\to\infty} \int_T^t \left[\delta(u) Q_2(u) - \frac{r(u)\left(\left(\delta^\Delta(u)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(u)} \right] \Delta u \geq \limsup_{t\to\infty} \int_T^t \frac{1}{u^{1/(\gamma+1)}} \Delta u = \infty. \tag{58}$$

Then, by Theorem 5, every solution of (55) is oscillatory.

Example 7. Consider the nonlinear dynamic equation

$$\left[(t\sigma(t))^\gamma \phi_\gamma\left(x^\Delta(t)\right)\right]^\Delta + \sum_{j=0}^n q_j(t)\phi_{\alpha_j}\left(x\left(g_j(t)\right)\right) = 0, \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{59}$$

where $0 < \gamma = \alpha_0 \leq 1$ is a positive real number, $q_0(t) := t^\gamma$, α_j, $j = 1, 2, \ldots, n$, are positive constants, q_j, $j = 1, 2, \ldots, n$, are nonnegative rd-continuous functions on $\mathbb{T}$, and g_j, $j = 0, 1, 2, \ldots, n$, are rd-continuous functions with $g_0(t) \leq t$ on $[t_0,\infty)_{\mathbb{T}}$. Assume

$$\int_{t_0}^\infty \frac{\Delta t}{t^{1-1/\alpha_0}\sigma(t)} = \infty, \quad 0 < \alpha_0 \leq 1. \tag{60}$$

It is clear that $r(t)$ satisfies

$$\int_{t_0}^\infty r^{-1/\gamma}(t)\Delta t < \infty \leq \int_{t_0}^\infty \frac{1}{t\sigma(t)}\Delta t = \int_{t_0}^\infty \left(\frac{-1}{t}\right)^\Delta \Delta t < \infty, \quad t \in [t_0,\infty)_{\mathbb{T}},\ t_0 > 0. \tag{61}$$

This holds for many time scales, for example, when $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$. To see that (6) holds note that

$$\begin{aligned}\int_{t_0}^\infty r^{-1/\gamma}(v)\left[\int_{t_0}^v Q_1(u)\Delta u\right]^{1/\gamma}\Delta v &= \int_{t_0}^\infty r^{-1/\alpha_0}(v)\left[\int_{t_0}^v \sum_{j=0}^n q_j(u)\lambda^{\alpha_j}\left(g_j(u)\right)\Delta u\right]^{1/\alpha_0}\Delta v \\ &\geq \int_{t_0}^\infty \frac{1}{v\sigma(v)}\left[\int_{t_0}^v u^{\alpha_0}\lambda^{\alpha_0}\left(g_0(u)\right)\Delta u\right]^{1/\alpha_0}\Delta v \\ &\geq \int_{t_0}^\infty \frac{\left(v-t_0\right)^{1/\alpha_0}}{v\sigma(v)}\Delta v.\end{aligned} \tag{62}$$

Since

$$\lambda\left(g_0(u)\right) = \int_{g_0(u)}^\infty r^{-1/\gamma}(w)\Delta w = \int_{g_0(u)}^\infty \frac{1}{w\sigma(w)}\Delta w = \int_{g_0(u)}^\infty \left(\frac{-1}{w}\right)^\Delta \Delta w = \frac{1}{g_0(u)} \geq \frac{1}{u}, \tag{63}$$

we can find $0 < k < 1$ such that $v - t_0 > kv$ for $v \geq t_k > t_0$. Therefore, we get

$$\int_{t_0}^\infty r^{-1/\gamma}(v)\left[\int_{t_0}^v Q_1(u)\Delta u\right]^{1/\gamma}\Delta v > k^{1/\alpha_0}\int_{t_K}^\infty \frac{\Delta v}{v^{1-1/\alpha_0}\sigma(v)} \overset{(60)}{=} \infty. \tag{64}$$

To apply Theorem 5, it remains to prove that condition (33) holds. By putting $\delta(t) \equiv 1$, we get

$$\limsup_{t\to\infty} \int_T^t \left[\delta(u) Q_2(u) - \frac{r(u)\left(\left(\delta^\Delta(u)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(u)} \right] \Delta u \geq \limsup_{t\to\infty} \int_T^t u^\gamma \Delta u = \infty. \tag{65}$$

We conclude that if $[t_0,\infty)_{\mathbb{T}}$, $t_0 > 0$, is a time scale, where $\int_{t_0}^\infty (\Delta t / t^{1-1/\gamma}\sigma(t)) = \infty$, then every solution of (59) is oscillatory by Theorem 5.

We are now ready to state and prove Philos-type oscillation criteria for (1). Its proof can be similarly done as [28] and hence is omitted.

Theorem 8. *Assume that one of conditions* (5) *and* (6) *holds. Furthermore, suppose that there exist functions* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t,u) : t \geq u \geq t_0\}$ *such that*

$$H(t,t) = 0, \quad t \geq t_0, \qquad H(t,u) > 0, \quad t > u \geq t_0, \tag{66}$$

and H *has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta_u}(t,u)$ *with respect to the second variable and satisfies*

$$H^{\Delta_u}(t,u) + H(t,u)\frac{\delta^\Delta(u)}{\delta^\sigma(u)} = -\frac{h(t,u)}{\delta^\sigma(u)}(H(t,u))^{\gamma/(\gamma+1)}, \tag{67}$$

and, for all sufficiently large T,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\delta(u) Q_2(u) H(t,u) - \frac{\left(h_-(t,u)\right)^{\gamma+1} r(u)}{(\gamma+1)^{\gamma+1}\delta^\gamma(u)} \right] \Delta u = \infty, \tag{68}$$

where $\delta(t)$ *is a positive* Δ*-differentiable function. Then every solution of* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Example 9. Consider the following dynamic equation:

$$\left[\phi_\gamma\left(x^\Delta(t)\right)\right]^\Delta + q_0(t)\phi_\gamma\left(g_0(t)\right) + \sum_{j=1}^n q_j(t)\phi_{\alpha_j}\left(x\left(g_j(t)\right)\right) = 0, \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{69}$$

where $r(t) = 1$, g_j, q_j, $j = 0,1,2,\ldots,n$, are rd-continuous functions with $g_0(t) \ge t$ and $q_j(t) \ge 0$ on $t \in [t_0,\infty)_{\mathbb{T}}$, and γ and α_j, $j = 1,2,\ldots,n$, are positive constants. It is easy to see that (5) holds. Choose an n-tuple $(\eta_1,\eta_2,\ldots,\eta_n)$ with $0 < \eta_j < 1$ satisfying (31). By the definition of φ_1, we know $\varphi_1(t) \equiv 1$. On the other hand, let $H(t,u) = (t-u)^2$ and $\delta(t) \equiv 1$. From (67), we obtain

$$H^{\Delta_u}(t,u) = \sigma(u) + u - 2t = -h(t,u)(H(t,u))^{\gamma/(\gamma+1)}. \quad (70)$$

We have that $h(t,u) \ge 0$ for $u \in [t_0,t)_{\mathbb{T}}$ and hence $h_-(t,u) \equiv 0$ for $u \in [t_0,t)_{\mathbb{T}}$. Therefore,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\Bigg[\delta(u)Q_2(u)H(t,u) &- \frac{(h_-(t,u))^{\gamma+1}r(u)}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(u)}\Bigg]\Delta u \\ &\ge \limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t\left[q_0(u)(t-u)^2\right]\Delta u.\end{aligned} \quad (71)$$

By Theorem 8, we can say that every solution of (69) is oscillatory if

$$\limsup_{t\to\infty}\frac{1}{(t-T)^2}\int_T^t\left[q_0(u)(t-u)^2\right]\Delta u = +\infty. \quad (72)$$

Theorem 10. *Assume that one of conditions* (5) *and* (6) *holds and*

$$\limsup_{t\to\infty}R^{\gamma}(t,T)\int_t^{\infty}Q_2(u)\Delta u > 1. \quad (73)$$

Then every solution of (18) *is oscillatory.*

Proof. Assume (1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that $x(t) > 0$ and $x(g(t,s)) > 0$ on $[T,\infty)_{\mathbb{T}} \times [a,b]_{\tilde{\mathbb{T}}}$. Then, by Lemma 1, we have, for $t \in [T,\infty)_{\mathbb{T}}$,

$$x^{\Delta}(t) > 0, \qquad \left[r(t)\phi_{\gamma}\left(x^{\Delta}(t)\right)\right]^{\Delta} < 0, \quad t \ge T. \quad (74)$$

Integrating both sides of the dynamic equation (18) from t to ∞, we obtain

$$\begin{aligned}r(t)\phi_{\gamma}\left(x^{\Delta}(t)\right) &\ge \int_t^{\infty}q_0(u)\phi_{\gamma}(x(h(u)))\Delta u \\ &\quad + \int_t^{\infty}\int_a^b q(u,s)\phi_{\alpha(s)}\left(x\left(g(u,s)\right)\right)\Delta\zeta(s)\Delta u \\ &\ge \int_t^{\infty}x^{\gamma}(u)\left\{q_0(u)\left[\frac{x(h(u))}{x(u)}\right]^{\gamma} + \int_a^b q(u,s)\frac{\left[x\left(g(u,s)\right)\right]^{\alpha(s)}}{x^{\gamma}(u)}\Delta\zeta(s)\right\}\Delta u.\end{aligned} \quad (75)$$

As shown in the proof of Theorem 5, we have

$$q_0(u)\left[\frac{x(h(u))}{x(u)}\right]^{\gamma} + \int_a^b q(u,s)\frac{\left[x\left(g(u,s)\right)\right]^{\alpha(s)}}{x^{\gamma}(u)}\Delta\zeta(s) \ge Q_2(u). \quad (76)$$

Then, from (75) and (76), we get

$$\begin{aligned}r(t)\phi_{\gamma}\left(x^{\Delta}(t)\right) &\ge \int_t^{\infty}x^{\gamma}(u)Q(u)\Delta u \\ &\ge x^{\gamma}(t)\int_t^{\infty}Q_2(u)\Delta u.\end{aligned} \quad (77)$$

Since $x^{\Delta}(t) > 0$ and $r(t) > 0$, we have

$$\frac{1}{r(t)}\int_t^{\infty}Q_2(u)\Delta u \le \left[\frac{x^{\Delta}(t)}{x(t)}\right]^{\gamma}. \quad (78)$$

Also, by using the fact that $r\phi_{\gamma}(x^{\Delta})$ is nonincreasing, we have

$$\begin{aligned}x(t) \ge x(t) - x(T) &= \int_T^t x^{\Delta}(s)\Delta s \\ &= \int_T^t\left[r(s)\phi_{\gamma}\left(x^{\Delta}(s)\right)\right]^{1/\gamma}r^{-1/\gamma}(s)\Delta s \\ &\ge \left[r(t)\phi_{\gamma}\left(x^{\Delta}(t)\right)\right]^{1/\gamma}\int_T^t r^{-1/\gamma}(s)\Delta s \\ &= \left[r(t)\phi_{\gamma}\left(x^{\Delta}(t)\right)\right]^{1/\gamma}R(t,T),\end{aligned} \quad (79)$$

or

$$\left[\frac{x^{\Delta}(t)}{x(t)}\right]^{\gamma} \le \frac{1}{r(t)R^{\gamma}(t,T)}. \quad (80)$$

In view of (78) and (80), we get

$$R^{\gamma}(t,T)\int_t^{\infty}Q_2(u)\Delta u \le 1, \quad (81)$$

which gives us the contradiction

$$\limsup_{t\to\infty}R^{\gamma}(t,T)\int_t^{\infty}Q_2(u)\Delta u \le 1. \quad (82)$$

This completes the proof. □

Example 11. For $t \in [t_0,\infty)_{\mathbb{T}}$, we consider the following dynamic equation:

$$\begin{aligned}\left[\phi_{\gamma}\left(x^{\Delta}(t)\right)\right]^{\Delta} &+ \frac{1}{t\sigma(t)}\phi_{\gamma}\left(x\left(g_0(t)\right)\right) \\ &+ \sum_{j=1}^{n}q_j(t)\phi_{\alpha_j}\left(x\left(g_j(t)\right)\right) = 0,\end{aligned} \quad (83)$$

where $r(t) = 1$, $q_0(t) = 1/t\sigma(t)$, g_j, $j = 0,1,2,\ldots,n$, are rd-continuous functions with $g_0(t) \ge t$ on $t \in [t_0,\infty)_{\mathbb{T}}$, q_j,

$j = 1, 2, \ldots, n$, are nonnegative rd-continuous functions on $\mathbb{T}$, $\gamma > 1$, and α_j, $j = 1, 2, \ldots, n$, are positive constants. It is obvious that (5) holds. Choose an n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ with $0 < \eta_j < 1$ satisfying (31). On the other hand, noting that $\varphi_1(t) = 1$ and $R(t,T) = \int_T^t r^{-1/\gamma}(s)\Delta s = t - T$, we can easily verify that

$$\begin{aligned} &\limsup_{t\to\infty} R^{\gamma}(t,T) \int_t^{\infty} Q_2(u)\,\Delta u \\ &\quad \geq \limsup_{t\to\infty} (t-T)^{\gamma} \int_t^{\infty} \frac{1}{u\sigma(u)} \Delta u = +\infty > 1. \end{aligned} \tag{84}$$

By Theorem 10, every solution of (83) is oscillatory.

The last theorem is under the assumption that $\int_{t_0}^{\infty} Q_2(u)\Delta u < \infty$. Its proof can be similarly done as in [28] and hence is omitted.

Theorem 12. *Assume that one of conditions* (5) *and* (6) *holds and $r(t)$ is a (delta) differentiable function with $r^{\Delta}(t) \geq 0$. Furthermore, assume that* $l = \liminf_{t\to\infty}(t/\sigma(t)) > 0$ *and*

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_{\sigma(t)}^{\infty} Q_2(u)\,\Delta u > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}. \tag{85}$$

Then every solution of (1) *is oscillatory.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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