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Research Article

Poincaré-Type Inequalities for the Composite Operator in $L^{\mathscr{A}}$-Averaging Domains

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We first establish the local Poincaré inequality with $L^{\mathscr{A}}$-averaging domains for the composition of the sharp maximal operator and potential operator, applied to the nonhomogenous A-harmonic equation. Then, according to the definition of $L^{\mathscr{A}}$-averaging domains and relative properties, we demonstrate the global Poincaré inequality with $L^{\mathscr{A}}$-averaging domains. Finally, we give some illustrations for these theorems.

1. Introduction

Poincaré inequality applied to differential forms has a vital role in PDEs, nonlinear analysis, and other related fields. With the further research conducted, we have established various versions of Poincaré inequality under different conditions. From [1–8], we have obtained the Poincaré inequality for the solution to the A-harmonic equation in uniformly bounded domain, John domains, and L^s-averaging domains. Nevertheless, most of these Poincaré inequalities are developed in L^s-averaging domains. In this paper, we will establish the Poincaré inequality for the composition of the sharp maximal operator and potential operator in $L^{\mathscr{A}}$-averaging domains. As we all know, both the uniformly bounded domain and John domains are special L^s-averaging domains, and the L^s-averaging domains are also particular $L^{\mathscr{A}}$-averaging domains, so the following results are the generalizations of the Poincaré inequality in L^s-averaging domains.

For convenience, we firstly introduce some notations and terminologies. Except for special instructions, $E \subseteq \mathbb{R}^m$ is a bounded domain, $|E|$ denotes the Lebesgue measure of E, and $m \geq 2$. The constant K and C can be varied at each step of the proof. Suppose that B_x^r is a ball, with a radius r, centered at x. For any $\rho > 0$, $B \subseteq E$ and $\rho B \subseteq E$ have the same center and satisfy $\operatorname{diam}(\rho B) = \rho \operatorname{diam}(B)$. Let $\Lambda^l(\mathbb{R}^m)$ be the space of all l-forms in $\mathbb{R}^m$, which is expanded by the exterior product of $e^{\mathscr{B}} = e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_l}$, where $\mathscr{B} = (i_1, \ldots, i_l)$, $1 \leq i_1 < \cdots < i_l \leq m$, $l = 1, 2, \ldots, m$. $C^{\infty}(\Lambda^l E)$ is the space of a smooth l-form on E. We use $D'(E, \Lambda^l)$ to denote the space of all differential l-forms on E; that is, $w(x)$ belongs to $D'(E, \Lambda^l)$ if and only if there exist some lth-differential functions $w_{\mathscr{B}}$ in E such that $w(x) = \sum_{\mathscr{B}} w_{\mathscr{B}}(x) dx_{\mathscr{B}} = \sum w_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$. $L^p(E, \Lambda^l)$ is a Banach space with the norm equipped by $\|w(x)\|_{p,E} = (\int_E |w(x)|^p dx)^{1/p}$, where $w(x) \in D'(E, \Lambda^l)$ and every coefficient function $w_{\mathscr{B}} \in L^p(E)$, $0 < p < \infty$. In fact, $w(x)$ on E is the Schwartz distribution. If $\omega(x) > 0$ a.e. and $\omega(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, $\omega(x)$ is called a weight. Let $d\mu = \omega(x)dx$; then $L^p(E, \Lambda^l, \omega)$ is a weighted Banach space with the norm expressed by $\|w(x)\|_{p,E,\omega} = (\int_E |w(x)|^p \omega(x) dx)^{1/p}$. In this notation, the exterior derivative is denoted by d and Hodge codifferential operator is expressed by $d^{\star}$. Search [9] for more details.

Considering our purpose, we intend to give a brief discussion about the A-harmonic equation for the differential form. The following equation is called a nonhomogeneous A-harmonic equation:

$$d^{\star} A(x, dw) = B(x, dw), \tag{1}$$

where $A : E \times \wedge^l(\mathbb{R}^m) \to \wedge^l(\mathbb{R}^m)$ and $B : E \times \wedge^l(\mathbb{R}^m) \to \wedge^{l-1}(\mathbb{R}^m)$ satisfy the conditions:

$$\left|A(x,\xi)\right| \le a|\xi|^{p-1}, \qquad A(x,\xi)\cdot\xi \ge |\xi|^p, \qquad \left|B(x,\xi)\right| \le b|\xi|^{p-1}, \tag{2}$$

for almost every $x \in E$ and all $\xi \in \wedge^l(\mathbb{R}^m)$. Here, $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1). If $B = 0$, the equation $d^\star A(x, dw) = 0$ is called a homogenous A-harmonic equation. See [9] for more information.

In order to describe it easily, we first give some definitions in this part.

Definition 1. Let $E \subseteq \mathbb{R}^m$ be a bounded domain and $w(x) \in L^p(E, \Lambda^l)$; the sharp maximal operator $\mathbb{M}_s^\sharp$ is equipped with

$$\mathbb{M}_s^\sharp(w) = \mathbb{M}_s^\sharp w = \mathbb{M}_s^\sharp w(x) = \sup_{r>0}\left(\frac{1}{|B_x^r|}\int_{B_x^r}\left|w(t) - w_{B_x^r}\right|^s dt\right)^{1/s}, \tag{3}$$

where B_x^r is the ball of radius r, centered at x, $1 \le s \le p$, $p \ge 1$.

Especially, if we take $s = 1$, denote $\mathbb{M}_s^\sharp \triangleq \mathbb{M}^\sharp$.

Definition 2 (see [10]). Suppose that $w(x)$ is a differential l-form; the potential operator P is expressed by

$$Pw(x) = \sum_{\mathscr{B}}\int_E K(x,y)\, w_{\mathscr{B}}(y)\, dy\, dx_\beta, \tag{4}$$

where the nonnegative and measurable function $K(x, y)$, defined on the set $\{(x, y) \mid x \ne y, x, y \in \mathbb{R}^m\}$, is a kernel function, and the summation is over all ordered l-tuple $\mathscr{B}$.

Definition 3. Take an increasingly continuous function $\mathscr{A} : [0,+\infty) \to [0,+\infty)$ as a convex function with $\mathscr{A}(0) = 0$, and $E \subseteq \mathbb{R}^m$ is a bounded domain, for any $w(x) \in L^p(E)$; the Orlicz norm for differential form is denoted by

$$\|w\|_{L^{\mathscr{A}}_{E,\mu}} = \inf\left\{\lambda > 0 \mid \frac{1}{\mu(E)}\int_E \mathscr{A}\left(\lambda^{-1}|u|\,d\mu\right) < 1\right\}, \tag{5}$$

where measure μ satisfies $d\mu = \omega(x)dx$, $\omega(x)$ is a weight.

We call $\mathscr{A}$ an Orlicz function if $\mathscr{A} : [0,+\infty) \to [0,+\infty)$ is an increasingly continuous function and satisfies $\mathscr{A}(0) = 0$ and $\mathscr{A}(\infty) = \infty$. Meanwhile, if the Orlicz function $\mathscr{A}(t)$ is a convex function, it is called a Young function.

Based on the above definition, we get the notation of $L^{\mathscr{A}}$-averaging domains.

Definition 4 (see [3]). Let $\mathscr{A}$ be a Young function; the proper domain $E \subseteq \mathbb{R}^m$ is called the $L^{\mathscr{A}}$-averaging domains if $\mu(E) < \infty$ and there exists a constant $C > 0$ such that for any $B_0 \subseteq E$ and $\mathscr{A}(|w|) \in L^1_{\mathrm{loc}}(E, \mu)$, w satisfies

$$\frac{1}{\mu(E)}\int_E \mathscr{A}\left(\tau\left|w - w_{B_0}\right|\right)d\mu \le C\sup_{4B\subset E}\frac{1}{\mu(B)}\int_B \mathscr{A}\left(\sigma\left|w - w_B\right|\right)d\mu, \tag{6}$$

where the measure μ is denoted by $d\mu = \omega(x)dx$, $\omega(x)$ is a weight, σ and τ are constants with $0 < \tau, \sigma < 1$, and the supremum is over all balls $B \subset E$ with $4B \subset E$.

Notice that if we let $\mathscr{A}(t) = t^s$, $L^{\mathscr{A}}$-averaging domains become the L^s-averaging domains, so $L^{\mathscr{A}}$-averaging domains are the generalization of L^s-averaging domains.

Definition 5 (see [11]). We call $w(x) \in D'(E, \Lambda^l)$ belongs to the $\mathrm{WRH}(\Lambda^l, E)$-class, $l = 0, 1, \ldots, m$, if for any constants $0 < s, t < \infty$ and any ball $B \subset E$ with $\rho B \subset E$, there exists a constant $C > 0$ such that $w(x)$ satisfies

$$\|w\|_{s,B} \le C|B|^{(t-s)/st}\|w\|_{t,\rho B}, \tag{7}$$

where $\rho > 1$ is a constant.

Remark 6. If $w(x)$ is a solution to the A-harmonic equation, we can prove that $w(x)$ belongs to the $\mathrm{WRH}(\Lambda^l, E)$-class.

2. Main Results

Before the main results are given, we need to impose some restrictions on the kernel function $K(x, y)$ and Young function $\mathscr{A}$. Firstly, let the kernel function satisfy the standard estimates; it is equal to say that if there exist $0 < \delta < 1$ and a constant $c > 0$ such that for any point $z \in \{z : |x - z| < (1/2)|x - y|, x, y \in \mathbb{R}^m\}$, the kernel function $K(x, y)$ satisfies that

(1) $K(x, y) \le c|x - y|^{-m}$, $x \ne y$;

(2) $|K(x, y) - K(z, y)| \le c|x - z|^\delta|x - y|^{-m-\delta}$, $x \ne y$;

(3) $|K(y, x) - K(y, z)| \le c|x - z|^\delta|x - y|^{-m-\delta}$, $x \ne y$,

where function $K(x, y) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $m \ge 1$.

With regard to the Young function $\mathscr{A}$, we let the Young function $\mathscr{A}$ belong to the $G(p, q, C)$-class $(1 \le p < q < \infty, C \ge 1)$; that is, for any $t > 0$, the Young function $\mathscr{A}$ satisfies that

(1) $1/C \le \mathscr{A}(t^{1/p})/f(t) \le C$;

(2) $1/C \le \mathscr{A}(t^{1/q})/g(t) \le C$,

where f and g are the increasingly convex and concave functions defined on $[0, \infty]$, respectively.

Now, we establish these two important theorems based on the above conditions.

Theorem 7. *Suppose that the Young function $\mathscr{A}$ belongs to the $G(p, q, C)$-class, $w \in C^\infty(\Lambda^l E)$ is a solution to the nonhomogenous A-harmonic equation, the sharp maximal* *operator is noted by $\mathbb{M}_s^\sharp$, P is the potential operator with its kernel function $K(x,y)$ satisfying the standard estimates, $1 \le s < p, q < \infty$, and the bounded subset $E \subseteq \mathbb{R}^m$ is the $L^{\mathscr{A}}$-averaging domains. Then, for any ball $B \subseteq E$, one gets*

$$\left\| \mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B \right\|_{L_B^{\mathscr{A}}} \le K \|w\|_{L_{\rho B}^{\mathscr{A}}}, \quad (8)$$

where B and $\rho B \subseteq E$ and the constant $\rho > 1$.

Based on the above theorem, we can establish the following theorem for the global Poincaré inequality in $L^{\mathscr{A}}$-averaging domains.

Theorem 8. *Suppose that the Young function $\mathscr{A}$ belongs to the $G(p,q,C)$-class, $w \in C^\infty(\Lambda^l E)$ is a solution to the nonhomogenous A-harmonic equation, the sharp maximal operator is denoted by $\mathbb{M}_s^\sharp$, P is the potential operator with its kernel function $K(x,y)$ satisfying the standard estimates, $1 \le s < p$, $q < \infty$, and the bounded subset $E \subseteq \mathbb{R}^m$ is the $L^{\mathscr{A}}$-averaging domains. Then, one has*

$$\left\| \mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_{B_0} \right\|_{L_E^{\mathscr{A}}} \le K \|w\|_{L_E^{\mathscr{A}}}, \quad (9)$$

where $B_0 \subseteq B$ is a fixed ball, which appears in Definition 4.

3. Preliminary Results

For proving the theorems in Section 2, we will show and demonstrate some lemmas in this part.

Lemma 9 (see [9]). *Let $0 < p, q < \infty$, and $1/t = (1/p)+(1/q)$, if f and g are the measurable functions defined on $\mathbb{R}^m$, then*

$$\|fg\|_{t,I} \le \|f\|_{p,I} \cdot \|g\|_{q,I}, \quad (10)$$

for any $I \subseteq \mathbb{R}^m$.

Lemma 10 (see [5]). *Let P be the potential operator applied on a differential form with $E \subseteq \mathbb{R}^m$, $w(x) \in WRH(\Lambda^l, E)$, and assume that the weight $\omega(x)$ belongs to $A(\alpha,\beta,E)$ with $\alpha, \beta > 0$. Then, there exists a constant C, independent of $w(x)$ such that*

$$\|P(w) - (P(w))_B\|_{s,B,\omega} \le C|B| \operatorname{diam}(B) \|w\|_{s,B,\omega}, \quad (11)$$

for any $B \subset E$, where $s > 1$ is a constant.

Remark 11. If we take $\omega(x) \equiv 1$, we get

$$\|P(w) - (P(w))_B\|_{s,B} \le C|B| \operatorname{diam}(B) \|w\|_{s,B}. \quad (12)$$

Lemma 12 (see [3]). *Take Ψ defined on $[0,+\infty)$ to be a strictly increasing convex function, $\Psi(0) = 0$, and $E \subset \mathbb{R}^m$ is a domain. Assume that $w(x) \in D'(E,\Lambda^l)$ satisfies $\Psi(|w|) \in L^1(E,\mu)$ and, for any constant c,*

$$\mu\{x \in E : |w - c| > 0\} > 0, \quad (13)$$

where μ is a Radon measure defined by $d\mu(x) = \omega(x)dx$ with a weight $\omega(x)$; then for any $a > 0$, one obtains

$$\int_E \Psi\left(\frac{a}{2}|w - w_E|\right) d\mu \le \int_E \Psi(a|w|)\, d\mu. \quad (14)$$

Lemma 13. *If $\omega(x) \in A_r(E)$, then there exist constants $\alpha > 1$ and K, not dependent on ω, such that*

$$\|\omega\|_{\alpha,B} \le K|B|^{(1/\alpha)-1} \|\omega\|_{1,B}, \quad (15)$$

for all balls B contained in E.

Lemma 14. *The sharp maximal operator $\mathbb{M}_s^\sharp$ is denoted by Definition 1, and the potential operator P is defined by Definition 2 with the kernel function $K(x,y)$ satisfying the standard estimates, $w(x) \in L^t(E,\Lambda^l) \cap C^\infty(\Lambda^l E)(l = 1,2,\ldots,m)$, $t \ge 1$. Then, there exists a constant $K > 0$, independent of w, such that*

$$\int_B \left|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|^t dx \le K|B|^{1+(1/m)} \|w\|_{t,B}, \quad (16)$$

for all balls $B \subset E$.

Proof. Let B be a ball in E, using Lemma 10 on any $B_x^r \subset B$, we have

$$\begin{aligned} &\left(\frac{1}{|B_x^r|}\int_{B_x^r} \left|P(w) - (P(w))_{B_x^r}\right|^s dx\right)^{1/s} \\ &\le K|B_x^r| \operatorname{diam}(B_x^r) |B_x^r|^{-1/s} \|w\|_{s,B_x^r} \\ &\le K|B_x^r|^{1-(1/s)} \operatorname{diam}(B_x^r) \|w\|_{s,B} \\ &\le K|B|^{1-(1/s)+(1/m)} \|w\|_{s,B}. \end{aligned} \quad (17)$$

From Lemma 14 in [7], it follows that

$$\|w\|_{s,B} \le |B|^{(1/s)-(1/t)} \|w\|_{t,B}, \quad (18)$$

where $0 < s \le t < \infty$. Substituting (18) into (17) yields

$$\begin{aligned} &\left(\frac{1}{|B_x^r|}\int_{B_x^r} \left|P(w) - (P(w))_{B_x^r}\right|^s dx\right)^{1/s} \\ &\le K|B|^{1-(1/t)+(1/m)} \|w\|_{t,B}. \end{aligned} \quad (19)$$

Taking the supremum for r, we get that

$$\begin{aligned} &\sup_{r>0}\left(\frac{1}{|B_x^r|}\int_{B_x^r} \left|P(w) - (P(w))_{B_x^r}\right|^s dx\right)^{1/s} \\ &\le \sup_{r>0} K|B|^{1-(1/t)+(1/m)} \|w\|_{t,B} \\ &= K|B|^{1-(1/t)+(1/m)} \|w\|_{t,B}. \end{aligned} \quad (20)$$

That is,

$$\mathbb{M}_s^\sharp (P(w)) \le K|B|^{1-(1/t)+(1/m)} \|w\|_{t,B}. \quad (21)$$

According to the definition of $L^t(E)$ norm and formula (21), it yields

$$\begin{aligned}\left\|\mathbb{M}_s^\sharp (P(w))\right\|_{t,B} &= \left(\int_B \left|\mathbb{M}_s^\sharp (P(w))\right|^t dx\right)^{1/t} \\ &\le \left(\int_B \left|K|B|^{1-(1/t)+(1/m)} \|w\|_{t,B}\right|^t dx\right)^{1/t} \\ &= K|B|^{1+(1/m)} \|w\|_{t,B}.\end{aligned} \tag{22}$$

Choosing $\Psi(t) = 2^t$, $a = 2$, and $\omega(x) \equiv 1$ in Lemma 12, we have

$$\begin{aligned}&\left\|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right\|_{t,B} \\ &\quad= \left(\int_B \left|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|^t dx\right)^{1/t} \\ &\quad\le \left(\int_B 2^t \left|\mathbb{M}_s^\sharp (P(w))\right|^t dx\right)^{1/t} \\ &\quad\le K|B|^{1+(1/m)} \|w\|_{t,B}.\end{aligned} \tag{23}$$

The proof of Lemma 14 has been completed. □

Lemma 15. *Suppose that $w(x) \in C^\infty(\Lambda^l E)$ is a solution to the A-harmonic equation, $E \subset \mathbb{R}^m$ is a bounded domain, P is a potential operator with the kernel function $K(x,y)$ satisfying the standard estimates, and the sharp maximal operator $\mathbb{M}_s^\sharp$ is expressed by Definition 1, $1 \le s < p$, $q < \infty$. Then, there exists a constant $K > 0$, such that*

$$\left\|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right\|_{q,B} \le K\|w\|_{p,\rho B}, \tag{24}$$

where the ball $B \subset E$ with $\rho B \subset E$, constant $\rho > 1$, the measure μ is defined by $d\mu = \omega(x)dx$, weight $\omega(x) \in A_r(E)$, $\omega(x) \ge \delta > 0$, for some $r > 1$ and a constant δ.

Proof. Because $1/q = ((\alpha - 1)/\alpha q) + (1/\alpha q)$, for any B with ρB contained in E, using Lemmas 9 and 14, we have

$$\begin{aligned}&\left(\int_B \left|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|^q d\mu\right)^{1/q} \\ &= \left(\int_B \left(\left|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right| \omega^{1/q}\right)^q dx\right)^{1/q} \\ &\le \left(\int_B \left(\left|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|^{\alpha q/(\alpha-1)}\right) dx\right)^{(\alpha-1)/\alpha q} \\ &\quad\times \left(\int_B \omega^\alpha dx\right)^{1/q\alpha} \\ &\le K|B|^{(1-\alpha)/\alpha q} \|\omega\|_{1,B}^{1/q} \left\|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right\|_{q\alpha/(\alpha-1),B}.\end{aligned} \tag{25}$$

According to Lemma 14 and Definition 5, letting $p = r \times z$, we get

$$\begin{aligned}&\left\|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right\|_{\alpha q/(\alpha-1),B} \\ &\quad\le K|B|^{1+(1/m)} \|w\|_{\alpha q/(\alpha-1),B} \\ &\quad= K|B|^{1+(1/m)+((z(\alpha-1)-\alpha q)/\alpha q z)} \|w\|_{z,\rho B}.\end{aligned} \tag{26}$$

Therefore, we know that

$$\begin{aligned}&\left(\int_B \left|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|^q d\mu\right)^{1/q} \\ &\le K|B|^{(1-\alpha)/q\alpha} \|\omega\|_{1,B}^{1/q} |B|^{1+(1/m)+((z(\alpha-1)-\alpha q)/\alpha q z)} \|w\|_{z,\rho B} \\ &= K|B|^{1+(1/m)-(1/z)} (\mu(B))^{1/q} \|w\|_{z,\rho B}.\end{aligned} \tag{27}$$

Because of $1/z = (1/p) + ((p-z)/zp)$, and using generalized Hölder's inequality, we get

$$\begin{aligned}\|w\|_{z,\rho B} &= \left(\int_{\rho B} \left(|w|\omega^{1/p}\omega^{-1/p}\right)^z dx\right)^{1/z} \\ &\le \left(\int_{\rho B} |w|^p \omega dx\right)^{1/p} \cdot \left(\int_{\rho B} \omega^{1/(r-1)} dx\right)^{(r-1)/p} \\ &= \|w\|_{p,\rho B,\omega} \left\|\omega^{-1}\right\|_{1/(r-1),\rho B}^{1/p}.\end{aligned} \tag{28}$$

In the light of $\omega \in A_r(E)$, finding details in [9], we know

$$\sup_{B \subset E} \left(\frac{1}{|B|}\int_B \omega dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{\omega}\right)^{1/(r-1)} dx\right)^{1/(r-1)} < \infty. \tag{29}$$

Therefore, we can see that

$$\begin{aligned}\left\|\frac{1}{\omega}\right\|_{1/(r-1),\rho B}^{1/p} &= \|\omega\|_{1,\rho B}^{-1/p} \cdot \|\omega\|_{1,\rho B}^{1/p} \cdot \left\|\frac{1}{\omega}\right\|_{1/(r-1),\rho B}^{1/p} \\ &\le K(\mu(\rho B))^{-1/p} |\rho B|^{1/z} \\ &\le K(\mu(\rho B))^{-1/p} |B|^{1/z}.\end{aligned} \tag{30}$$

In addition, considering $\omega \ge \delta > 0$, so we have that

$$\mu(\rho B) = \int_{\rho B} d\mu \ge \int_{\rho B} \delta dx = \delta |\rho B|. \tag{31}$$

Combining (27), (28), and (31), we obtain

$$\begin{aligned}&\left(\int_B \left|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|^q d\mu\right)^{1/q} \\ &\quad\le K|\rho B|^{1+(1/m)-(1/p)} (\mu(B))^{1/q} \left(\int_{\rho B} |w|^{1/p} d\mu\right)^{1/q} \\ &\quad\le K|E|^{1+(1/m)-(1/p)} (\mu(E))^{1/q} \left(\int_{\rho B} |w|^{1/p} d\mu\right)^{1/q} \\ &\quad\le K\|w\|_{p,\rho B}.\end{aligned} \tag{32}$$

Therefore, we finish the proof of this lemma. □

4. Demonstration of Main Results

According to the above definitions and lemmas, we will prove these two theorems in detail. Firstly, let us prove Theorem 7.

Proof of Theorem 7. Let B and $\rho B \subseteq E$, f and g are, respectively, convex and concave increasing function, use Lemma 15, and take $\omega(x) \equiv 1$; then

$$\begin{aligned}
&\mathscr{A}\left(\lambda^{-1}\left(\int_B \left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|^q dx\right)\right)^{1/q} \\
&\quad \leq \mathscr{A}\left(\lambda^{-1}K\left(\int_{\rho B}|w|^p dx\right)^{1/p}\right) \\
&\quad = \mathscr{A}\left(\left(\lambda^{-p}K^p\int_{\rho B}|w|^p dx\right)^{1/p}\right) \\
&\quad \leq Cf\left(\lambda^{-p}K^p\int_{\rho B}|w|^p dx\right) \\
&\quad \leq C\int_{\rho B} f\left(\lambda^{-p}K^p|w|^p\right)dx.
\end{aligned} \tag{33}$$

Because $f(t) \leq C\mathscr{A}(t^{1/p})$, we know that

$$\int_{\rho B} f\left(\lambda^{-p}K^p|w|^p\right)dx \leq C\int_{\rho B}\mathscr{A}\left(K\lambda^{-1}|w|\right)dx. \tag{34}$$

Furthermore, we obtain

$$\begin{aligned}
&\mathscr{A}\left(\lambda^{-1}\left(\int_B \left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|^q dx\right)\right)^{1/q} \\
&\quad \leq C\int_{\rho B}\mathscr{A}\left(K\lambda^{-1}|w|\right)dx.
\end{aligned} \tag{35}$$

For function g, using Jensen's inequality, we get

$$\begin{aligned}
&\int_B \mathscr{A}\left(\lambda^{-1}\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|\right)dx \\
&\leq g\left(\int_B g^{-1}\left(\mathscr{A}\left(\lambda^{-1}\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|\right)\right)dx\right) \\
&\leq g\left(C\int_B\left(\lambda^{-1}\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|\right)^q dx\right) \\
&\leq C\mathscr{A}\left(\left(C\int_B\left(\lambda^{-1}\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|\right)^q dx\right)^{1/q}\right) \\
&= C\mathscr{A}\left(\lambda^{-1}\left(C\int_B\left(\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|\right)^q dx\right)^{1/q}\right).
\end{aligned} \tag{36}$$

Using the doubling property of $\mathscr{A}$ for the above the formula, we have

$$\begin{aligned}
&\int_B \mathscr{A}\left(\lambda^{-1}\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|\right)dx \\
&\quad \leq K\mathscr{A}\left(\lambda^{-1}\left(\int_B\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|^q dx\right)^{1/q}\right) \\
&\quad \leq K\int_{\rho B}\mathscr{A}\left(\lambda^{-1}|w|\right)dx.
\end{aligned} \tag{37}$$

The proof of Theorem 7 has been finished. □

Now, we will use Definition 4 and Theorem 7 to prove Theorem 8.

Proof of Theorem 8. According to Definition 4, we can know

$$\begin{aligned}
&\int_E \mathscr{A}\left(\lambda^{-1}\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_{B_0}\right|\right)dx \\
&\quad \leq K\sup_{B\subseteq E}\int_B \mathscr{A}\left(\lambda^{-1}\left|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_B\right|\right)dx \\
&\quad \leq K\sup_{B\subseteq E}\int_{\rho B}\mathscr{A}\left(\lambda^{-1}|w|\right)dx \\
&\quad \leq K\sup_{B\subseteq E}\int_E\mathscr{A}\left(\lambda^{-1}|w|\right)dx.
\end{aligned} \tag{38}$$

Because $\sup_{B\subseteq E}\int_E \mathscr{A}(\lambda^{-1}|w|)dx$ is independent on the ball B, we obtain that

$$\left\|\mathbb{M}_s^\sharp(P(w)) - \left(\mathbb{M}_s^\sharp(P(w))\right)_{B_0}\right\|_{L_E^{\mathscr{A}}} \leq K\|w\|_{L_E^{\mathscr{A}}}. \tag{39}$$

We finish the proof of Theorem 8. □

5. Applications

In this part, we firstly use Theorem 8 to do an estimate for a solution to the Laplace equation $\Delta u = 0$.

Example 16. Let u be a differential 2-form in $\mathbb{R}^m$, and

$$u = \frac{1}{\vartheta}\left(x_1dx_2\wedge dx_3 + x_2dx_1\wedge dx_3 + x_3dx_1\wedge dx_2\right), \tag{40}$$

where $\vartheta = \sqrt{x_1^2+x_2^2+x_3^2}$. It is very easy to obtain that $|u| = 1$ and $du = 0$, so u is a solution for the Laplace equation $\Delta u = 0$. If we take

$$\mathscr{A}(t) = t\log_+^t = \begin{cases} t, & t \leq e \\ t\log^t, & t > e,\end{cases} \tag{41}$$

then $\mathscr{A}(t)$ is a Young function and belongs to the $G(p,q,C)$-class, with $\mathscr{A}(|u|) \in L^1(E)$. According to Theorem 8, we get that, for any fixed $B_0 \subset E$, there exists a constant $K > 0$ such that

$$\left\|\mathbb{M}_s^\sharp (P(u)) - \left(\mathbb{M}_s^\sharp (P(u))\right)_{B_0}\right\|_{L_E^{\mathscr{A}}} \le K\|1\|_{L_E^{\mathscr{A}}}, \tag{42}$$

where $\|1\|_{L_E^{\mathscr{A}}} = \inf\{\lambda > 0 \mid (1/|E|) \int_E \mathscr{A}(\lambda^{-1}\log_+^{\lambda})dt < 1\}$.

Now, our aim is to prove the following corollary by using Theorem 7.

Corollary 17. *Suppose that the Young function $\mathscr{A}$ belongs to the $G(p,q,C)$-class, and $w \in C^\infty(\Lambda^l E)$ is a solution to the nonhomogenous A-harmonic equation. The sharp maximal operator is noted by $M_s^\sharp$, P is the potential operator with its kernel function $K(x,y)$ satisfying the standard estimates ($1 \le s < p, q < \infty$), and the bounded $E \subseteq \mathfrak{R}^m$ is the $L^{\mathscr{A}}$-averaging domains. Then, there exists a constant $K > 0$, such that*

$$\left\|\mathbb{M}_s^\sharp (P(w))\right\|_{L_B^{\mathscr{A}}} \le K\|w\|_{L_{\rho B}^{\mathscr{A}}}, \tag{43}$$

where $B \subset E$ with $\rho B \subseteq E$, and the constant $\rho > 1$.

Proof. By using Minkowski inequality, we know that

$$\begin{aligned}\left\|\mathbb{M}_s^\sharp (P(w))\right\|_{L_B^{\mathscr{A}}} &\le \left\|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right\|_{L_B^{\mathscr{A}}} \\ &\quad + \left\|\left(\mathbb{M}_s^\sharp (P(w))\right)_B\right\|_{L_B^{\mathscr{A}}}.\end{aligned} \tag{44}$$

From [12] and formula (22), we have

$$\begin{aligned}&\int_B \mathscr{A}\left(\lambda^{-1}\left|\left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|\right) dt \\ &\quad = f \cdot f^{-1}\left(\int_B \mathscr{A}\left(\lambda^{-1}\left|\left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|\right) dt\right) \\ &\quad \le f\left(K\int_B f^{-1}\left(\mathscr{A}\left(\lambda^{-1}\left|\left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|\right)\right) dt\right) \\ &\quad \le f\left(K\int_B \lambda^{-q}\left|\left(\mathbb{M}_s^\sharp (P(w))\right)_B\right|^q dt\right) \\ &\quad \le f\left(K\int_B \lambda^{-q}\left|\left(\mathbb{M}_s^\sharp (P(w))\right)\right|^q dt\right) \\ &\quad \le f\left(K\lambda^{-q}\int_B |w|^q dt\right) \\ &\quad \le K\int_B f\left(\lambda^{-q}|w|^q\right) dt \\ &\quad \le K\int_B \mathscr{A}\left(\lambda^{-1}|w|\right) dt \\ &\quad \le K\|w\|_{L_{\rho B}^{\mathscr{A}}},\end{aligned} \tag{45}$$

where $\rho > 1$. In addition, according to Theorem 7, there exists a constant K such that

$$\left\|\mathbb{M}_s^\sharp (P(w)) - \left(\mathbb{M}_s^\sharp (P(w))\right)_B\right\|_{L_B^{\mathscr{A}}} \le K\|w\|_{L_{\rho B}^{\mathscr{A}}}. \tag{46}$$

Substituting (45) and (46) into (44), we conclude that there exists a constant $K > 0$, independent of w, such that

$$\left\|\mathbb{M}_s^\sharp (P(w))\right\|_{L_B^{\mathscr{A}}} \le K\|w\|_{L_{\rho B}^{\mathscr{A}}}. \tag{47}$$

The proof of Corollary 17 has been completed. □

Virtually, we can obtain a global estimate about the composition operator by using Definition 4.

Corollary 18. *Suppose that the Young function $\mathscr{A}$ belongs to the $G(p,q,C)$-class, and $w \in C^\infty(\Lambda^l E)$ is a solution to the nonhomogenous A-harmonic equation. Let the sharp maximal operator be noted by $M_s^\sharp$, P is the potential operator with its kernel function $K(x,y)$ satisfying standard estimates ($1 \le s < p, q < \infty$), and the bounded $E \subseteq \mathfrak{R}^m$ is the $L^{\mathscr{A}}$-averaging domains. Then, there exists a constant $K > 0$ such that*

$$\left\|\mathbb{M}_s^\sharp (P(w))\right\|_{L_E^{\mathscr{A}}} \le K\|w\|_{L_E^{\mathscr{A}}}, \tag{48}$$

where B_0 is a fixed ball.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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