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Research Article

Computation of the Eigenpairs of Two-Parameter Sturm-Liouville Problems Using the *Regularized Sampling Method*

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This paper deals with the computation of the eigenvalues of two-parameter Sturm-Liouville (SL) problems using the *Regularized Sampling Method*, a method which has been effective in computing the eigenvalues of broad classes of SL problems (singular, non-self-adjoint, nonlocal, impulsive, etc.). We have shown, in this work that it can tackle two-parameter SL problems with equal ease. An example was provided to illustrate the effectiveness of the method.

1. Introduction

In an interesting paper published in 1963, Arscott [1] showed that the method of separation of variables used in solving boundary value problems for Laplace's equation leads to a two-parameter eigenvalue problem for ordinary differential equations with the auxiliary requirement that the solutions satisfy boundary conditions at several points. This has led to an extensive development of multiparameter spectral theory for linear operators (see, e.g., [2–8]). In [9], the authors give an overview of results on two-parameter eigenvalue problems for second order linear differential equations. Several properties of corresponding eigencurves are given. In [5], the authors have obtained interesting geometric properties of the eigencurves (e.g., the condition of transversal intersections is equivalent to the simplicity of the eigenvalues in the sense of Chow and Hale). All the above works are concerned with the theoretical aspect of the existence of eigenvalues. Also, several authors have dealt with the theoretical numerical analysis of two-parameter eigenvalue problems, [10, 11] and the references therein as well as the works of Plestenjak and his collaborators [12–15]. Eigenvalue problems have played a major role in the applied sciences. Consequently, the problem of computing eigenvalues of one-parameter problems has attracted many researchers (see, e.g., [16, 17] and the references therein).

Concerning the computations of eigenvalues of one-parameter Sturm-Liouville problems, the authors in [18] introduced a new method based on Shannon's sampling theory. It uses the analytic properties of the boundary function. The method has been generalized to a class of singular problems of Bessel type [19]. The books by Atkinson [20], Chow and Hale [21], Faierman [22], McGhee and Picard [23], Sleeman [24], Volkmer [25], and the long awaited monograph by Atkinson and Mingarelli [26] contain several results on eigenvalues of multiparameter Sturm-Liouville problems and the corresponding bifurcation problems. However, no attempt has been made to compute the eigenvalues of two-parameter Sturm-Liouville problems using the approach based on the *Regularized Sampling Method* introduced recently by the first author in [27] to compute the eigenvalues of general Sturm-Liouville problems and extended to the case of singular [28], non-self-adjoint [29], nonlocal [30], and impulsive SLPs [31]. We will consider, in this paper, the computation of the eigenpairs of two-parameter Sturm-Liouville problems with three-point boundary conditions using the *Regularized Sampling Method*.

2. The Characteristic Function

Consider the two-parameter Sturm-Liouville problem as follows,

$$\begin{aligned} -y'' + qy &= \left(\mu_1^2 w_1 + \mu_2^2 w_2\right) y, \quad 0 < x < 1, \\ y(0) = 0, &\qquad y(c) = 0, \qquad y(1) = 0, \end{aligned} \tag{1}$$

where w_1 and w_2 are positive and in $\mathscr{C}^2[0,1]$, $q \in L[0,1]$, and $c \in (0,1)$ some given constant.

By an eigenvalue of (1) we mean a value of the couple (μ_1, μ_2) for which problem (1) has a nontrivial solution. Conditions that insure the existence of eigenvalue are given in [2–4, 7, 32, 33]. In fact, under fairly general conditions, it has been shown (see [21, 34]) that there are smooth curves of eigenvalues (actually eigenpairs). Our objective is to effectively localize the eigencurves in the parameter (μ_1, μ_2)-plane. We should point out that we have restricted our attention to Dirichlet boundary conditions in order to eliminate technical details that might obscure the ideas.

We will associate with (1) the initial value problem as follows,

$$\begin{aligned} y'' + \left(\mu_1^2 w_1 + \mu_2^2 w_2\right) y &= qy, \quad 0 < x < 1, \\ y(0) = 0, &\qquad y'(0) = 1 \end{aligned} \tag{2}$$

and deal first with the unperturbed case ($q = 0$) then with the perturbed case ($q \neq 0$).

2.1. The Unperturbed Case ($q = 0$). In this case, (2) reduces to

$$\begin{aligned} \varphi_1'' + w\varphi_1 &= 0, \quad 0 < x < 1, \\ \varphi_1(0) = 0, &\qquad \varphi_1'(0) = 1, \end{aligned} \tag{3}$$

where $w = \mu_1^2 w_1 + \mu_2^2 w_2$.

Theorem 1. *The solution φ_1 of (3) is an entire function of $(\mu_1, \mu_2) \in \mathbb{C}^2$ for each fixed $x \in (0,1]$ of order $(1,1)$ and type $(\sigma_1(x), \sigma_2(x))$ and satisfies the following estimate,*

$$\left|\varphi_1(x)\right| \leq K_1 \exp\left[\sigma_1(x)\left|\mu_1\right| + \sigma_2(x)\left|\mu_2\right|\right], \quad (\mu_1, \mu_2) \in \mathbb{C}^2 \tag{4}$$

for each fixed $x \in (0,1]$, where $\sigma_i(x) = 2\{x\int_0^x w_i(\xi)d\xi\}^{1/2}$ for $i = 1, 2$.

Proof. From (3), we get the following integral equation,

$$\varphi_1(x) = x - \int_0^x (x-\xi)\left[\mu_1^2 w_1(\xi) + \mu_2^2 w_2(\xi)\right]\varphi_1(\xi)\,d\xi. \tag{5}$$

Let

$$\varphi_{1,0}(x) = x,$$

$$\varphi_{1,n+1}(x) = \int_0^x (x-\xi)\left[\mu_1^2 w_1(\xi) + \mu_2^2 w_2(\xi)\right]\varphi_{1,n}(\xi)\,d\xi, \quad n \geq 0. \tag{6}$$

We will show, by induction on n, that

$$\left|\varphi_{1,n}(x)\right| \leq \frac{x}{n!\,(n+1)!}\left\{x\int_0^x \left[\left|\mu_1\right|^2 w_1(\xi) + \left|\mu_2\right|^2 w_2(\xi)\right]d\xi\right\}^n, \quad n \geq 0. \tag{7}$$

It is true for $n = 0$. Assume it is true for n. We will show that it is true for $n+1$. Indeed, from (6), we have

$$\begin{aligned} &\left|\varphi_{1,n+1}(x)\right| \\ &\leq \int_0^x (x-\xi)\left[\left|\mu_1\right|^2 w_1(\xi) + \left|\mu_2\right|^2 w_2(\xi)\right] \\ &\quad \times \frac{\xi}{n!\,(n+1)!}\left\{\xi\int_0^\xi \left[\left|\mu_1\right|^2 w_1(\tau) + \left|\mu_2\right|^2 w_2(\tau)\right]d\tau\right\}^n d\xi. \end{aligned} \tag{8}$$

Using the fact that the expression $(x-\xi)\xi^{n+1}$ attains its maximum at $\xi = ((n+1)/(n+2))x$ over $[0,x]$, we get

$$\begin{aligned} &\left|\varphi_{1,n+1}(x)\right| \\ &\leq \frac{x}{n!\,(n+1)!}\frac{1}{n+1}\frac{1}{n+2}\left(\frac{n+1}{n+2}x\right)^{n+1} \\ &\quad \times \left\{\int_0^x \left[\left|\mu_1\right|^2 w_1(\tau) + \left|\mu_2\right|^2 w_2(\tau)\right]d\tau\right\}^{n+1} \end{aligned} \tag{9}$$

and using the fact that $(n+1)/(n+2) < 1$, we obtain

$$\begin{aligned} \left|\varphi_{1,n+1}(x)\right| &\leq \frac{x}{(n+1)!\,(n+2)!} \\ &\quad \times \left\{x\int_0^x \left[\left|\mu_1\right|^2 w_1(\tau) + \left|\mu_2\right|^2 w_2(\tau)\right]d\tau\right\}^{n+1}; \end{aligned} \tag{10}$$

that is, (7) is true for $n+1$. Hence, it is true for all $n \geq 0$.

Now, $\varphi_1(x) = \sum_{n\geq 0}(-1)^n \varphi_{1,n}(x)$ and the series is absolutely and uniformly convergent since

$$\begin{aligned} \left|\varphi_1(x)\right| &= \left|\sum_{n\geq 0}(-1)^n \varphi_{1,n}(x)\right| \leq \sum_{n\geq 0}\left|\varphi_{1,n}(x)\right| \\ &\leq x\sum_{n\geq 0}\frac{1}{n!\,(n+1)!} \\ &\quad \times \left\{x\int_0^x \left[\left|\mu_1\right|^2 w_1(\tau) + \left|\mu_2\right|^2 w_2(\tau)\right]d\tau\right\}^n \\ &= xI_1\left(\left\{2\sqrt{x\int_0^x \left[\left|\mu_1\right|^2 w_1(\tau) + \left|\mu_2\right|^2 w_2(\tau)\right]d\tau}\right\}\right) \\ &\quad \times \left\{x\int_0^x \left[\left|\mu_1\right|^2 w_1(\tau) + \left|\mu_2\right|^2 w_2(\tau)\right]d\tau\right\}^{-1/2}, \end{aligned} \tag{11}$$

where I_1 is the modified Bessel function of the first kind order 1 as follows,

$$I_1(z) = \sum_{n\geq 0} \frac{1}{n!\,(n+1)!}\left(\frac{z}{2}\right)^{2n+1}. \tag{12}$$

Using the fact that $I_1(z) \sim e^z/\sqrt{2\pi z}$ as $z \to \infty$, we get

$$\begin{aligned}&|\varphi_1(x)|\\ &\leq K_1 \exp\left[2\left\{x\int_0^x \left[|\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau)\right] d\tau\right\}^{1/2}\right]\\ &\leq K_1 \exp\left[\sigma_1(x)|\mu_1| + \sigma_2(x)|\mu_2|\right].\end{aligned} \tag{13}$$

Therefore, φ_1 is an entire function of $(\mu_1, \mu_2) \in \mathbb{C}^2$, as a uniformly convergent series of entire functions, for each fixed $x \in (0,1]$, of order $(1,1)$ and type $(\sigma_1(x), \sigma_2(x))$. This concludes the proof. □

We will make use of Liouville-Green's transformation

$$\begin{aligned}t(x) &= \int_0^x \sqrt{w(\xi)}d\xi,\\ z(t) &= \{w(x)\}^{1/4}\varphi_1(x)\end{aligned} \tag{14}$$

to bring (2) to the following form,

$$\begin{aligned}&\frac{d^2z}{dt^2} + \{1 + r(t)\}z = 0, \quad 0 < t < \int_0^1 \sqrt{w(\xi)}d\xi,\\ &z(0) = 0, \qquad \frac{dz}{dt}(0) = \{w(0)\}^{-1/4},\end{aligned} \tag{15}$$

which can be written as an integral equation as follows,

$$z(t) = \{w(0)\}^{-1/4}\sin t - \int_0^t \sin(t-\tau)\, r(\tau)\, z(\tau)\, d\tau, \tag{16}$$

where $r(t) = [\{w(x)\}^{-3/4}(d^2/dx^2)\{w(x)\}^{-1/4}]|_{x=x(t)}$.

Returning to the original variables, we deduce that φ_1 satisfies the following integral equation,

$$\begin{aligned}\varphi_1(x) &= \{w(0)\,w(x)\}^{-1/4}\sin\left\{\int_0^x \sqrt{w(\xi)}d\xi\right\}\\ &\quad - \int_0^x \sin\left(\int_{\overline{x}}^x \sqrt{w(\xi)}d\xi\right)\psi(x,\overline{x})\,\varphi_1(\overline{x})\,d\overline{x},\end{aligned} \tag{17}$$

where

$$\begin{aligned}\psi(x,\overline{x}) &= \{w(x)\}^{-1/4}\{w(\overline{x})\}^{-3/4}\\ &\quad\times\left[-\frac{1}{4}w''(\overline{x})\{w(\overline{x})\}^{-1/2}\right.\\ &\quad\left.+\frac{5}{16}\left\{w'(\overline{x})\right\}^2\{w(\overline{x})\}^{-3/2}\right].\end{aligned} \tag{18}$$

We will present next some estimates whose proofs are immediate and left to the reader.

Lemma 2. *The function $\psi(x,\overline{x})$ satisfies the following estimate,*

$$\begin{aligned}&|\psi(x,\overline{x})| \sim \frac{K_2}{|\mu_1| + |\mu_2|},\\ &\text{as } |\mu_1| + |\mu_2| \longrightarrow \infty, \quad (\mu_1,\mu_2) \in \mathbb{R}^2.\end{aligned} \tag{19}$$

Lemma 3. *The function α defined by*

$$\alpha(x) = \left(\sin c\left\{\sigma_1(x)\,\mu_1 + \sigma_2(x)\,\mu_2\right\}\right)^m, \tag{20}$$

where $\sin c(z) = z^{-1}\sin z$ and m is a positive integer, is an entire function of $(\mu_1,\mu_2) \in \mathbb{C}^2$ for each fixed $x \in (0,1]$ of order $(1,1)$ and type $(\sigma_1(x), \sigma_2(x))$. Furthermore, α satisfies the following estimate,

$$|\alpha(1)| \sim \frac{K_3}{|\mu_1| + |\mu_2|}, \quad \text{as } |\mu_1| + |\mu_2| \longrightarrow \infty,\ (\mu_1,\mu_2) \in \mathbb{R}^2. \tag{21}$$

Lemma 4. *The function φ_1 satisfies the following estimate,*

$$|\varphi_1(1)| \sim \frac{K_4}{|\mu_1| + |\mu_2|}, \quad \text{as } |\mu_1| + |\mu_2| \longrightarrow \infty,\ (\mu_1,\mu_2) \in \mathbb{R}^2. \tag{22}$$

Combining the above results, we obtain the following theorem.

Theorem 5. *The function $\alpha\varphi_1$ is an entire function of $(\mu_1,\mu_2) \in \mathbb{C}^2$ for each fixed $x \in (0,1]$ of order $(1,1)$ and type $((m+1)\sigma_1(x), (m+1)\sigma_2(x))$ and satisfies the estimate*

$$\begin{aligned}&|\alpha(x)\,\varphi_1(x)| \sim \frac{K(x)}{\left(|\mu_1| + |\mu_2|\right)^{m+1}},\\ &\text{as } |\mu_1| + |\mu_2| \longrightarrow \infty, \quad (\mu_1,\mu_2) \in \mathbb{R}^2,\end{aligned} \tag{23}$$

where K depends on $x \in (0,1]$ but is independent of (μ_1,μ_2).

Let $\mathrm{PW}_{\beta_1,\beta_2}$ denote the Paley-Wiener space as follows,

$$\mathrm{PW}_{\beta_1,\beta_2} = \left\{\begin{array}{c} h(z_1,z_2) \text{ entire, } |h(z_1,z_2)| \leq C\exp\{\beta_1|z_1| + \beta_2|z_2|\},\\ \iint_{-\infty}^{\infty} |h(z_1,z_2)|^2 dz_1 dz_2 < \infty\end{array}\right\}; \tag{24}$$

we have the following.

Theorem 6. *$\alpha(x)\varphi_1(x)$, as a function of (μ_1,μ_2), belongs to the Paley-Wiener space PW_{β_1,β_2}, where $(\beta_1,\beta_2) = ((m+1)\sigma_1(x), (m+1)\sigma_2(x))$ for each fixed $x \in (0,1]$.*

2.2. The Perturbed Case ($q \neq 0$). Let φ_1 and φ_2 be two linearly independent solutions of $\varphi'' + w\varphi = 0$ satisfying $\varphi_1(0) = \varphi_2'(0) = 0$ and $\varphi_1'(0) = \varphi_2(0) = 1$; then, the method of variation of parameters shows that (2) can be written as the integral equation as follows,

$$y(x) = \varphi_1(x) + \int_0^x \Phi(x,\xi)\, q(\xi)\, y(\xi)\, d\xi, \tag{25}$$

where $\Phi(x,\xi) = \varphi_1(\xi)\varphi_2(x) - \varphi_2(\xi)\varphi_1(x)$.

Here again, it is not hard to show that $y(x)$ is an entire function of (μ_1, μ_2) for each $x \in (0,1]$, of order $(1,1)$ and type $(\sigma_1(x), \sigma_2(x))$. Multiplication by α gives a function $\alpha(x)y(x)$ of (μ_1, μ_2) in a Paley-Wiener space $\mathrm{PW}_{\beta_1,\beta_2}$ for each $x \in (0,1]$. More specifically, we have the following.

Theorem 7. *The function $\tilde{y}$, defined by $\tilde{y}(x) = \alpha(x)y(x)$, belongs to PW_{β_1,β_2}, where $(\beta_1, \beta_2) = ((m+1)\sigma_1(x), (m+1)\sigma_2(x))$ as a function of $(\mu_1, \mu_2) \in \mathbb{C}$ for each $x \in (0,1]$, and satisfies the following estimate,*

$$\begin{aligned} &|\alpha(x)\, y(x)| \sim \frac{K(x)}{\left(|\mu_1| + |\mu_2|\right)^{m+1}}, \\ &\text{as } \ |\mu_1| + |\mu_2| \longrightarrow \infty, \quad (\mu_1, \mu_2) \in \mathbb{R}^2, \end{aligned} \tag{26}$$

where K depends on $x \in (0,1]$ but is independent of (μ_1, μ_2).

Proof. Since $\Phi_{xx} + w\Phi = 0$, $\Phi(t,t) = 0$, and $\Phi_x(t,t) = 1$, we have

$$\begin{aligned} \Phi(x,t) = {} & \{w(t)\, w(x)\}^{-1/4} \sin\left(\int_t^x \sqrt{w(\xi)}d\xi\right) \\ & - \int_t^x \sin\left(\int_{\overline{x}}^x \sqrt{w(\xi)}d\xi\right) \psi(x,\overline{x})\, \Phi(\overline{x},t)\, d\overline{x} \end{aligned} \tag{27}$$

so that

$$\begin{aligned} &|\Phi(x,t)| \sim \frac{K_4}{|\mu_1| + |\mu_2|} \le K_5, \\ &\text{as } \ |\mu_1| + |\mu_2| \longrightarrow \infty, \quad (\mu_1, \mu_2) \in \mathbb{R}^2, \end{aligned} \tag{28}$$

from which we get, after using Gronwall's lemma [35] on (25) and the estimate for φ_1,

$$\begin{aligned} &|y(x)| \\ &\quad \le K_1 \exp\left[\sigma_1(x)\,|\mu_1| + \sigma_2(x)\,|\mu_2|\right] \exp\left\{K_5 \int_0^x |q(t)|\, dt\right\}, \\ &|y(1)| \le K_7 \exp\left[\sigma_1(1)\,|\mu_1| + \sigma_2(1)\,|\mu_2|\right], \\ &\quad \text{as } \ |\mu_1| + |\mu_2| \longrightarrow \infty, \quad (\mu_1, \mu_2) \in \mathbb{R}^2, \\ &|\alpha(1)\, y(1)| \\ &\quad \le K_8 \exp\left[(m+1)\,\sigma_1(1)\,|\mu_1| + (m+1)\,\sigma_2(1)\,|\mu_2|\right], \\ &\qquad \text{as } \ |\mu_1| + |\mu_2| \longrightarrow \infty, \quad (\mu_1, \mu_2) \in \mathbb{R}^2. \end{aligned} \tag{29}$$

Furthermore, we have

$$\begin{aligned} &|\alpha(x)\, y(x)| \sim \frac{K(x)}{\left(|\mu_1| + |\mu_2|\right)^{m+1}}, \\ &\text{as } \ |\mu_1| + |\mu_2| \longrightarrow \infty, \quad (\mu_1, \mu_2) \in \mathbb{R}^2, \end{aligned} \tag{30}$$

where K depends on $x \in (0,1]$ but is independent of (μ_1, μ_2). □

To summarize, in both cases, unperturbed and perturbed, the transform $\tilde{y}(x; \mu_1, \mu_2)$ of the solution $y(x; \mu_1, \mu_2)$ of (2) is in a Paley-Wiener space $\mathrm{PW}_{\beta_1,\beta_2}$, where $(\beta_1, \beta_2) = ((m+1)\sigma_1(x), (m+1)\sigma_2(x))$. Thus $\tilde{y}(x; \mu_1, \mu_2)$ can be recovered at each $x \in (0,1]$ from its samples at the lattice points $(\mu_{1j}, \mu_{2k}) = (j(\pi/(m+1)\sigma_1(x)), k(\pi/(m+1)\sigma_2(x)))$, $(j,k) \in \mathbb{Z}^2$ using the rectangular cardinal series [36, 37].

Theorem 8. *Let $f \in PW_{\beta_1,\beta_2}$; then,*

$$\begin{aligned} &f(\mu_1, \mu_2) \\ &\quad = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(\mu_{1j}, \mu_{2k}) \\ &\qquad \times \frac{\sin\beta_1(\mu_1 - \mu_{1j})}{\beta_1(\mu_1 - \mu_{1j})} \frac{\sin\beta_2(\mu_2 - \mu_{2k})}{\beta_2(\mu_2 - \mu_{2k})} \end{aligned} \tag{31}$$

with the convergence of the series being uniform and in $L^2_{d\mu_1 d\mu_2}(\mathbb{R}^2)$, and $\mu_{mn} = n\pi/\beta_m$, $m = 1, 2$, $n \in \mathbb{Z}$.

Let $\sigma_{11} = \sigma_1(1)$, $\sigma_{21} = \sigma_2(1)$, $\sigma_{12} = \sigma_1(c)$, and $\sigma_{22} = \sigma_2(c)$.

The eigenpairs are therefore (μ_1^2, μ_2^2), where (μ_1, μ_2) solve the nonlinear system as follows,

$$\begin{aligned} B_1(\mu_1, \mu_2) &= 0, \\ B_2(\mu_1, \mu_2) &= 0, \end{aligned} \tag{32}$$

where

$$\begin{aligned} &B_1(\mu_1, \mu_2) \\ &\quad \triangleq \frac{1}{\alpha(1)} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \tilde{y}(1; \mu_{1j}, \mu_{2k}) \\ &\qquad \times \frac{\sin 2(m+1)\,\sigma_{11}(\mu_1 - \mu_{1j})}{2(m+1)\,\sigma_{11}(\mu_1 - \mu_{1j})} \\ &\qquad \times \frac{\sin 2(m+1)\,\sigma_{21}(\mu_2 - \mu_{2k})}{2(m+1)\,\sigma_{21}(\mu_2 - \mu_{2k})}, \\ &B_1(\mu_1, \mu_2) \\ &\quad \triangleq \frac{1}{\alpha(c)} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \tilde{y}(c; \mu_{1j}, \mu_{2k}) \\ &\qquad \times \frac{\sin 2(m+1)\,\sigma_{12}(\mu_1 - \mu_{1j})}{2(m+1)\,\sigma_{12}(\mu_1 - \mu_{1j})} \\ &\qquad \times \frac{\sin 2(m+1)\,\sigma_{22}(\mu_2 - \mu_{2k})}{2(m+1)\,\sigma_{22}(\mu_2 - \mu_{2k})}. \end{aligned} \tag{33}$$

Table 1

μ_1 (exact)	μ_2 (exact)	μ_1 (RSM)	μ_2 (RSM)
7.788149097670813	7.5908224365786845	7.78814883048352	7.590823605399021
4.194056047341936	22.273796542861913	4.194057357011064	22.273795699572364
15.597607295800684	15.165889997583099	15.597605904429917	15.1658929384375
13.87615754699624	30.597837626955204	13.876157718920586	30.59783713797321
23.40242244972004	22.744341450749697	23.40242640980619	22.744329536815076
9.51130982713563	44.296491367611544	9.511333341150682	44.29647446214639
39.31833491621106	15.670248767892955	39.31833152483279	15.670257662031721
47.18632817417246	24.620489734469363	47.186345272524804	24.62038504864854
46.81208951678993	45.483251580157955	46.813097550330646	45.480149444567346
30.03437464767369	46.558148101315844	30.034392317076748	46.55811263809656

3. A Numerical Example

We will consider in this section the two-parameter Sturm-Liouville problem with three-point boundary conditions given by

$$-y'' = \left(\mu_1^2 + \mu_2^2 x\right) y, \quad 0 < x < 1,$$
$$y(0) = y(0.7) = y(1). \tag{34}$$

The general solution y of the first differential equation can be expressed in terms of Ai and Bi functions [38] and their first derivatives as

$$\begin{aligned} &y(x;\mu_1,\mu_2) \\ &= (-1)^{2/3}\left(\mathrm{Ai}\left(\frac{\sqrt[3]{-1}\mu_1^2}{\mu_2^{4/3}}\right)\mathrm{Bi}\left(\frac{\sqrt[3]{-1}\left(\mu_1^2+\mu_2^2 x\right)}{\mu_2^{4/3}}\right)\right. \\ &\qquad \left.-\mathrm{Ai}\left(\frac{\sqrt[3]{-1}\left(\mu_1^2+\mu_2^2 x\right)}{\mu_2^{4/3}}\right)\mathrm{Bi}\left(\frac{\sqrt[3]{-1}\mu_1^2}{\mu_2^{4/3}}\right)\right) \\ &\times\left(\mu_2^{2/3}\left(\mathrm{Ai}'\left(\frac{\sqrt[3]{-1}\mu_1^2}{\mu_2^{4/3}}\right)\mathrm{Bi}\left(\frac{\sqrt[3]{-1}\mu_1^2}{\mu_2^{4/3}}\right)\right.\right. \\ &\qquad \left.\left.-\mathrm{Ai}\left(\frac{\sqrt[3]{-1}\mu_1^2}{\mu_2^{4/3}}\right)\mathrm{Bi}'\left(\frac{\sqrt[3]{-1}\mu_1^2}{\mu_2^{4/3}}\right)\right)\right)^{-1}. \end{aligned} \tag{35}$$

Thus the eigenpairs (μ_1^2, μ_2^2) can be obtained from the solutions (μ_1, μ_2) of the following system,

$$\widetilde{y}(1;\mu_1,\mu_2) = 0,$$
$$\widetilde{y}(c;\mu_1,\mu_2) = 0. \tag{36}$$

For numerical purposes, we have truncated the associated series to $|j|, |k| \le N = 50$ and took $m = 5$ in the function α. Thus, the approximate eigenpairs are seen as solutions of the following system,

$$\widetilde{y}^{[N]}(1;\mu_1,\mu_2) = 0,$$
$$\widetilde{y}^{[N]}(c;\mu_1,\mu_2) = 0. \tag{37}$$

Table 1 shows the exact eigenpairs together with their approximations using the *Regularized Sampling Method* (RSM).

4. Conclusion

In this paper, we have successfully computed the eigenpairs of two-parameter Sturm-Liouville problems using the regularized sampling method, a method which has been very efficient in computing the eigenvalues of broad classes of Sturm-Liouville problems (singular, non-self-adjoint, nonlocal, impulsive, etc.). We have shown in this work that it can tackle two-parameter SL problems with equal ease. An example was provided to illustrate the effectiveness of the method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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