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Research Article

The Solutions of Sturm-Liouville Boundary-Value Problem for Fourth-Order Impulsive Differential Equation via Variational Methods

Yu Tian and Dongpo Sun

School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China

Correspondence should be addressed to Yu Tian; tianyu2992@163.com and Dongpo Sun; sundongpolove@163.com

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The Sturm-Liouville boundary-value problem for fourth-order impulsive differential equations is studied. The existence results for one solution and multiple solutions are obtained. The main ideas involve variational methods and three critical points theory.

1. Introduction

The aim of the present paper is to study the following Sturm-Liouville boundary-value problem for the fourth-order impulsive differential equation:

$$\begin{gathered} u^{(4)}(t)+u(t)=\lambda f(t,u(t)), \quad t\neq t_i,\ t\in[0,T], \\ \Delta u'''(t_i)=\lambda I_{1i}(u(t_i)), \quad i=1,2,\ldots,l, \\ -\Delta u''(t_i)=\lambda I_{2i}(u'(t_i)), \quad i=1,2,\ldots,l, \\ au(0)+bu'''(0)=0, \qquad au(T)-bu'''(T)=0, \\ cu'(0)-du''(0)=0, \qquad cu'(T)+du''(T)=0, \end{gathered} \tag{1}$$

where a, b, c, and d are real constants, λ is a positive parameter, $0=t_0<t_1<\cdots<t_l<t_{l+1}=T$, $a,b,c,d>0$, $\Delta u'''(t_i)=u'''(t_i^+)-u'''(t_i^-)$, $\Delta u''(t_i)=u''(t_i^+)-u''(t_i^-)$, $u'''(t_i^+),u''(t_i^+)(u'''(t_i^-),u''(t_i^-))$ denote the right (left) limits, respectively, of $u'''(t),u''(t)$ at $t=t_i$, and $f\in C([0,T]\times R,R)$, $I_{1i},I_{2i}\in C(R;R)$, $i=1,2,\ldots,l$.

Recently, many authors have studied the existence of solutions for boundary-value problems with impulsive effects [1–16]. Variational methods are powerful tools for them. We refer the readers to [17–19] for related basic information.

In [10], the authors studied the following equation with impulsive effects:

$$\begin{gathered} -u''(t)=\lambda u(t)+f(t,u(t)), \quad t\neq t_i,\ t\in[0,T], \\ -\Delta u'(t_i)=I_i(u(t_i)), \quad i=1,2,\ldots,l, \\ u'(0)=0, \qquad u(T)=0. \end{gathered} \tag{2}$$

By applying critical point theory to (2), several existence results are obtained when f is imposed some assumptions and λ lies in suitable interval. In [8], the authors studied the existence of solutions for the following problem:

$$\begin{gathered} u^{(iv)}(t)+Au''(t)+Bu(t)=f(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ -\Delta u''(t_j)=I_{1j}(u'(t_j)), \quad j=1,2,\ldots,l, \\ -\Delta u'''(t_j)=I_{2j}(u(t_j)), \quad j=1,2,\ldots,l, \\ u(0)=u(T)=u''(0^+)=u''(T^-)=0. \end{gathered} \tag{3}$$

They essentially proved that when f, I_{1j}, and I_{2j} satisfy some conditions, (3) has at least one solution or infinitely many classical solutions via variational methods.

To the best of our knowledge, besides [12, 13] for second-order differential equations, [8] for fourth-order differential equation, limited work has been done in the Sturm-Liouville boundary-value problem, let alone higher order. Motivated by the above facts, we study the existence of solutions for problem (1) by applying variational methods. With the impulse effects and the Sturm-Liouville boundary conditions taken into consideration, the corresponding variational functional J will be more complicated than the ones of any fourth-order boundary-value problems before. In our study, some difficulties such as how to prove that the critical points of J are just the solutions of problem (1) and how to prove the space X and the functional J to satisfy the conditions of the related theorems must be overcome. To verify that the weak solution of problem (1) is just the classical solution of (1), we construct a Fundamental Lemma 5, by which we can easily prove that the critical point of the functional is just the solution of problem (1).

This paper is organized as follows. In Section 2, we present some preliminaries and establish the variational structure. In Section 3, we discuss the existence results for one solution and multiple solutions. In Section 4, we discuss the existence results for positive solutions. In Section 5, we will give some examples.

2. Preliminaries and Variational Structure

First we present some theorems that will be needed in the proof of main results.

Theorem 1 (see Theorem 2.2 [19]). *Let E be a real Banach space and $I \in C^1(E, R)$ satisfying the Palais-Smale condition (PS). Suppose $I(0) = 0$ and*

(C1) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$,*

(C2) *there is an $e \in E \backslash B_\rho$ such that $I(e) \leq 0$. Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as*

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u), \tag{4}$$

where

$$\Gamma = \{g \in (C[0,1], E) : g(0) = 0,\ g(1) = e\}. \tag{5}$$

Theorem 2 (see Theorem 9.12 [19]). *Let E be an infinite dimensional Banach space and let $I \in C^1(E, R)$ be even, satisfy (PS), and $I(0) = 0$. If $E = V \bigoplus X$, where V is finite dimensional, and I satisfies that*

(C3) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$,*

(C4) *for each finite dimensional subspace $W \subset E$, there is an $R = R(W)$ such that $I \leq 0$ on $W \setminus B_{R(W)}$.*

Then I has unbounded sequence of critical values.

Let X be a nonempty set and $\Phi, \Psi : X \to R$ two functionals. For all $r, r_1, r_2 > \inf_X \Phi$, $r_2 > r_1$, $r_3 > 0$, we define

$$\begin{aligned}
\varphi(r) &:= \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)\right) - \Psi(u)}{r - \Phi(u)}, \\
\beta(r_1, r_2) &:= \inf_{u \in \Phi^{-1}(]-\infty, r_1[)} \sup_{v \in \Phi^{-1}([r_1, r_2[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}, \\
\gamma(r_2, r_3) &:= \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2 + r_3[)} \Psi(u)}{r_3}, \\
\alpha(r_1, r_2, r_3) &:= \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.
\end{aligned} \tag{6}$$

Theorem 3 (see Theorem 2.1 [2]). *Let X be a reflexive real Banach space, $\Phi : X \to R$ a convex, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

(1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(2) *for every u_1, u_2 such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$ one has*

$$\inf_{t \in [0,1]} \Psi(tu_1 + (1-t)u_2) \geq 0. \tag{7}$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(i) $\varphi(r_1) < \beta(r_1, r_2)$;

(ii) $\varphi(r_2) < \beta(r_1, r_2)$;

(iii) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]1/\beta(r_1, r_2), 1/\alpha(r_1, r_2, r_3)[$, the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(]-\infty, r_1[)$, $u_2 \in \Phi^{-1}([r_1, r_2[)$, and $u_3 \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

Let us define the space $X = W^{2,2}([0,T], R)$ equipped with the norm

$$\|u\| = \left(\int_0^T |u(t)|^2 + \left|u'(t)\right|^2 + \left|u''(t)\right|^2 dt\right)^{1/2}. \tag{8}$$

We set the functional $J : X \to R$ defined by

$$\begin{aligned}
J(u) = \frac{1}{2}\|u\|_X^2 &- \lambda \int_0^T F(t, u(t))\, dt - \lambda \sum_{i=1}^{l} \int_0^{u(t_i)} I_{1i}(s)\, ds \\
&- \lambda \sum_{i=1}^{l} \int_0^{u'(t_i)} I_{2i}(s)\, ds,
\end{aligned} \tag{9}$$

where $\|u\|_X = (\int_0^T |u''(t)|^2 + |u(t)|^2 dt + (a/b)u^2(T) + (a/b)u^2(0) + (c/d)|u'(T)|^2 + (c/d)|u'(0)|^2)^{1/2}$, $F(t,u) = \int_0^u f(t,s)ds$. J is differentiable for any $u \in X$ and

$$\begin{aligned} J'(u)(v) &= \int_0^T (u''v'' + uv)\,dt - \lambda \sum_{i=1}^{l} I_{1i}(u(t_i))\,v(t_i) \\ &\quad - \lambda \sum_{i=1}^{l} I_{2i}(u'(t_i))\,v'(t_i) - \lambda \int_0^T f(t,u(t))\,v(t)\,dt \\ &\quad + \frac{a}{b}u(T)\,v(T) + \frac{a}{b}u(0)\,v(0) + \frac{c}{d}u'(T)\,v'(T) \\ &\quad + \frac{c}{d}u'(0)\,v'(0). \end{aligned} \tag{10}$$

Set the usual norm of $C^1([0,T])$, $L^2(0,T)$, respectively, as follows:

$$\begin{aligned} \|u\|_{C^1} &= \max\left\{\max_{t\in[0,T]} |u(t)|,\ \max_{t\in[0,T]} |u'(t)|\right\}, \\ \|u\|_{L^2} &= \left(\int_0^T u^2(t)\,dt\right)^{1/2}. \end{aligned} \tag{11}$$

Lemma 4. *The norm $\|u\|_X$ is equivalent to the usual norm $\|u\|$.*

Proof. First, we will show that there exists $M_1 > 0$ such that $\|u\|^2 \le M_1\|u\|_X^2$. Since u' is absolutely continuous in X, we have $u'(t) = u'(0) + \int_0^t u''(s)ds$. So

$$\begin{aligned} \int_0^T |u'(t)|^2 dt &= \int_0^T \left|u'(0) + \int_0^t u''(s)\,ds\right|^2 dt \\ &\le 2T|u'(0)|^2 + 2T^2 \int_0^T |u''(s)|^2 ds, \end{aligned} \tag{12}$$

which implies

$$\begin{aligned} &\|u\|^2 \\ &\le \int_0^T |u(t)|^2 + |u''(t)|^2 dt + 2T|u'(0)|^2 + 2T^2 \int_0^T |u''(t)|^2 dt \\ &\le (1 + 2T^2) \int_0^T |u''(t)|^2 ds + \int_0^T |u(t)|^2 dt + 2T|u'(0)|^2 \\ &\le \max\left\{1 + 2T^2, \frac{2Td}{c}\right\}\left(\int_0^T |u''(t)|^2 + |u(t)|^2 dt + \frac{c}{d}|u'(0)|^2\right) \\ &:= M_1\|u\|_X^2, \end{aligned} \tag{13}$$

where $M_1 = \max\{1 + 2T^2, 2Td/c\}$.

Next, we prove that there exists $M_2 > 0$ such that $\|u\|_X^2 \le M_2\|u\|^2$.

Obviously, $\max_{t\in[0,T]}|u(t)| = u(\xi) = \overline{u} + \int_\eta^\xi u'(s)ds \le \overline{u} + \int_0^T |u'(s)|ds$, where $\overline{u} = \int_0^T u(s)ds/T = u(\eta)$. Thus, we have

$$\begin{aligned} \left(\max_{t\in[0,T]} |u(t)|\right)^2 &\le 2\left(|\overline{u}|^2 + \left(\int_0^T |u'(s)|\,ds\right)^2\right) \\ &\le \frac{2}{T}\int_0^T |u(s)|^2 ds + 2T \int_0^T |u'(s)|^2 ds. \end{aligned} \tag{14}$$

Similar to the above proof, we have

$$\left(\max_{t\in[0,T]} |u'(t)|\right)^2 \le \frac{2}{T}\int_0^T |u'(s)|^2 ds + 2T \int_0^T |u''(s)|^2 ds. \tag{15}$$

By (14) and (15), we have

$$\begin{aligned} \|u\|_X^2 &\le \int_0^T |u(s)|^2 + |u''(s)|^2 ds \\ &\quad + \frac{2a}{b}\left(\frac{2}{T}\int_0^T |u(s)|^2 ds + 2T\int_0^T |u'(s)|^2 ds\right) \\ &\quad + \frac{2c}{d}\left(\frac{2}{T}\int_0^T |u'(s)|^2 ds + 2T\int_0^T |u''(s)|^2 ds\right) \\ &\le \left(1 + \frac{4a}{bT}\right)\int_0^T |u(s)|^2 ds + \left(1 + \frac{4cT}{d}\right)\int_0^T |u''(s)|^2 ds \\ &\quad + \left(\frac{4aT}{b} + \frac{4c}{dT}\right)\int_0^T |u'(s)|^2 ds := M_2\|u\|^2, \end{aligned} \tag{16}$$

where $M_2 = \max\{1 + 4a/bT, 1 + 4cT/d, 4aT/b + 4c/dT\}$. By (13) and (16), the proof is complete. □

Lemma 5 (Fundamental Lemma). *Let $u, v \in L^1([0,T];R)$. If for every $f \in C^1[0,T]$ with $f'' \in L^1[0,T]$, $f(0) = f(T) = f'(0) = f'(T) = 0$ satisfying $\int_0^T u(t)f''(t)dt = \int_0^T v(t)f(t)dt$, then there exist $C_1, C_2 \in R$ such that $u(t) = \int_0^t \int_0^s v(\theta)d\theta ds + C_1 t + C_2$ a.e. on $[0,T]$.*

Proof. Define $w(t) \in C([0,T];R)$ by $w(t) = \int_0^t \int_0^s v(\theta)d\theta ds$; we have

$$\int_0^T w(t)\,f''(t)\,dt = \int_0^T \int_0^t \int_0^s v(\theta)\,d\theta ds f''(t)\,dt. \tag{17}$$

By the Fubini theorem, we have

$$\begin{aligned}\int_0^T w(t) f''(t)\,dt &= \int_0^T \int_s^T \int_0^s v(\theta)\,d\theta f''(t)\,dt\,ds \\ &= \int_0^T \int_0^s v(\theta)\,d\theta \left(f'(T) - f'(s)\right) ds \\ &= -\int_0^T \int_0^s v(\theta)\,d\theta f'(s)\,ds \\ &= -\int_0^T \int_\theta^T v(\theta) f'(s)\,ds\,d\theta \\ &= -\int_0^T v(\theta)\left(f(T) - f(\theta)\right) d\theta \\ &= \int_0^T v(\theta) f(\theta)\,d\theta. \end{aligned} \tag{18}$$

So

$$\int_0^T (u(t) - w(t)) f''(t)\,dt = 0. \tag{19}$$

In particular, we choose $f(t) = \int_0^t \int_0^s (u(\theta) - w(\theta) - C_1\theta - C_2) d\theta ds$, where

$$\begin{aligned} C_1 &= \frac{12}{T^3}\left(\int_0^T t(u(t) - w(t))\,dt - \frac{T}{2}\int_0^T u(s) - w(s)\,ds\right), \\ C_2 &= \frac{4}{T}\int_0^T u(s) - w(s)\,ds - \frac{6}{T^2}\int_0^T t(u - w)\,dt. \end{aligned} \tag{20}$$

By computation, $f \in C_0^1[0,T]$, $f'' \in L^1[0,T]$, $f(0) = f(T) = f'(0) = f'(T) = 0$, and

$$\int_0^T (C_1 t + C_2) f''(t)\,dt = 0. \tag{21}$$

By (19) and (21), $\int_0^T (u(t) - w(t) - C_1 t - C_2) f''(t) dt = 0$; that is, $\int_0^T |u(t) - w(t) - C_1 t - C_2|^2 dt = 0$, which means $u(t) = w(t) + C_1 t + C_2$. The proof is complete. □

Definition 6. A function $u \in X$ is said to be a weak solution of (1), if u satisfies $J'(u)(v) = 0$ for all $v \in X$.

Definition 7. A function $u \in X$ is said to be a classical solution of problem (1) if u satisfies the equation in (1) for a.e. $t \in [0,T] \setminus \{t_1, t_2, \ldots, t_l\}$ and the impulsive condition and boundary condition in (1). Moreover, u is said to be a positive classical solution of problem (1) if $u(t) \geq 0$, $u(t) \not\equiv 0$, $t \in [0,T]$.

Lemma 8. *If $u \in X$ is a weak solution of problem* (1)*, then u is a classical solution of problem* (1).

Proof. By Definition 6, if $u \in X$ is a weak solution of (1), then $J'(u)(v) = 0$ holds for all $v \in X$ and hence for all $v \in C_0^\infty(t_i, t_{i+1})$, $v' \in C_0^\infty(t_i, t_{i+1})$, $v(t) \equiv 0$, $t \in [0, t_i] \cup [t_{i+1}, T]$. So $\int_{t_i}^{t_{i+1}} u''v'' + uv - \lambda f(t,u) v dt = 0$. By Lemma 5, we have

$$u''(t) = \int_{t_i}^t \left(\int_0^s -u(\theta) + \lambda f(\theta, u(\theta))\,d\theta\right) ds + C_1 t + C_2 \tag{22}$$

for a.e. $t \in [t_i, t_{i+1}]$ and some $C_1, C_2 \in R$. So $u^{(4)} + u - \lambda f(t,u) = 0$, a.e. $t \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, l$. Thus u satisfies the equation in problem (1) and $u \in C^4([0,T] \setminus \{t_1, t_2, \ldots, t_l\})$. By integration by parts for two times, we have that

$$\begin{aligned} &\int_0^T \left(u''v'' + uv\right) dt - \lambda \int_0^T f(t, u(t)) v(t)\,dt \\ &\quad - \lambda \sum_{i=1}^l I_{1i}(u(t_i)) v(t_i) \\ &\quad - \lambda \sum_{i=1}^l I_{2i}\left(u'(t_i)\right) v'(t_i) + \frac{a}{b} u(T) v(T) + \frac{a}{b} u(0) v(0) \\ &\quad + \frac{c}{d} u'(T) v'(T) + \frac{c}{d} u'(0) v'(0) \\ &= \sum_{i=1}^l u''(t) v'(t) \Big|_{t=t_i^+}^{t_{i+1}} - \sum_{i=1}^l u'''(t) v(t) \Big|_{t=t_i^+}^{t_{i+1}} \\ &\quad + \int_0^T \left[u^{(4)}(t) + u(t) - \lambda f(t, u(t))\right] v(t)\,dt \\ &\quad - \lambda \sum_{i=1}^l I_{1i}(u(t_i)) v(t_i) - \lambda \sum_{i=1}^l I_{2i}\left(u'(t_i)\right) v'(t_i) \\ &\quad + \frac{a}{b} u(T) v(T) \\ &\quad + \frac{a}{b} u(0) v(0) + \frac{c}{d} u'(T) v'(T) + \frac{c}{d} u'(0) v'(0) \\ &= \sum_{i=1}^l \left(\Delta u'''(t_i) - \lambda I_{1i}(u(t_i))\right) v(t_i) \\ &\quad + \int_0^T \left[u^{(4)}(t) + u(t) - \lambda f(t, u(t))\right] v(t)\,dt \\ &\quad - \sum_{i=1}^l \left(\Delta u''(t_i) + \lambda I_{2i}\left(u'(t_i)\right)\right) v'(t_i) + u''(T) v'(T) \\ &\quad - u''(0) v'(0) - u'''(T) v(T) \\ &\quad + u'''(0) v(0) + \frac{a}{b} u(T) v(T) + \frac{a}{b} u(0) v(0) \\ &\quad + \frac{c}{d} u'(T) v'(T) + \frac{c}{d} u'(0) v'(0) \end{aligned} \tag{23}$$

holds for all $v \in X$. Since u satisfies the equation of problem (1), (23) becomes

$$\sum_{i=1}^{l}\left(\Delta u'''(t_i)-\lambda I_{1i}(u(t_i))\right)v(t_i) - \sum_{i=1}^{l}\left(\Delta u''(t_i)+\lambda I_{2i}(u'(t_i))\right)v'(t_i) + \left[u''(T)+\frac{c}{d}u'(T)\right]v'(T) - \left[u''(0)-\frac{c}{d}u'(0)\right]v'(0) - \left[u'''(T)-\frac{a}{b}u(T)\right]v(T) + \left[u'''(0)+\frac{a}{b}u(0)\right]v(0)=0 \tag{24}$$

for all $v \in X$.

Next we will verify that u satisfies impulsive condition in (1). If not, without loss of generality, we assume (24) holds for $v(t) \equiv 0$ for $t \in [0,t_i] \cup [t_{i+1},T]$, $v'(t_i) \neq 0$. So $(\Delta u''(t_i) + \lambda I_{2i}(u'(t_i)))v'(t_i) = 0$, and then $\Delta u''(t_i) + \lambda I_{2i}(u'(t_i)) = 0$. We assume $v(t) \equiv 0$ for $t \in [0,t_i] \cup [t_i + (t_{i+1} - t_i)/2, T]$, $v'(t_i) \neq 0$. So (24) becomes $(\Delta u''(t_i) + \lambda I_{2i}(u'(t_i)))v'(t_i) = 0$, which means $\Delta u''(t_i) + \lambda I_{2i}(u'(t_i) = 0$. Similarly, by choosing particular $v \in X$, we can show that u satisfies boundary conditions in problem (1). □

Lemma 9. *Let $u \in X$; then $\|u\|_{C^1} \leq M\|u\|_X$, where $M = \max\{1/\sqrt{T}, \sqrt{T}\}\max\{1 + 2T^2, 2Td/c\}$.*

Proof. For any $u \in X$, it follows from the mean-value theorem that

$$u(\tau)=\frac{1}{T}\int_0^T u(s)\,ds \tag{25}$$

for some $\tau \in [0,T]$. Hence, for $t \in [0,T]$, using (25) and Hölder's inequality, we have

$$\begin{aligned} |u(t)| &= \left|u(\tau)+\int_\tau^t u'(s)\,ds\right| \\ &\leq \frac{1}{\sqrt{T}}\left(\int_0^T |u(s)|^2 ds\right)^{1/2} + \sqrt{T}\left(\int_0^T \left|u'(s)\right|^2 ds\right)^{1/2} \\ &\leq \max\left\{\frac{1}{\sqrt{T}},\sqrt{T}\right\}\|u\|. \end{aligned} \tag{26}$$

Similarly, we have $|u'(t)| \leq \max\{1/\sqrt{T}, \sqrt{T}\}\|u\|$. So $\|u\|_{C^1} \leq \max\{1/\sqrt{T}, \sqrt{T}\}\|u\|$, which together with (13) yields the results. □

Lemma 10. *Suppose the following conditions hold.*

(H1) *There exist constants $\mu > 2$ and $r \geq 0$ such that, for $|\xi| \geq r$,*

$$0<\mu F(t,\xi)\leq \xi f(t,\xi). \tag{27}$$

(H2) *The impulsive functions I_{1i} satisfy sublinear growth; that is, there exist constants $\alpha_i > 0$, $\beta_i > 0$, and $\gamma_i \in [0,1)$, $i = 1,2,\ldots,l$, such that*

$$\left|I_{1i}(u)\right| \leq \alpha_i + \beta_i|u|^{\gamma_i}. \tag{28}$$

(H3) *The impulsive functions I_{2i}, $i = 1,2,\ldots,l$, are bounded.*

(H4) *$f(t,u) = o(|u|)$, $I_{1i}(u) = o(|u|)$, $I_{2i}(u) = o(|u|)$ as $|u| \to 0$, $i = 1,2,\ldots,l$.*

Then the functional J defined by (9) is continuously differentiable. Moreover, it satisfies the Palais-Smale (PS) condition.

Proof. By the continuity of f, I_{1i} and I_{2i}, $i = 1,2,\ldots,l$, we know that J is continuously differentiable. Next, we will prove that J satisfies the Palais-Smale condition. Let $\{J(u_k)\}$ be a bounded sequence such that $|J'(u_k)| \to 0$ as $k \to \infty$. Then there exist two constants $k, C_1 > 0$ such that for k sufficiently large

$$\left|J(u_k)\right| \leq C_1. \tag{29}$$

By (H3), there exists a constant $C_2 > 0$ such that

$$\left|I_{2i}\right| \leq C_2, \quad i=1,2,\ldots,l. \tag{30}$$

Then for k sufficiently large, by (H1) (H2) and the definitions of J, J', we have

$$\begin{aligned} \|u_k\|_X^2 &= 2J(u_k) + 2\lambda\sum_{i=1}^{l}\int_0^{u_k(t_i)} I_{1i}(s)\,ds + 2\lambda\sum_{i=1}^{l}\int_0^{u_k'(t_i)} I_{2i}(s)\,ds + 2\lambda\int_0^T F(t,u_k(t))\,dt \\ &\leq 2J(u_k) + 2\lambda\sum_{i=1}^{l}\int_0^{u_k(t_i)} I_{1i}(s)\,ds + 2\lambda\sum_{i=1}^{l}\int_0^{u_k'(t_i)} I_{2i}(s)\,ds + \frac{2\lambda}{\mu}\int_0^T f(t,u_k(t))\,u_k(t)\,dt \\ &= 2J(u_k) + 2\lambda\sum_{i=1}^{l}\int_0^{u_k(t_i)} I_{1i}(t)\,dt + 2\lambda\sum_{i=1}^{l}\int_0^{u_k'(t_i)} I_{2i}(s)\,ds + \frac{2}{\mu}\|u_k\|_X^2 \\ &\quad - \frac{2\lambda}{\mu}\sum_{i=1}^{l}I_{1i}(u_k(t_i))\,u_k(t_i) - \frac{2\lambda}{\mu}\sum_{i=1}^{l}I_{2i}(u_k'(t_i))\,u_k'(t_i) \\ &\quad - \frac{2}{\mu}J'(u_k)(u_k). \end{aligned} \tag{31}$$

By Lemma 8, (29), (30), (31), (H2), and (H3), one has

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_k\|_X^2 \\
&= J(u_k)-\frac{1}{\mu}J'(u_k)(u_k)+\lambda\sum_{i=1}^{l}\int_0^{u_k(t_i)} I_{1i}(s)\,ds \\
&\quad+\lambda\sum_{i=1}^{l}\int_0^{u_k'(t_i)} I_{2i}(s)\,ds \\
&\quad-\frac{\lambda}{\mu}\sum_{i=1}^{l}I_{1i}(u_k(t_i))u_k(t_i)-\frac{\lambda}{\mu}\sum_{i=1}^{l}I_{2i}(u_k'(t_i))u_k'(t_i) \\
&\le C_1+\frac{C_1}{\mu}\|u_k\|+\lambda\|u_k\|_{C^1}\sum_{i=1}^{l}\left(\alpha_i+\beta_i\|u_k\|_{C^1}^{\gamma_i}\right)+l\lambda C_2\|u_k\|_{C^1} \\
&\quad+\frac{\lambda}{\mu}\|u_k\|_{C^1}\sum_{i=1}^{l}\left(\alpha_i+\beta_i\|u_k\|_{C^1}^{\gamma_i}\right)+\frac{l\lambda C_2}{\mu}\|u_k\|_{C^1} \\
&= C_1+\frac{C_1}{\mu}\|u_k\|+\lambda\left(1+\frac{1}{\mu}\right)M\|u_k\|\sum_{i=1}^{l}\left(\alpha_i+\beta_i M\|u_k\|^{\gamma_i}\right) \\
&\quad+l\lambda C_2\left(1+\frac{1}{\mu}\right)M\|u_k\|.
\end{aligned} \tag{32}$$

So $\{u_k\}$ is bounded in X, which implies that the sequence $\{u_k\}$ weakly converges to u.

Next we show that $\{u_k\}$ strongly converges to u in X:

$$\begin{aligned}
&\left(J'(u_k)-J'(u)\right)(u_k-u) \\
&= \|u_k-u\|_X^2-\lambda\sum_{i=1}^{l}[I_{1i}(u_k(t_i))-I_{1i}(u(t_i))][u_k(t_i)-u(t_i)] \\
&\quad-\lambda\sum_{i=1}^{l}\left[I_{2i}(u_k'(t_i))-I_{2i}(u'(t_i))\right]\left[u_k'(t_i)-u'(t_i)\right] \\
&\quad-\lambda\int_0^T[f(t,u_k(t))-f(t,u(t))][u_k(t_i)-u(t_i)]\,dt.
\end{aligned} \tag{33}$$

Similar to the proof of Proposition 1.2 in [18], the weak convergence $u_k \rightharpoonup u$ implies that $\{u_k\}$ uniformly converges to u in $C([0,T])$. Since $u_k \in X$, u_k' converges to u' in $C[0,T]$. Thus

$$\begin{gathered}
\left(J'(u_k)-J'(u)\right)(u_k-u)\longrightarrow 0, \\
\lambda\sum_{i=1}^{l}[I_{1i}(u_k(t_i))-I_{1i}(u(t_i))][u_k(t_i)-u(t_i)]\longrightarrow 0, \\
\lambda\sum_{i=1}^{l}\left[I_{2i}(u_k'(t_i))-I_{2i}(u'(t_i))\right]\left[u_k'(t_i)-u'(t_i)\right]\longrightarrow 0, \\
\lambda\int_0^T[f(t,u_k(t))-f(t,u(t))][u_k(t_i)-u(t_i)]\,dt\longrightarrow 0.
\end{gathered} \tag{34}$$

So $\|u_k-u\|_X \to 0$ as $k \to \infty$. In other words, $\{u_k\}$ converges strongly to u in X. □

Remark 11. By (H1), there exist $a_1, a_2 > 0$ such that

$$F(t,\xi)\ge a_1|\xi|^{\mu}-a_2, \tag{35}$$

for all $t\in[0,T]$, $\xi\in R$.

3. Existence Results for One Solution and Infinitely Many Solutions

Theorem 12. *Suppose that (H1)–(H3) hold. Furthermore, we assume (H4) holds. Then problem* (1) *has at least one nontrivial solution.*

Proof. We will use Theorem 1 to prove the theorem. By Lemma 10, we have known that J satisfies the (PS) condition and it is obvious that $J(0)=0$. By (H4), for any $\varepsilon>0$, there exists a $\delta>0$ such that $|\xi|\le\delta$, which implies

$$\begin{gathered}
|F(t,\xi)|\le\frac{1}{2}\varepsilon|\xi|^2,\qquad \left|\sum_{i=1}^{l}\int_0^{\xi}I_{1i}(s)\,ds\right|\le\frac{1}{2}l\varepsilon|\xi|^2, \\
\left|\sum_{i=1}^{l}\int_0^{\xi}I_{2i}(s)\,ds\right|\le\frac{1}{2}l\varepsilon|\xi|^2
\end{gathered} \tag{36}$$

for all $t\in[0,T]$. Consequently, by Lemma 9, one has, for $\|u\|_X\le\delta/M$,

$$\begin{gathered}
\left|\int_0^T F(t,u(t))\,dt\right|\le\frac{1}{2}M^2T\varepsilon\|u\|_X^2, \\
\left|\sum_{i=1}^{l}\int_0^{u(t_i)}I_{1i}(s)\,ds\right|\le\frac{1}{2}M^2l\varepsilon\|u\|_X^2, \\
\left|\sum_{i=1}^{l}\int_0^{u'(t_i)}I_{2i}(s)\,ds\right|\le\frac{1}{2}M^2l\varepsilon\|u\|_X^2.
\end{gathered} \tag{37}$$

Thus

$$\begin{aligned}
&\left|\int_0^T F(t,u(t))\,dt\right|+\left|\sum_{i=1}^{l}\int_0^{u(t_i)}I_{1i}(s)\,ds\right|+\left|\sum_{i=1}^{l}\int_0^{u'(t_i)}I_{2i}(s)\,ds\right| \\
&\le\frac{1}{2}M^2(T+2l)\,\varepsilon\|u\|_X^2=o\left(\|u\|_X^2\right)
\end{aligned} \tag{38}$$

as $\|u\|_X \to 0$. Therefore,

$$J(u) = \frac{1}{2}\|u\|_X^2 - \lambda\sum_{i=1}^{l}\int_0^{u(t_i)} I_{1i}(s)\,ds - \lambda\sum_{i=1}^{l}\int_0^{u'(t_i)} I_{2i}(s)\,ds - \lambda\int_0^T F(t,u(t))\,dt \geq \frac{1}{2}\|u\|_X^2 + o\left(\|u\|_X^2\right) \tag{39}$$

as $u \to 0$. Thus (C1) holds.

To verify (C2), we choose $e(t) \in X$, $\kappa \in R$ such that $K_1 = \|e\|_X > 0$, $K_2 = \|e\|_{L^2} > 0$, K_1, K_2 are constants. Then by (H1), (H2), Remark 11, Lemma 8, and (30), one has

$$\begin{aligned} J(\kappa e) &= \frac{\kappa^2}{2}\|e\|_X^2 - \lambda\sum_{i=1}^{l}\int_0^{\kappa e(t_i)} I_{1i}(s)\,ds - \lambda\sum_{i=1}^{l}\int_0^{\kappa e'(t_i)} I_{2i}(s)\,ds - \lambda\int_0^T F(t,\kappa e(t))\,dt \\ &\leq \frac{\kappa^2}{2}\|e\|_X^2 + \lambda\sum_{i=1}^{l}\left(\alpha_i\|\kappa e\|_{C^1} + \frac{\beta_i\|\kappa e\|_{C^1}^{\gamma_i+1}}{\gamma_i+1}\right) + \lambda l C_2|\kappa|\,\|e\|_{C^1} - \lambda\int_0^T\left(a_1|\kappa|^\mu|e(t)|^\mu - a_2\right)dt \\ &\leq \frac{\kappa^2}{2}K_1^2 + \lambda\sum_{i=1}^{l}\left(\alpha_i|\kappa|MK_1 + \frac{\beta_i|\kappa|^{\gamma_i+1}M^{\gamma_i+1}K_1^{\gamma_i+1}}{\gamma_i+1}\right) + \lambda l C_2 M K_1|\kappa| - \lambda\int_0^T\left(a_1|\kappa|^\mu|e(t)|^\mu - a_2\right)dt. \end{aligned} \tag{40}$$

By Hölder's inequality, we have

$$\int_0^T|e(t)|^\mu dt \geq \left[\int_0^T e^2(t)\,dt T^{(2-\mu)/\mu}\right]^{\mu/2} = \|e\|_{L^2}^\mu T^{(2-\mu)/2} = K_2^\mu T^{(2-\mu)/2}. \tag{41}$$

Substituting (41) into (40), we have

$$J(\kappa e) \leq \frac{\kappa^2}{2}K_1^2 + \lambda\left[MK_1\sum_{i=1}^{l}\left(\alpha_i + \frac{\beta_i|\kappa e|^{\gamma_i+1}M^{\gamma_i+1}K_1^{\gamma_i+1}}{\gamma_i+1}\right) + lC_2K_1M\right]|\kappa| - a_1\lambda|\kappa|^\mu K_2^\mu T^{(2-\mu)/2} + a_2\lambda T \longrightarrow -\infty \tag{42}$$

as $|\kappa| \to +\infty$. Hence (C2) holds. Therefore, applying Theorem 1, we deduce that J admits a critical value $c > 0$ characterized as in the statement of Theorem 1 to J, J possesses critical value $c > 0$ given by

$$c = \inf_{g\in\Gamma}\max_{u\in g([0,1])} J(u), \tag{43}$$

where

$$\Gamma = \{g \in C([0,1],E) : g(0) = 0,\ g(1) = e\}. \tag{44}$$

Let $u^* \in X$ be a critical point associated with the critical value c of J (i.e., $J(u^*) = c$). Condition $c > 0$ implies that $u^* \not\equiv 0$. Lemma 8 means that *IBVP* (1) has at least one nontrivial solution. □

Theorem 13. *Suppose that (H1)–(H4) hold. Moreover, assume that the nonlinearity $f(t,u)$ and impulsive functions I_{1i}, I_{2i} are all odd in u. Then IBVP (1) has infinitely many classical solutions.*

Proof. We apply Theorem 2 to complete the proof. Clearly $J \in C^1(X,R)$ is even and $J(0) = 0$. Lemma 10 shows that J satisfies (PS) condition. The arguments of Theorem 12 show that J satisfies (C3) in Theorem 2. To verify (C4), let W be any finite dimensional space in X. For any $u \in W$, by (H1), (H2), Remark 11, Lemma 4, and (30), one has

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_X^2 - \lambda\sum_{i=1}^{l}\int_0^{u(t_i)} I_{1i}(s)\,ds - \lambda\sum_{i=1}^{l}\int_0^{u'(t_i)} I_{2i}(s)\,ds - \lambda\int_0^T F(t,u(t))\,dt \\ &\leq \frac{1}{2}\|u\|_X^2 + \lambda\sum_{i=1}^{l}\left(\alpha_i + \frac{\beta_i|u|^{\gamma_i+1}}{\gamma_i+1}\right) + l\lambda C_2\|u\|_{C^1} - \lambda\int_0^T\left(a_1|u(t)|^\mu - a_2\right)dt \\ &\leq \frac{1}{2}\|u\|_X^2 + \lambda M\|u\|_X\sum_{i=1}^{l}\left(\alpha_i + \frac{\beta_i M^{\gamma_i+1}\|u\|_X^{\gamma_i+1}}{\gamma_i+1}\right) + l\lambda C_2M\|u\|_X - \lambda\int_0^T\left(a_1|u(t)|^\mu - a_2\right)dt. \end{aligned} \tag{45}$$

For finite dimensional space W, the norm $\|\cdot\|_X$ is equivalent to $\|\cdot\|_W$.

So there exists $c_0 > 0$ satisfying

$$\|u\|_X \leq c_0\|u\|_{L^\mu} \tag{46}$$

for $u \in W$. Thus,

$$\int_0^T a_1|u(t)|^\mu dt \geq a_1c_0^{-\mu}\|u\|_X^\mu. \tag{47}$$

By (45) (46) (47) we have

$$J(u) \leq \frac{1}{2}\|u\|_X^2 + \lambda\sum_{i=1}^{l}\left(\alpha_i + \frac{\beta_i M^{\gamma_i+1}\|u\|_X^{\gamma_i+1}}{\gamma_i+1}\right) + l\lambda C_2M\|u\|_X - \lambda a_1c_0^{-\mu}\|u\|_X^\mu + \lambda a_2 T \longrightarrow -\infty \tag{48}$$

as $\|u\|_X \to +\infty$. That is, there exists $R > 0$ such that $J(u) < 0$ for $u \in W \setminus B_{R(W)}$. The proof is complete. □

4. Existence Result for Three Nonnegative Solutions

In this part, we need the following conditions.

(H5) For I_{1i}, I_{2i}, and f,

(C5) $f \in C([0,T]\times[0,+\infty);[0,+\infty))$, $I_{1i}\in C(R;R)$, $I_{2i}\in C(R;R)$, $i=1,2,\ldots,l$;

(C6) $f(t,0)=I_{1i}(0)=I_{2i}(0)=0$ for almost every $t\in[0,T]$ and $I_{2i}(x)x\geq 0$ for all $x\in R$.

Lemma 14 (see Lemma 2.2 [12]). *For $u\in X$, let $u^{\pm}=\max\{\pm u,0\}$. Then the following five properties hold:*

(i) $u\in X \Rightarrow u^+, u^-\in X$;

(ii) $u=u^+-u^-$;

(iii) $\|u^+\|_X\leq\|u\|_X$;

(iv) *if (u_n) uniformly converges to u in $C([0,T])$, then (u_n^+) uniformly converges to u^+ in $C([0,T])$;*

(v) $u^+(t)u^-(t)=0$, $(u^+)'(t)(u^-)'(t)=0$ *for a.e.* $t\in[0,T]$.

Lemma 15. *If $u\in C([0,T])$ is a classical solution of problem*

$$u^{(4)}(t)+u(t)=\lambda f(t,u^+(t)),\quad t\neq t_i,\ t\in[0,T],$$
$$\Delta u'''(t_i)=\lambda I_{1i}(u^+(t_i)),\quad i=1,2,\ldots,l,$$
$$-\Delta u''(t_i)=\lambda I_{2i}\left((u^+)'(t_i)\right),\quad i=1,2,\ldots,l, \tag{49}$$
$$au(0)+bu'''(0)=0,\qquad au(T)-bu'''(T)=0,$$
$$cu'(0)-du''(0)=0,\qquad cu'(T)+du''(T)=0,$$

then $u(t)\geq 0$ for $t\in[0,T]$, and hence it is a nonnegative classical solution of (1).

Proof. Since $u\in C[0,T]$ and $f\in C([0,T]\times[0,+\infty);[0,+\infty))$, we have $u^{(4)}\in C[0,T]\setminus\{t_1,t_2,\ldots,t_l\}$. If $u\in C([0,T])$ is a classical solution of problem (49), by Lemma 14, (H5) and boundary conditions, we have

$$\begin{aligned}
0&=\int_0^T\left(u^{(4)}(t)+u(t)-\lambda f(t,u^+(t))\right)u^-(t)\,dt\\
&=-\lambda\sum_{i=1}^{l}I_{1i}(u^+(t_i))u^-(t_i)-\lambda\sum_{i=1}^{l}I_{2i}\left((u^+)'(t_i)\right)(u^-)'(t_i)\\
&\quad+u'''(T)u^-(T)\\
&\quad-u'''(0)u^-(0)-u''(T)(u^-)'(T)+u''(0)(u^-)'(0)\\
&\quad+\int_0^T u''(t)(u^-)''(t)\,dt\\
&\quad+\int_0^T u(t)(u^-)(t)\,dt-\lambda\int_0^T f(t,u^+(t))u^-(t)\,dt\\
&=\frac{a}{b}u(T)u^-(T)+\frac{a}{b}u(0)u^-(0)+\frac{c}{d}u'(T)(u^-)'(T)\\
&\quad+\frac{c}{d}u'(0)(u^-)'(0)\\
&\quad+\int_0^T u''(t)(u^-)''(t)+u(t)(u^-)(t)\,dt\leq-\|u^-\|_X^2.
\end{aligned} \tag{50}$$

So $u^-(t)=0$ for $t\in[0,T]$; that is, $u(t)\geq 0$. The proof is complete. □

Remark 16. By Lemmas 14 and 15, in order to obtain the nonnegative solutions of (1), it is sufficient to show the existence of solutions of (49).

For each $u\in X$, set

$$\Phi(u)=\frac{1}{2}\|u\|_X^2, \tag{51}$$

$$\begin{aligned}
\Psi(u)&=\sum_{i=1}^{l}\int_0^{u^+(t_i)}I_{1i}(s)\,ds+\sum_{i=1}^{l}\int_0^{(u^+)'(t_i)}I_{2i}(s)\,ds\\
&\quad+\int_0^T F(t,u^+(t))\,dt,\\
J(u)&=\Phi(u)-\lambda\Psi(u).
\end{aligned} \tag{52}$$

It is obvious that Φ, Ψ, and J are differentiable for any $u\in X$. Then we have

$$\begin{aligned}
\Phi'(u)(v)&=\int_0^T\left(u''v''+uv\right)dt+\frac{a}{b}u(T)v(T)+\frac{a}{b}u(0)v(0)\\
&\quad+\frac{c}{d}u'(T)v'(T)+\frac{c}{d}u'(0)v'(0),
\end{aligned} \tag{53}$$

$$\begin{aligned}
\Psi'(u)(v)&=\sum_{i=1}^{l}I_{1i}(u^+(t_i))v(t_i)+\sum_{i=1}^{l}I_{2i}\left((u^+)'(t_i)\right)v'(t_i)\\
&\quad+\int_0^T f(t,u^+(t))v(t)\,dt.
\end{aligned} \tag{54}$$

Definition 17. A function $u\in X$ is said to be a weak solution of (49), if u satisfies $J'(u)(v)=0$ for all $v\in X$.

Lemma 18. *If $u\in X$ is a weak solution of* (49), *then u is a classical solution of* (49).

Proof. It is similar to the proof of Lemma 8, so we omit it here. □

Lemma 19. $\Phi' : X \to X^*$ *admits a continuous inverse on* X^*.

Proof. First, for every $u \in X \setminus \{0\}$, by (53), we have

$$\lim_{\|u\|_X \to +\infty} \frac{\Phi'(u)(u)}{\|u\|_X} = \lim_{\|u\|_X \to +\infty} \frac{\|u\|_X^2}{\|u\|_X} = +\infty, \quad (55)$$

which means that Φ' is coercive. Furthermore, given $u, v \in X$, one has

$$\left(\Phi'(u) - \Phi'(v)\right)(u - v) \geq \|u - v\|_X^2, \quad (56)$$

so Φ' is uniformly monotone. By Theorem 26.A(d) of [20], we have that $(\Phi')^{-1}$ exists and $(\Phi')^{-1}$ is continuous on X^*. Thus, $\Phi' : X \to X^*$ admits a continuous inverse on X^*. The proof is complete. □

Lemma 20. $\Psi' : X \to X^*$ *is a continuous and compact operator.*

Proof. First we will show that Ψ' is strongly continuous on X. Let $u_n \rightharpoonup u$ as $n \to \infty$ on X. By [20], we have (u_n) converges uniformly to u on $[0, T]$ as $n \to \infty$. Since f is continuous, one has $f(t, u_n) \to f(t, u)$ as $n \to \infty$. Furthermore, I_{1i}, I_{2i} are all continuous. So $\Psi'(u_n) \to \Psi'(u)$, which implies that Ψ' is continuous and that Ψ' is a compact operator by Proposition 26.2 of [20]. The proof is complete. □

Theorem 21. *Suppose that the condition (H5) holds. Let* $k := \{2M^2(1024/T^3 + (83/240)T)\}^{-1}$, $I(s) := \sum_{i=1}^{l} |I_{1i}(s)| + \sum_{i=1}^{l} |I_{2i}(s)|$. *There exist four positive constants m, n, p, q, with* $\sqrt{k}m < n < \sqrt{k}p < \sqrt{k}q$ *such that*

(H6) (C7) $(\int_0^m I(s)ds + \int_0^T F(t,m)dt)/m^2 < k((-\int_0^m I(s)ds + \int_{T/4}^{3T/4} F(t,n)dt - \int_0^T F(t,m)dt)/n^2)$,

(C8) $(\int_0^p I(s)ds + \int_0^T F(t,p)dt)/p^2 < k((-\int_0^m I(s)ds + \int_{T/4}^{3T/4} F(t,n)dt - \int_0^T F(t,m)dt)/n^2)$,

(C9) $(\int_0^q I(s)ds + \int_0^T F(t,q)dt)/(q^2 - p^2) < k((-\int_0^m I(s)ds + \int_{T/4}^{3T/4} F(t,n)dt - \int_0^T F(t,m)dt)/n^2)$.

Then, for every

$$\lambda \in \left] \frac{n^2}{2M^2k} \left(-\int_0^m I(s)\,ds + \int_{T/4}^{3T/4} F(t,n)\,dt - \int_0^T F(t,m)\,dt \right)^{-1}, \right.$$
$$\min \left\{ \frac{m^2}{2M^2} \left(\int_0^m I(s)\,ds + \int_0^T F(t,m)\,dt \right)^{-1}, \frac{p^2}{2M^2} \left(\int_0^p I(s)\,ds + \int_0^T F(t,p)\,dt \right)^{-1}, \frac{q^2 - p^2}{2M^2} \left(\int_0^q I(s)\,ds + \int_0^T F(t,q)\,dt \right)^{-1} \right\} \left[, \right. \quad (57)$$

the problem (1) *has at least three distinct nonnegative classical solutions* u_i $(i = 1, 2, 3)$, *such that* $\|u_i\|_{C^1} < q$, *which means that the problem* (1) *has at least two distinct positive classical solutions.*

Proof. The proof is based on Theorem 3. First, we will prove that Φ and Ψ satisfy the hypotheses in Theorem 3. On the one hand, Φ is coercive and its Gâteaux derivative admits a continuous inverse by Lemma 19. On the other hand, Φ is obviously convex. Ψ's Gâteaux derivative is continuous and compact by Lemma 20. In addition, $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. By (H5), we have

$$\int_0^s I_{1i}(s)\,ds \geq 0, \qquad \int_0^s I_{2i}(s)\,ds \geq 0,$$
$$F(t,s) = \int_0^s f(t,s)\,ds \geq 0, \quad (58)$$

which deduces that $\Psi(u) \geq 0$ for all $u \in X$.

Next, we will verify the conditions (i), (ii), (iii) in Theorem 3. First, we define

$$\overline{v}(t) = \begin{cases} \frac{32n}{T^2} t^2, & t \in \left[0, \frac{T}{8}\right], \\ -\frac{32n}{T^2}\left(t - \frac{T}{4}\right)^2 + n, & t \in \left]\frac{T}{8}, \frac{T}{4}\right], \\ n, & t \in \left]\frac{T}{4}, \frac{3T}{4}\right], \\ -\frac{32n}{T^2}\left(t - \frac{3T}{4}\right)^2 + n, & t \in \left]\frac{3T}{4}, \frac{7T}{8}\right], \\ \frac{32n}{T^2}(t - T)^2, & t \in \left]\frac{7T}{8}, T\right]. \end{cases} \quad (59)$$

It is easy to verify that

$$\overline{v}^+ = \overline{v}, \qquad \overline{v}^- = 0. \quad (60)$$

By computing,

$$\Phi(\overline{v}) = \left(\frac{1024}{T^3} + \frac{83}{240}T\right) n^2 = \frac{n^2}{2M^2k}. \quad (61)$$

Let $r_1 = m^2/2M^2$, $r_2 = p^2/2M^2$, and $r_3 = (q^2 - p^2)/2M^2$. By $\sqrt{k}m < n < \sqrt{k}p < \sqrt{k}q$, one has $r_1 < \Phi(\overline{v}) < r_2$, which means that $\overline{v} \in \Phi^{-1}(]r_1, r_2[)$ and $r_3 > 0$. When $\Phi(u) < r_1$, by Lemma 9 and (51), we have

$$\max\left\{\max_{t\in[0,T]} \left|u^+(t)\right|, \max_{t\in[0,T]} \left|(u^+)'(t)\right|\right\} \leq M\|u^+\|_X \leq M\|u\|_X \leq \sqrt{2M^2\Phi(u)} < m. \quad (62)$$

In view of (51) and $I(s)$, we have

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}&\Psi(u)\\ &\leq \max_{|\xi|\leq m}\sum_{i=1}^{l}\int_0^{\xi}\left|I_{1i}(s)\right|ds+\max_{|\xi|\leq m}\sum_{i=1}^{l}\int_0^{\xi}\left|I_{2i}(s)\right|ds\\ &\quad+\int_0^T\max_{|\xi|\leq m}F(t,\xi)\,dt\\ &=\int_0^m I(s)\,ds+\int_0^T F(t,m)\,dt.\end{aligned} \tag{63}$$

Similarly, we have

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)&\leq\int_0^p I(s)\,ds+\int_0^T F(t,p)\,dt,\\ \sup_{u\in\Phi^{-1}(]-\infty,r_2+r_3[)}\Psi(u)&\leq\int_0^q I(s)\,ds+\int_0^T F(t,q)\,dt.\end{aligned} \tag{64}$$

Therefore, taking into consideration that $0\in\Phi^{-1}(]-\infty,r_1[)$ and $0\in\Phi^{-1}(]-\infty,r_2[)$, by (63) and (64), we have

$$\begin{aligned}\varphi(r_1)&\leq\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1}\\ &\leq\frac{2M^2}{m^2}\left(\int_0^m I(s)\,ds+\int_0^T F(t,m)\,dt\right),\\ \varphi(r_2)&\leq\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2}\\ &\leq\frac{2M^2}{p^2}\left(\int_0^p I(s)\,ds+\int_0^T F(t,p)\,dt\right),\\ \gamma(r_2,r_3)&=\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2+r_3[)}\Psi(u)}{r_3}\\ &\leq\frac{2M^2}{q^2-p^2}\left(\int_0^q I(s)\,ds+\int_0^T F(t,q)\,dt\right).\end{aligned} \tag{65}$$

Furthermore, by (60) and the definition of $\overline{v}$,

$$\begin{aligned}\Psi(\overline{v})&=\sum_{i=1}^{l}\int_0^{\overline{v}(t_i)}I_{1i}(s)\,ds+\sum_{i=1}^{l}\int_0^{\overline{v}'(t_i)}I_{2i}(s)\,ds\\ &\quad+\int_0^T F(t,\overline{v}(t))\,dt\\ &\geq\sum_{i=1}^{l}\int_0^{\min_{t\in[0,T]}\overline{v}(t)}I_{1i}(s)\,ds+\sum_{i=1}^{l}\int_0^{\min_{t\in[0,T]}|\overline{v}'(t)|}\left|I_{2i}(s)\right|ds\\ &\quad+\int_{T/4}^{3T/4}F(t,\overline{v}(t))\,dt=\int_{T/4}^{3T/4}F(t,n)\,dt.\end{aligned} \tag{66}$$

Taking $\overline{v}\in\Phi^{-1}([r_1,r_2[)$ into consideration, by (61), (63), (66), (C9), and $\Phi(u)\geq 0$, one has

$$\begin{aligned}&\beta(r_1,r_2)\\ &\geq\inf_{u\in\Phi^{-1}(]-\infty,r_1[)}\frac{\Psi(\overline{v})-\Psi(u)}{\Phi(\overline{v})-\Phi(u)}\\ &\geq\frac{\int_{T/4}^{3T/4}F(t,n)\,dt-\left(\int_0^m I(s)\,ds+\int_0^T F(t,m)\,dt\right)}{n^2/2M^2k}\\ &=\frac{2M^2k}{n^2}\left(-\int_0^m I(s)\,ds+\int_{T/4}^{3T/4}F(t,n)\,dt-\int_0^T F(t,m)\,dt\right).\end{aligned} \tag{67}$$

By (65), (67), and (C7)–(C9) of (H6), we have

$$\alpha(r_1,r_2,r_3)<\beta(r_1,r_2), \tag{68}$$

which yields the conditions in Theorem 3. By Theorem 3, it follows that, for each

$$\begin{aligned}\lambda\in\Bigg]&\frac{n^2}{2M^2k}\left(-\int_0^m I(s)\,ds+\int_{T/4}^{3T/4}F(t,n)\,dt\right.\\ &\qquad\left.-\int_0^T F(t,m)\,dt\right)^{-1},\\ &\min\left\{\frac{m^2}{2M^2}\left(\int_0^m I(s)\,ds+\int_0^T F(t,m)\,d\right)^{-1},\right.\\ &\qquad\frac{p^2}{2M^2}\left(\int_0^p I(s)\,ds+\int_0^T F(t,p)\,dt\right)^{-1},\\ &\qquad\left.\frac{q^2-p^2}{2M^2}\left(\int_0^q I(s)\,ds+\int_0^T F(t,q)\,dt\right)^{-1}\right\}\Bigg[,\end{aligned} \tag{69}$$

the functional $J=\Phi-\lambda\Psi$ has three distinct critical points u_i $(i=1,2,3)$ in X with $\Phi(u_i)<r_2+r_3$. By Lemma 9 and (51)

$$\begin{aligned}\|u_i^+\|_{C^1}&=\max\left\{\max_{t\in[0,T]}\left|u_i^+(t)\right|,\max_{t\in[0.T]}\left|(u_i^+)'(t)\right|\right\}\\ &\leq M\|u_i^+\|_X\leq M\|u_i\|_X\leq\sqrt{2M^2\Phi(u_i)}<q.\end{aligned} \tag{70}$$

By Remark 16, u_i $(i=1,2,3)$ are three positive solutions of (1). The proof is complete. □

Remark 22. If we choose different $\overline{v}$, then the constrictions on F, I_i are different.

5. Examples

Example 1. Let $T>0$, $t_i\in(0,T)$, $a,b,c,d>0$, $x_1,x_2,x_3,y_i,z_i\in C([0,T],R^+)$, $i=1,2,\ldots,l$. Consider the following Sturm-Liouville boundary-value problem with impulse:

$$\begin{gathered} u^{(4)}(t)+u(t)=\lambda f_1(t,u), \quad t\in[0,T]\setminus\{t_1,t_2,\ldots,t_l\}, \\ \Delta u'''(t_i)=\lambda I_i(u(t_i)), \quad i=1,2,\ldots,l, \\ -\Delta u''(t_i)=-\lambda z_i(t)\sin u'^3(t_i), \quad i=1,2,\ldots,l, \\ au(0)+bu'''(0)=0, \qquad au(T)-bu'''(T)=0, \\ cu'(0)-du''(0)=0, \qquad cu'(T)+du''(T)=0, \end{gathered} \tag{71}$$

where

$$\begin{gathered} f_1(t,u)=x_1(t)u^3+x_2(t)\sin u+x_3(t)\exp|u|, \\ I_i(u)=\begin{cases} y_i(t)u^3, & |u|\le 1, \\ y_i(t)u^{1/3}, & |u|>1. \end{cases} \end{gathered} \tag{72}$$

By computing, $F(t,u) = (1/4)x_1(t)u^4 - x_2(t)\cos u + x_3(t)\operatorname{sgn} u\exp|u|$. Let $\mu = 6$; there exists $r \ge 0$ such that, for $|\xi|\ge r$, $0<\mu F(t,\xi)\le \xi f(t,\xi)$. The conditions (H1)–(H4) are satisfied. Applying Theorem 1, problem (71) has at least one nontrivial solution.

Furthermore, the functions $f(t,u)$, I_{1i}, I_{2i} are all odd in u. Applying Theorem 2, problem (71) has infinitely many classical solutions.

Example 2. Let $T = 1$, $t_i \in (0,1)$, $i = 1,2,\ldots,l$, $a,b,c,d > 0$. Consider the following Sturm-Liouville boundary-value problem:

$$\begin{gathered} u^{(4)}(t)+u(t)=\lambda f_2(t,u(t)), \quad t\in[0,1]\setminus\{t_1,t_2,\ldots,t_l\}, \\ \Delta u'''(t_i)=\lambda I_{1i}(u(t_i)), \quad i=1,2,\ldots,l, \\ -\Delta u''(t_i)=\lambda I_{2i}(u'(t_i)), \quad i=1,2,\ldots,l, \\ au(0)+bu'''(0)=0, \qquad au(1)-bu'''(1)=0, \\ cu'(0)-du''(0)=0, \qquad cu'(1)+du''(1)=0, \end{gathered} \tag{73}$$

where

$$\begin{gathered} f_2(t,s)=I_{1i}(s)=I_{2i}(s)=\begin{cases} 0, & 0<s\le 1, \\ t(s-1), & 1<s\le 2, \\ -t(s-3), & 2<s\le 3, \\ 0, & s>3, \end{cases} \\ \text{for every } \lambda\in\left[\frac{7680}{245843},+\infty\right), \end{gathered} \tag{74}$$

problem (73) has at least two distinct positive classical solutions.

In fact, (H5) is fulfilled. By computing, $M = \max\{3, 2d/c\}$, so $k = \min\{40/737529, 30c^2/245843d^2\}$. Considering $\sqrt{k}m < n < \sqrt{k}p < \sqrt{k}q$, without loss of generality, we choose $m = 1/2$, $n = 2$ and sufficiently large p, q. Then we have

$$\begin{gathered} \frac{n^2}{2M^2k}\left(-\int_0^m I(s)\,ds+\int_{T/4}^{3T/4}F(t,n)\,dt-\int_0^T F(t,m)\,dt\right)^{-1} \\ =\frac{161280}{245843}, \\ \frac{m^2}{2M^2}\left(\int_0^m I(s)\,ds+\int_0^T F(t,m)\,dt\right)^{-1}=\infty, \\ \frac{p^2}{2M^2}\left(\int_0^p I(s)\,ds+\int_0^T F(t,p)\,dt\right)^{-1} \text{ is sufficiently large}, \\ \frac{q^2-p^2}{2M^2}\left(\int_0^q I(s)\,ds+\int_0^T F(t,q)\,dt\right)^{-1} \text{ is sufficiently large}, \end{gathered} \tag{75}$$

and that (H6) is satisfied. Applying Theorem 21, problem (73) has at least two distinct positive classical solutions for every $\lambda \in [161280/245843, +\infty)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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