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Research Article

Existence Results for a Coupled System of Nonlinear Fractional Hybrid Differential Equations with Homogeneous Boundary Conditions

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We study an existence result for the following coupled system of nonlinear fractional hybrid differential equations with homogeneous boundary conditions $D^{\alpha}_{0^+}[x(t)/f(t,x(t),y(t))] = g(t,x(t),y(t)), D^{\alpha}_{0^+}[y(t)/f(t,y(t),x(t))] = g(t,y(t),x(t)), 0 < t < 1$, and $x(0) = y(0) = 0$, where $\alpha \in (0,1)$ and $D^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional derivative. The main tools in our study are the techniques associated to measures of noncompactness in the Banach algebras and a fixed point theorem of Darbo type.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of a great number of processes which appear in physics, chemistry, aerodynamics, and so forth and involve also derivatives of fractional order. For details, see [1–5] and the references therein.

On the other hand, about the theory of hybrid differential equations, we refer to the paper [6] where the authors studied the hybrid differential equation of first order:

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J = [0,T), \qquad x(t_0) = x_0 \in \mathbb{R}, \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.

In [7], the authors studied the fractional version of the abovementioned problem, that is,

$$D^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J, \ 0 < \alpha < 1, \qquad x(0) = 0, \tag{2}$$

under the same assumptions on f and g in [6].

Recently, in [8], the authors studied the following fractional hybrid initial value problem with supremum:

$$D^{\alpha}_{0^+}\left[\frac{x(t)}{f\left(t,x(t),\max_{0\le\tau\le t}|x(\tau)|\right)}\right] = g(t,x(t)), \quad 0 < t < 1, \qquad x(0) = 0, \tag{3}$$

where $0 < \alpha < 1$, $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $g \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

The coupled systems involving fractional differential equations are very important because they occur in numerous problems of applied nature; for instance, see [9–13].

In this paper, we consider the following coupled system:

$$D_{0^+}^{\alpha}\left[\frac{x(t)}{f(t,x(t),y(t))}\right]=g(t,x(t),y(t)),$$
$$D_{0^+}^{\alpha}\left[\frac{y(t)}{f(t,y(t),x(t))}\right]=g(t,y(t),x(t)), \quad 0<t<1, \tag{4}$$
$$x(0)=y(0)=0,$$

where $\alpha\in(0,1)$ and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative.

The main tool in our study is a fixed point theorem of Darbo type associated to measures of noncompactness.

2. Preliminaries

We begin this section with some definitions and results about fractional calculus.

Let $\alpha>0$ and $n=[\alpha]+1$, where $[\alpha]$ denotes the integer part of α. For a function $f:(0,\infty)\to\mathbb{R}$, the Riemann-Liouville fractional integral of order $\alpha>0$ of f is defined as

$$I_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds, \tag{5}$$

provided that the right side is pointwise defined on $(0,\infty)$.

The Riemann-Liouville fractional derivative of order α of a continuous function f is defined by

$$D_{0^+}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x\frac{f(s)}{(x-s)^{\alpha-n+1}}ds, \tag{6}$$

provided that the right side is pointwise defined on $(0,\infty)$.

The following lemma will be useful for our study, [14].

Lemma 1. *Let $h\in L^1(0,1)$ and $0<\alpha<1$. Then,*

(a)
$$D_{0^+}^{\alpha}I_{0^+}^{\alpha}h(x)=h(x); \tag{7}$$

(b)
$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}h(x)=h(x)-\frac{I_{0^+}^{1-\alpha}h(x)\big|_{x=0}}{\Gamma(\alpha)}x^{\alpha-1} \quad \text{a.e. on } (0,1). \tag{8}$$

Lemma 2. *Let $0<\alpha<1$ and suppose that $f\in C([0,1],\mathbb{R}\setminus\{0\})$ and $y\in C[0,1]$. Then, the unique solution of the fractional hybrid initial value problem*

$$D_{0^+}^{\alpha}\left[\frac{x(t)}{f(t)}\right]=y(t), \quad 0<t<1$$
$$x(0)=0, \tag{9}$$

is given by

$$x(t)=\frac{f(t)}{\Gamma(\alpha)}\int_0^t\frac{y(s)}{(t-s)^{1-\alpha}}ds, \quad t\in[0,1]. \tag{10}$$

Proof. Suppose that $x(t)$ is a solution of problem (9). Using the operator $I_{0^+}^{\alpha}$ and taking into account Lemma 1, we get

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}\left[\frac{x(t)}{f(t)}\right]=I_{0^+}^{\alpha}y(t), \tag{11}$$

or, equivalently,

$$\frac{x(t)}{f(t)}-\frac{I_{0^+}^{1-\alpha}(x(t)/f(t))\big|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}=I_{0^+}^{\alpha}y(t). \tag{12}$$

Since $x(t)/f(t)|_{t=0}=x(0)/f(0)=0/f(0)=0$ (because $f(0)\neq0$), we have

$$x(t)=f(t)I_{0^+}^{\alpha}y(t). \tag{13}$$

This means that

$$x(t)=\frac{f(t)}{\Gamma(\alpha)}\int_0^t\frac{y(s)}{(t-s)^{1-\alpha}}ds. \tag{14}$$

Conversely, suppose that $x(t)$ is given by

$$x(t)=\frac{f(t)}{\Gamma(\alpha)}\int_0^t\frac{y(s)}{(t-s)^{1-\alpha}}ds, \quad t\in[0,1]. \tag{15}$$

This means that

$$x(t)=f(t)I_{0^+}^{\alpha}y(t), \quad t\in[0,1]. \tag{16}$$

Applying $D_{0^+}^{\alpha}$ and taking into account Lemma 1 and that $f(t)\neq0$ for $t\in[0,1]$, we obtain

$$D_{0^+}^{\alpha}\left[\frac{x(t)}{f(t)}\right]=D_{0^+}^{\alpha}I_{0^+}^{\alpha}y(t)=y(t), \quad 0<t<1. \tag{17}$$

Moreover, for $t=0$ in (16), we have $x(0)=f(0)\cdot0=0$. This completes the proof. □

In the sequel, we recall some definitions and basic facts about measures of noncompactness.

Assume that E is a real Banach space with norm $\|\cdot\|$ and zero element θ. By $B(x,r)$ we denote the closed ball in E centered at x with radius r. By B_r we denote the ball $B(\theta,r)$. If X is a nonempty subset of E, by the symbols $\overline{X}$ and $\text{Conv}X$ we denote the closure and the convex closure of X, respectively. By $\|X\|$ we denote the quantity $\|X\|=\sup\{\|x\|:x\in X\}$. Finally, by $\mathfrak{M}_E$ we will denote the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ we denote its subfamily consisting of all relatively compact subsets of E.

Definition 3. A mapping $\mu:\mathfrak{M}_E\to\mathbb{R}_+=[0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions.

(a) The family $\ker\mu=\{X\in\mathfrak{M}_E:\mu(X)=0\}$ is nonempty and $\ker\mu\subset\mathfrak{N}_E$.

(b) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(c) $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$.

(d) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

(e) If (X_n) is a sequence of closed subsets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n \geq 1)$ and $\lim_{n\to\infty}\mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker\mu$ appearing in (a) is called the kernel of the measure of noncompactness μ. Notice that the set X_∞ appearing in (e) is an element of $\ker\mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, \ldots$, we infer that $\mu(X_\infty) = 0$ and this says that $X_\infty \in \ker\mu$.

An important theorem about fixed point theorem in the context of measures of noncompactness is the following Darbo's fixed point theorem [15].

Theorem 4. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \to C$ be a continuous mapping. Suppose that there exists a constant $k \in [0,1)$ such that*

$$\mu(T(X)) \leq k\mu(X), \tag{18}$$

for any nonempty subset X of C.

Then, T has a fixed point.

A generalization of Theorem 4 which will be very useful in our study is the following theorem, due to Sadovskiĭ [16].

Theorem 5. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \to C$ be a continuous operator satisfying*

$$\mu(T(X)) < \mu(X), \tag{19}$$

for any nonempty subset X of C with $\mu(X) > 0$.

Then, T has a fixed point.

Next, we will assume that the space E has structure of Banach algebra. By xy we will denote the product of two elements $x, y \in X$ and by XY we will denote the set defined by $XY = \{xy : x \in X, y \in Y\}$.

Definition 6. Let E be a Banach algebra. We will say that a measure of noncompactness μ defined on E satisfies condition (m) if

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X), \tag{20}$$

for any $X, Y \in \mathfrak{M}_E$.

This definition appears in [17].

In this paper, we will work in the space $C[0,1]$ consisting of all real functions defined and continuous on $[0,1]$ with the standard supremum norm

$$\|x\| = \sup\{|x(t)| : t \in [0,1]\}, \tag{21}$$

for $x \in C[0,1]$. It is clear that $(C[0,1], \|\cdot\|)$ is a Banach algebra, where the multiplication is defined as the usual product of real functions.

Next, we present the measure of noncompactness in $C[0,1]$ which will be used later. Let us fix $X \in \mathfrak{M}_{C[0,1]}$ and $\varepsilon > 0$. For $x \in X$, we denote by $\omega(x,\varepsilon)$ the modulus of continuity of x; that is,

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0,1], |t-s| \leq \varepsilon\}. \tag{22}$$

Put

$$\begin{aligned} \omega(X,\varepsilon) &= \sup\{\omega(x,\varepsilon) : x \in X\}, \\ \omega_0(X) &= \lim_{\varepsilon\to 0}\omega(X,\varepsilon). \end{aligned} \tag{23}$$

In [15], it is proved that $\omega_0(X)$ is a measure of noncompactness in $C[0,1]$.

Proposition 7. *The measure of noncompactness ω_0 on $C[0,1]$ satisfies condition (m).*

Proof. Fix $X, Y \in \mathfrak{M}_{C[0,1]}$, $\varepsilon > 0$, and $t, s \in [0,1]$ with $|t-s| \leq \varepsilon$. Then, we have

$$\begin{aligned} |x(t)y(t) - x(s)y(s)| &\leq |x(t)y(t) - x(t)y(s)| \\ &\quad + |x(t)y(s) - x(s)y(s)| \\ &= |x(t)||y(t) - y(s)| \\ &\quad + |y(s)||x(t) - x(s)| \\ &\leq \|x\|\omega(y,\varepsilon) + \|y\|\omega(x,\varepsilon). \end{aligned} \tag{24}$$

This means that

$$\omega(xy,\varepsilon) \leq \|x\|\omega(y,\varepsilon) + \|y\|\omega(x,\varepsilon), \tag{25}$$

and, therefore,

$$\omega(XY,\varepsilon) \leq \|X\|\omega(Y,\varepsilon) + \|Y\|\omega(X,\varepsilon). \tag{26}$$

Taking $\varepsilon \to 0$, we get

$$\omega_0(XY) \leq \|X\|\omega_0(Y) + \|Y\|\omega_0(X). \tag{27}$$

This completes the proof. □

Proposition 7 appears in [17] and we have given the proof for the paper is self-contained.

3. Main Results

We begin this section introducing the following class $\mathscr{A}$ of functions:

$$\begin{aligned} \mathscr{A} = \Big\{\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ &: \varphi \text{ is nondecreasing} \\ &\text{and } \lim_{n\to\infty}\varphi^n(t) = 0 \text{ for any } t > 0\Big\}, \end{aligned} \tag{28}$$

where φ^n denotes the n-iteration of φ.

Remark 8. Notice that if $\varphi \in \mathcal{A}$, then $\varphi(t) < t$, for any $t > 0$. Indeed, in contrary case, we can find $t_0 > 0$ and $t_0 \leq \varphi(t_0)$. Since φ is nondecreasing, we have

$$0 < t_0 \leq \varphi(t_0) \leq \varphi^2(t_0) \leq \cdots \leq \varphi^n(t_0) \leq \cdots, \tag{29}$$

and, therefore, $0 < t_0 \leq \lim_{n\to\infty}\varphi^n(t_0)$ and this contradicts the fact that $\varphi \in \mathcal{A}$.

Moreover, the fact that $\varphi(t) < t$ for any $t > 0$ proves that if $\varphi \in \mathcal{A}$, then φ is continuous at $t_0 = 0$.

Using Remark 8 and Theorem 5, we have the following fixed point theorem.

Theorem 9. *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \to C$ be a continuous operator satisfying*

$$\mu(T(X)) \leq \varphi(\mu(X)), \tag{30}$$

for any nonempty subset X of C, where $\varphi \in \mathcal{A}$.

Then, T has a fixed point.

Theorem 9 appears in [18], where the authors present a proof without using Theorem 5.

The following result which appears in [19] will be interesting in our study.

Theorem 10. *Let $\mu_1, \mu_2, \ldots, \mu_n$ be measures of noncompactness in the Banach spaces $E_1, E_2, \ldots, E_n$, respectively. Suppose that $F : [0,\infty)^n \to [0,\infty)$ is a convex function such that $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$.*

Then,

$$\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n)) \tag{31}$$

defines a measure compactness in $E_1 \times E_2 \times \cdots \times E_n$, where X_i denotes the natural projection of X into E_i, for $i = 1, 2, \ldots, n$.

Remark 11. As a consequence of Theorem 10, we have that if μ is a measure of noncompactness on a Banach space E and we consider the function $F : [0,\infty) \times [0,\infty) \to [0,\infty)$ defined by $F(x, y) = \max(x, y)$, then, since F is convex and $F(x, y) = 0$ if and only if $x = y = 0$, $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ defines a measure of noncompactness in the space $E \times E$.

Next, we present the definition of a coupled fixed point.

Definition 12. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of a mapping $G : X \times X \to X$ if $G(x, y) = x$ and $G(y, x) = y$.

The following result is crucial for our study.

Theorem 13. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E, and let μ be a measure of noncompactness in E. Suppose that $G : \Omega \times \Omega \to \Omega$ is a continuous operator satisfying*

$$\mu(G(X_1 \times X_2)) \leq \varphi(\max(\mu(X_1), \mu(X_2))) \tag{32}$$

for all nonempty subsets X_1 and X_2 of Ω, where $\varphi \in \mathcal{A}$.

Then, G has at least a coupled fixed point.

Proof. Notice that, by Remark 11, $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $E \times E$, where X_1 and X_2 are the projections of X into E.

Now, we consider the mapping $\widetilde{G} : \Omega \times \Omega \to \Omega \times \Omega$ defined by $\widetilde{G}(x, y) = (G(x, y), G(y, x))$. It is easily seen that $\Omega \times \Omega$ is a nonempty, bounded, closed, and convex subset of $E \times E$. Since G is continuous, it is clear that $\widetilde{G}$ is also continuous.

Next, we take a nonempty X of $\Omega \times \Omega$. Then,

$$\begin{aligned}
\widetilde{\mu}(\widetilde{G}(X)) &= \widetilde{\mu}(\widetilde{G}(X_1 \times X_2)) \\
&= \widetilde{\mu}(G(X_1 \times X_2) \times G(X_2 \times X_1)) \\
&= \max\{\mu(G(X_1 \times X_2)), \mu(G(X_2 \times X_1))\} \\
&\leq \max\{\varphi(\max(\mu(X_1), \mu(X_2))), \\
&\qquad \varphi(\max(\mu(X_2), \mu(X_1)))\} \\
&= \varphi(\max(\mu(X_1), \mu(X_2))) \\
&= \varphi(\widetilde{\mu}(X_1 \times X_2)).
\end{aligned} \tag{33}$$

Since $\varphi \in \mathcal{A}$, by Theorem 9, the mapping $\widetilde{G}$ has at least one fixed point. This means that there exists $(x_0, y_0) \in \Omega \times \Omega$ such that $\widetilde{G}(x_0, y_0) = (x_0, y_0)$ or, equivalently, $G(x_0, y_0) = x_0$ and $G(y_0, x_0) = y_0$. This proves that G has at least a coupled fixed point. □

Now, we consider the coupled system of integral equations:

$$\begin{aligned}
x(t) &= \frac{f(t, x(t), y(t))}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s), y(s))}{(t-s)^{1-\alpha}} ds, \\
y(t) &= \frac{f(t, y(t), x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(s, y(s), x(s))}{(t-s)^{1-\alpha}} ds, \quad t \in [0, 1].
\end{aligned} \tag{34}$$

Lemma 14. *Assume that $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then, $(x, y) \in C[0, 1] \times C[0, 1]$ is a solution of* (34) *if and only if $(x, y) \in C[0, 1] \times C[0, 1]$ is a solution of* (4).

Proof. The proof is an immediate consequence of Lemma 2, so we omit it. □

Next, we will study problem (4) under the following assumptions.

(H_1) $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

(H_2) The functions f and g satisfy

$$\begin{aligned}
&|f(t, x_1, y_1) - f(t, x_2, y_2)| \\
&\quad \leq \varphi_1(\max(|x_1 - x_2|, |y_1 - y_2|)), \\
&|g(t, x_1, y_2) - g(t, x_2, y_2)| \\
&\quad \leq \varphi_2(\max(|x_1 - y_1|, |x_2 - y_2|)),
\end{aligned} \tag{35}$$

respectively, for any $t \in [0,1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $\varphi_1, \varphi_2 \in \mathscr{A}$ and φ_1 is continuous.

Notice that assumption (H_1) gives us the existence of two nonnegative constants k_1 and k_2 such that $|f(t,0,0)| \le k_1$ and $|g(t,0,0)| \le k_2$, for any $t \in [0,1]$.

(H_3) There exists $r_0 > 0$ satisfying the inequalities

$$\begin{gathered}(\varphi_1(r) + k_1) \cdot (\varphi_2(r) + k_2) \le r\Gamma(\alpha+1), \\ \varphi_2(r) + k_2 \le \Gamma(\alpha+1).\end{gathered} \tag{36}$$

Theorem 15. *Under assumptions (H_1)–(H_3), problem (4) has at least one solution in $C[0,1] \times C[0,1]$.*

Proof. In virtue of Lemma 14, a solution $(x,y) \in C[0,1] \times C[0,1]$ of problem (4) satisfies

$$\begin{gathered}x(t) = \frac{f(t,x(t),y(t))}{\Gamma(\alpha)} \int_0^t \frac{g(s,x(s),y(s))}{(t-s)^{1-\alpha}} ds, \\ y(t) = \frac{f(t,y(t),x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(s,y(s),x(s))}{(t-s)^{1-\alpha}} ds, \\ t \in [0,1].\end{gathered} \tag{37}$$

We consider the space $C[0,1] \times C[0,1]$ equipped with the norm $\|(x,y)\|_{C[0,1]\times C[0,1]} = \max\{\|x\|, \|y\|\}$, for any $(x,y) \in C[0,1] \times C[0,1]$.

In $C[0,1] \times C[0,1]$, we define the operator

$$G(x,y)(t) = \frac{f(t,x(t),y(t))}{\Gamma(\alpha)} \int_0^t \frac{g(s,x(s),y(s))}{(t-s)^{1-\alpha}} ds, \quad t \in [0,1]. \tag{38}$$

Let $\mathscr{F}$ and $\mathscr{G}$ be the operators given by

$$\begin{gathered}\mathscr{F}(x,y)(t) = f(t,x(t),y(t)), \\ \mathscr{G}(x,y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s,x(s),y(s))}{(t-s)^{1-\alpha}} ds\end{gathered} \tag{39}$$

for any $(x,y) \in C[0,1] \times C[0,1]$ and any $t \in [0,1]$. Then,

$$G(x,y) = \mathscr{F}(x,y) \cdot \mathscr{G}(x,y). \tag{40}$$

Firstly, we will prove that G applies $C[0,1] \times C[0,1]$ into $C[0,1]$. To do this, it is sufficient to prove that $\mathscr{F}(x,y)$, $\mathscr{G}(x,y) \in C[0,1]$ for any $(x,y) \in C[0,1] \times C[0,1]$ since the product of continuous functions is continuous.

In virtue of assumption (H_1), it is clear that $\mathscr{F}(x,y) \in C[0,1]$ for $(x,y) \in C[0,1] \times C[0,1]$. In order to prove that $\mathscr{G}(x,y) \in C[0,1]$ for $(x,y) \in C[0,1] \times C[0,1]$, we fix $t_0 \in [0,1]$ and we consider a sequence $(t_n) \in [0,1]$ such that $t_n \to t_0$, and we have to prove that $\mathscr{G}(x,y)(t_n) \to \mathscr{G}(x,y)(t_0)$. Without loss of generality, we can suppose that $t_n > t_0$. Then, we have

$$\begin{aligned}&|\mathscr{G}(x,y)(t_n) - \mathscr{G}(x,y)(t_0)| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_n} \frac{g(s,x(s),y(s))}{(t_n-s)^{1-\alpha}} ds - \int_0^{t_0} \frac{g(s,x(s),y(s))}{(t_0-s)^{1-\alpha}} ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_n} \frac{g(s,x(s),y(s))}{(t_n-s)^{1-\alpha}} ds - \int_0^{t_n} \frac{g(s,x(s),y(s))}{(t_0-s)^{1-\alpha}} ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_n} \frac{g(s,x(s),y(s))}{(t_0-s)^{1-\alpha}} ds - \int_0^{t_0} \frac{g(s,x(s),y(s))}{(t_0-s)^{1-\alpha}} ds \right| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^{t_n} \left| (t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1} \right| \times |g(s,x(s),y(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} \left| (t_0-s)^{\alpha-1} \right| |g(s,x(s),y(s))| ds.\end{aligned} \tag{41}$$

By assumption (H_1), since $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, g is bounded on the compact set $[0,1] \times [-\|x\|, \|x\|] \times [-\|y\|, \|y\|]$. Denote by

$$M = \sup\{|g(s,x_1,y_1)| : s \in [0,1], x_1 \in [-\|x\|, \|x\|], y_1 \in [-\|y\|, \|y\|]\}. \tag{42}$$

From the last estimate, we obtain

$$\begin{aligned}&|\mathscr{G}(x,y)(t_n) - \mathscr{G}(x,y)(t_0)| \\ &\le \frac{M}{\Gamma(\alpha)} \int_0^{t_n} \left| (t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1} \right| ds \\ &\quad + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t_n} \left| (t_0-s)^{\alpha-1} \right| ds.\end{aligned} \tag{43}$$

As $0 < \alpha < 1$ and $t_n > t_0$, we infer that

$$\begin{aligned}
&\left|\mathcal{G}(x,y)(t_n) - \mathcal{G}(x,y)(t_0)\right| \\
&\quad \le \frac{M}{\Gamma(\alpha)}\left[\int_0^{t_0}\left|(t_n-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right|ds \right.\\
&\qquad \left. + \int_{t_0}^{t_n}\left|(t_n-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right|ds\right]\\
&\qquad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t_n}\frac{1}{(s-t_0)^{1-\alpha}}ds\\
&\quad = \frac{M}{\Gamma(\alpha)}\left[\int_0^{t_0}\left[(t_0-s)^{\alpha-1}-(t_n-s)^{\alpha-1}\right]ds \right.\\
&\qquad \left. + \int_{t_0}^{t_n}\frac{ds}{(t_n-s)^{1-\alpha}} + \int_{t_0}^{t_n}\frac{ds}{(s-t_0)^{1-\alpha}}\right]\\
&\qquad + \frac{M}{\Gamma(\alpha)}\int_{t_0}^{t_n}\frac{1}{(s-t_0)^{1-\alpha}}ds\\
&\quad \le \frac{M}{\Gamma(\alpha+1)}\left[(t_n-t_0)^{\alpha}+t_0^{\alpha}-t_n^{\alpha}\right.\\
&\qquad \left. + (t_n-t_0)^{\alpha}+(t_n-t_0)^{\alpha}\right]\\
&\qquad + \frac{M}{\Gamma(\alpha+1)}(t_n-t_0)^{\alpha}\\
&\quad = \frac{4M}{\Gamma(\alpha+1)}(t_n-t_0)^{\alpha} + \frac{M}{\Gamma(\alpha+1)}\left(t_0^{\alpha}-t_n^{\alpha}\right)\\
&\quad < \frac{4M}{\Gamma(\alpha+1)}(t_n-t_0)^{\alpha},
\end{aligned} \tag{44}$$

where the last inequality has been obtained by using the fact that $t_0^{\alpha} - t_n^{\alpha} < 0$.

Therefore, since $t_n \to t_0$, from the last estimate, we deduce that $\mathcal{G}(x,y)(t_n) \to \mathcal{G}(x,y)(t_0)$. This proves that $\mathcal{G}(x,y) \in C[0,1]$. Consequently, $\mathcal{G} : C[0,1]\times C[0,1] \to C[0,1]$. On the other hand, for $(x,y) \in C[0,1]\times C[0,1]$ and $t \in C[0,1]$, we have

$$\begin{aligned}
&\left|G(x,y)(t)\right| \\
&\quad = \left|\mathcal{F}(x,y)(t)\cdot\mathcal{G}(x,y)(t)\right|\\
&\quad = \left|f(t,x(t),y(t))\right|\left|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{g(s,x(s),y(s))}{(t-s)^{1-\alpha}}ds\right|\\
&\quad \le \left[\left|f(t,x(t),y(t))-f(t,0,0)\right|+\left|f(t,0,0)\right|\right]\\
&\qquad \times\left[\frac{1}{\Gamma(\alpha)}\left|\int_0^t\frac{g(s,x(s),y(s))-g(s,0,0)}{(t-s)^{1-\alpha}}ds\right.\right.\\
&\qquad \left.\left. + \int_0^t\frac{g(s,0,0)}{(t-s)^{1-\alpha}}ds\right|\right]\\
&\quad \le \frac{1}{\Gamma(\alpha)}\left[\varphi_1\left(\max\left(|x(t)|,|y(t)|\right)\right)+k_1\right]\\
&\qquad \times\left[\int_0^t\frac{\left|g(s,x(s),y(s))-g(s,0,0)\right|}{(t-s)^{1-\alpha}}ds\right.\\
&\qquad \left. + \int_0^t\frac{\left|g(s,0,0)\right|}{(t-s)^{1-\alpha}}ds\right]\\
&\quad \le \frac{1}{\Gamma(\alpha)}\left[\varphi_1\left(\max\left(\|x\|,\|y\|\right)\right)+k_1\right]\\
&\qquad \times\left[\int_0^t\frac{\varphi_2\left(\max\left(|x(s)|,|y(s)|\right)\right)}{(t-s)^{1-\alpha}}ds\right.\\
&\qquad \left. + k_2\int_0^t\frac{ds}{(t-s)^{1-\alpha}}\right]\\
&\quad \le \frac{1}{\Gamma(\alpha)}\left[\varphi_1\left(\max\left(\|x\|,\|y\|\right)\right)+k_1\right]\\
&\qquad \cdot\left[\varphi_2\left(\max\left(\|x\|,\|y\|\right)\right)+k_2\right]\int_0^t\frac{ds}{(t-s)^{1-\alpha}}\\
&\quad \le \frac{1}{\Gamma(\alpha+1)}\left(\varphi_1\left(\|(x,y)\|\right)+k_1\right)\\
&\qquad \cdot\left(\varphi_2\left(\|(x,y)\|\right)+k_2\right).
\end{aligned} \tag{45}$$

Now, taking into account assumption (H_3), we infer that the operator G applies $B_{r_0}\times B_{r_0}$ into B_{r_0}. Moreover, from the last estimates, it follows that

$$\begin{aligned}
&\left\|\mathcal{F}\left(B_{r_0}\times B_{r_0}\right)\right\| \le \varphi_1(r_0)+k_1,\\
&\left\|\mathcal{G}\left(B_{r_0}\times B_{r_0}\right)\right\| \le \frac{\varphi_2(r_0)+k_2}{\Gamma(\alpha+1)}.
\end{aligned} \tag{46}$$

Next, we will prove that the operators $\mathcal{F}$ and $\mathcal{G}$ are continuous on the ball $B_{r_0}\times B_{r_0}$ and, consequently, G will be also continuous.

In fact, we fix $\varepsilon > 0$ and we take $(x_0,y_0), (x,y) \in B_{r_0}\times B_{r_0}$ with $\|(x,y)-(x_0,y_0)\| = \|(x-x_0, y-y_0)\| = \max\{\|x-x_0\|, \|y-y_0\|\} \le \varepsilon$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
&\left|\mathcal{F}(x,y)(t)-\mathcal{F}(x_0,y_0)(t)\right|\\
&\quad = \left|f(t,x(t),y(t))-f(t,x_0(t),y_0(t))\right|\\
&\quad \le \varphi_1\left(\max\left(|x(t)-x_0(t)|,|y(t),y_0(t)|\right)\right)\\
&\quad \le \varphi_1\left(\max\left(\|x-x_0\|,\|y-y_0\|\right)\right)\\
&\quad \le \varphi_1(\varepsilon) < \varepsilon,
\end{aligned} \tag{47}$$

where we have used Remark 8. This proves the continuity of $\mathcal{F}$ on $B_{r_0}\times B_{r_0}$.

In order to prove the continuity of $\mathscr{G}$ on $B_{r_0} \times B_{r_0}$, we have

$$\begin{aligned}
&\left|\mathscr{G}(x,y)(t) - \mathscr{G}(x_0,y_0)(t)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^t \frac{g(s,x(s),y(s))}{(t-s)^{1-\alpha}}ds - \int_0^t \frac{g(s,x_0(s),y_0(s))}{(t-s)^{1-\alpha}}ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\left|g(s,x(s),y(s)) - g(s,x_0(s),y_0(s))\right|}{(t-s)^{1-\alpha}}ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\varphi_2(\max(\left|x(s)-x_0(s)\right|,\left|y(s)-y_0(s)\right|))}{(t-s)^{1-\alpha}}ds \\
&\leq \frac{1}{\Gamma(\alpha)}\varphi_2(\max(\|x-x_0\|,\|y-y_0\|))\int_0^t \frac{ds}{(t-s)^{1-\alpha}} \\
&\leq \frac{1}{\Gamma(\alpha+1)}\varphi_2(\varepsilon) \\
&< \frac{\varepsilon}{\Gamma(\alpha+1)}.
\end{aligned} \tag{48}$$

Therefore,

$$\|\mathscr{G}(x,y) - \mathscr{G}(x_0,y_0)\| < \frac{\varepsilon}{\Gamma(\alpha+1)} \tag{49}$$

and, consequently, $\mathscr{G}$ is a continuous operator on $B_{r_0} \times B_{r_0}$.

In order to prove that $\mathscr{G}$ satisfies assumptions of Theorem 13, only we have to check the condition

$$\mu(G(X_1 \times X_2)) \leq \varphi(\max(\mu(X_1),\mu(X_2))) \tag{50}$$

for any subsets X_1 and X_2 of B_{r_0}.

To do this, we fix $\varepsilon > 0$, $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \leq \varepsilon$ and $(x,y) \in X_1 \times X_2$; then, we have

$$\begin{aligned}
&\left|\mathscr{F}(x,y)(t_1) - \mathscr{F}(x,y)(t_2)\right| \\
&\quad = \left|f(t_1,x(t_1),y(t_1)) - f(t_2,x(t_2),y(t_2))\right| \\
&\quad \leq \left|f(t_1,x(t_1),y(t_1)) - f(t_1,x(t_2),y(t_2))\right| \\
&\qquad + \left|f(t_1,x(t_2),y(t_2)) - f(t_2,x(t_2),y(t_2))\right| \\
&\quad \leq \varphi_1(\max(\left|x(t_1)-x(t_2)\right|,\left|y(t_1)-y(t_2)\right|)) \\
&\qquad + \omega(f,\varepsilon) \\
&\quad \leq \varphi_1(\max(\omega(x,\varepsilon),\omega(y,\varepsilon))) + \omega(f,\varepsilon),
\end{aligned} \tag{51}$$

where $\omega(f,\varepsilon)$ denotes the quantity

$$\begin{aligned}
\omega(f,\varepsilon) = \sup\{&\left|f(t,x,y) - f(s,x,y)\right| : t,s \in [0,1], \\
&|t-s| \leq \varepsilon, x, y \in [-r_0,r_0]\}.
\end{aligned} \tag{52}$$

From the last estimate, we infer that

$$\begin{aligned}
&\omega(\mathscr{F}(X_1 \times X_2),\varepsilon) \\
&\quad \leq \varphi_1(\max(\omega(X_1,\varepsilon),\omega(X_2,\varepsilon))) + \omega(f,\varepsilon).
\end{aligned} \tag{53}$$

Since $f(t,x,y)$ is uniformly continuous on bounded subsets of $[0,1] \times \mathbb{R} \times \mathbb{R}$, we deduce that $\omega(f,\varepsilon) \to 0$ as $\varepsilon \to 0$ and, therefore,

$$\omega_0(\mathscr{F}(X_1 \times X_2)) \leq \lim_{\varepsilon \to 0}\varphi_1(\max(\omega(X_1,\varepsilon),\omega(X_2,\varepsilon))). \tag{54}$$

By assumption (H_2), since φ_1 is continuous, we infer

$$\omega_0(\mathscr{F}(X_1 \times X_2)) \leq \varphi_1(\max(\omega_0(X_1),\omega_0(X_2))). \tag{55}$$

Now, we estimate the quantity $\omega_0(\mathscr{G}(X_1 \times X_2))$.

Fix $\varepsilon > 0$, $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \leq \varepsilon$ and $(x,y) \in X_1 \times X_2$. Without loss of generality, we can suppose that $t_1 < t_2$; then, we have

$$\begin{aligned}
&\left|\mathscr{G}(x,y)(t_2) - \mathscr{G}(x,y)(t_1)\right| \\
&\quad = \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{g(s,x(s),y(s))}{(t_2-s)^{1-\alpha}}ds - \int_0^{t_1} \frac{g(s,x(s),y(s))}{(t_1-s)^{1-\alpha}}ds\right| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}\left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| \times \left|g(s,x(s),x(\sigma s))\right|ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|g(s,x(s),y(s))\right|ds\right] \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\left[\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] \times \left|g(s,x(s),x(\sigma s))\right|ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|g(s,x(s),y(s))\right|ds\right].
\end{aligned} \tag{56}$$

Since $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \times\mathbb{R})$ is bounded on the compact subsets of $[0,1] \times \mathbb{R} \times \mathbb{R}$, particularly, on $[0,1] \times [-r_0,r_0] \times [-r_0,r_0]$. Put $L = \sup\{|g(t,x,y)| : t \in [0,1], x, y \in [-r_0,r_0]\}$. Then, from the last inequality, we infer that

$$\begin{aligned}
&\left|\mathscr{G}(x,y)(t_2) - \mathscr{G}(x,y)(t_1)\right| \\
&\quad \leq \frac{L}{\Gamma(\alpha)}\left[\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right] \\
&\quad \leq \frac{L}{\Gamma(\alpha+1)}\left[(t_2-t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha} + (t_2-t_1)^{\alpha}\right] \\
&\quad \leq \frac{2L}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \\
&\quad \leq \frac{2L}{\Gamma(\alpha+1)}\varepsilon^{\alpha},
\end{aligned} \tag{57}$$

where we have used the fact that $t_1^\alpha - t_2^\alpha \le 0$. Therefore,

$$\omega\left(\mathscr{G}\left(X_1 \times X_2\right), \varepsilon\right) \le \frac{2L}{\Gamma(\alpha+1)} \varepsilon^\alpha. \quad (58)$$

From this, it follows that

$$\omega_0\left(\mathscr{G}\left(X_1 \times X_2\right)\right) = 0. \quad (59)$$

Next, by Proposition 7, (46), (55), and (59), we have

$$\begin{aligned} &\omega_0\left(G\left(X_1 \times X_2\right)\right) \\ &\quad = \omega_0\left(\mathscr{F}\left(X_1 \times X_2\right) \cdot \mathscr{G}\left(X_1 \times X_2\right)\right) \\ &\quad \le \left\|\mathscr{F}\left(X_1 \times X_2\right)\right\| \omega_0\left(\mathscr{G}\left(X_1 \times X_2\right)\right) \\ &\qquad + \left\|\mathscr{G}\left(X_1 \times X_2\right)\right\| \omega_0\left(\mathscr{F}\left(X_1 \times X_2\right)\right) \\ &\quad \le \left\|\mathscr{F}\left(B_{r_0} \times B_{r_0}\right)\right\| \omega_0\left(\mathscr{G}\left(X_1 \times X_2\right)\right) \\ &\qquad + \left\|\mathscr{G}\left(B_{r_0} \times B_{r_0}\right)\right\| \omega_0\left(\mathscr{F}\left(X_1 \times X_2\right)\right) \\ &\quad \le \frac{\varphi_2\left(r_0\right) + k_2}{\Gamma(\alpha+1)} \varphi_1\left(\max\left(\omega_0\left(X_1\right), \omega_0\left(X_2\right)\right)\right). \end{aligned} \quad (60)$$

By assumption (H_3), since $\varphi_2(r_0) + k_2 \le \Gamma(\alpha+1)$ and since it is easily proved that if $\alpha \in [0,1]$ and $\varphi \in \mathscr{A}$, then $\alpha\varphi \in \mathscr{A}$, we deduce that

$$\omega_0\left(G\left(X_1 \times X_2\right)\right) \le \varphi\left(\max\left(\omega_0\left(X_1\right), \omega_0\left(X_2\right)\right)\right), \quad (61)$$

where $\varphi \in \mathscr{A}$.

Finally, by Theorem 13, the operator $\mathscr{G}$ has at least a coupled fixed point and this is the desired result. This completes the proof. □

The nonoscillary character of the solutions of problem (4) seems to be an interesting question from the practical point of view. This means that the solutions of problem (4) have a constant sign on the interval $(0,1)$. In connection with this question, we notice that if $f(t,x,y)$ and $g(t,x,y)$ have constant sign and are equal (this means that $f(t,x,y) > 0$ and $g(t,x,y) \ge 0$ or $f(t,x,y) < 0$ and $g(t,x,y) \le 0$ for any $t \in [0,1]$ and $x, y \in \mathbb{R}$) and under assumptions of Theorem 15, then the solution $(x,y) \in C[0,1] \times C[0,1]$ of problem (4) given by Theorem 15 satisfies $x(t) \ge 0$ and $y(t) \ge 0$ for $t \in [0,1]$, since the solution (x,y) satisfies the system of integral equations

$$\begin{aligned} x(t) &= \frac{f(t,x(t),y(t))}{\Gamma(\alpha)} \int_0^t \frac{g(s,x(s),y(s))}{(t-s)^{1-\alpha}} ds, \\ y(t) &= \frac{f(t,y(t),x(t))}{\Gamma(\alpha)} \int_0^t \frac{g(s,y(s),x(s))}{(t-s)^{1-\alpha}} ds, \quad 0 \le t \le 1. \end{aligned} \quad (62)$$

On the other hand, if we perturb the data function in problem (4) of the following manner:

$$\begin{aligned} D_{0^+}^\alpha\left[\frac{x(t)}{f(t,x(t),y(t))}\right] &= g(t,x(t),y(t)) + \eta(t), \\ D_{0^+}^\alpha\left[\frac{y(t)}{f(t,y(t),x(t))}\right] &= g(t,y(t),x(t)) + \eta(t), \\ &\qquad 0 < t < 1, \end{aligned} \quad (63)$$

where $0 < \alpha < 1$, $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $\eta \in C[0,1]$, then, under assumptions of Theorem 15, problem (63) can be studied by using Theorem 15, where assumptions (H_1) and (H_2) are automatically satisfied and we only have to check assumption (H_3). This fact gives a great applicability to Theorem 15.

Before presenting an example illustrating our results, we need some facts about the functions involving this example. The following lemma appears in [18].

Lemma 16. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then, the following conditions are equivalent:*

(i) $\lim_{n\to\infty} \varphi^n(t) = 0$ *for any* $t \ge 0$;

(ii) $\varphi(t) < t$ *for any* $t > 0$.

Particularly, the functions $\alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\alpha_1(t) = \arctan t$ and $\alpha_2(t) = t/(1+t)$ belong to the class $\mathscr{A}$, since they are nondecreasing and continuous, and, as it is easily seen, they satisfy (ii) of Lemma 16.

On the other hand, since the function $\alpha_1(t) = \arctan t$ is concave (because $\alpha''(t) \le 0$) and $\alpha_1(0) = 0$, we infer that α_1 is subadditive and, therefore, for any $t, t' \in \mathbb{R}_+$, we have

$$\begin{aligned} \left|\alpha_1(t) - \alpha_1\left(t'\right)\right| &= \left|\arctan t - \arctan t'\right| \\ &\le \arctan\left|t - t'\right|. \end{aligned} \quad (64)$$

Moreover, it is easily seen that $\max(\alpha_1, \alpha_2)$ is a nondecreasing and continuous function because α_1 and α_2 are nondecreasing and continuous and $\max(\alpha_1, \alpha_2)$ satisfies (ii) of Lemma 16. Therefore, $\max(\alpha_1, \alpha_2) \in \mathscr{A}$.

Now, we are ready to present an example where our results can be applied.

Example 17. Consider the following coupled system of fractional hybrid differential equations:

$$\begin{aligned} D_{0^+}^{1/2}&\left[x(t) \times \left(\frac{1}{4} + \left(\frac{1}{10}\right)\arctan|x(t)|\right.\right. \\ &\qquad \left.\left. + \left(\frac{1}{20}\right)\left(\frac{|y(t)|}{(1+|y(t)|)}\right)\right)^{-1}\right] \\ &= \frac{1}{7} + \frac{1}{9}x(t) + \frac{1}{10}y(t), \end{aligned}$$

$$D_{0^+}^{1/2}\left[y(t)\times\left(\frac{1}{4}+\left(\frac{1}{10}\right)\arctan|y(t)|+\left(\frac{1}{20}\right)\left(\frac{|x(t)|}{(1+|x(t)|)}\right)\right)^{-1}\right]=\frac{1}{7}+\frac{1}{9}y(t)+\frac{1}{10}x(t),$$
$$0<t<1,$$
$$x(0)=y(0)=0. \tag{65}$$

Notice that problem (17) is a particular case of problem (4), where $\alpha = 1/2$, $f(t,x,y) = 1/4 + (1/10)\arctan|x| + (1/20)(|y|/(1+|y|))$, and $g(t,x,y) = 1/7 + (1/9)x + (1/10)y$.

It is clear that $f \in C([0,1]\times\mathbb{R}\times\mathbb{R}, \mathbb{R}\setminus\{0\})$ and $g \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and, moreover, $k_1 = \sup\{|f(t,0,0)| : t \in [0,1]\} = 1/4$ and $k_2 = \sup\{|g(t,0,0)| : t \in [0,1]\} = 1/7$. Therefore, assumption (H_1) of Theorem 15 is satisfied.

Moreover, for $t \in [0,1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&|f(t,x_1,y_1)-f(t,x_2,y_2)|\\
&\le\frac{1}{10}\left|\arctan|x_1|-\arctan|x_2|\right|+\frac{1}{20}\left|\frac{|y_1|}{1+|y_1|}-\frac{|y_2|}{1+|y_2|}\right|\\
&\le\frac{1}{10}\arctan\left||x_1|-|x_2|\right|+\frac{1}{20}\left|\frac{|y_1|-|y_2|}{(1+|y_1|)(1+|y_2|)}\right|\\
&\le\frac{1}{10}\arctan(|x_1-x_2|)+\frac{1}{20}\frac{|y_1-y_2|}{1+|y_1-y_2|}\\
&=\frac{1}{10}\alpha_1(|x_1-x_2|)+\frac{1}{20}\alpha_2(|y_1-y_2|)\\
&\le\frac{1}{10}\max(\alpha_1,\alpha_2)(|x_1-x_2|)+\frac{1}{10}\max(\alpha_1,\alpha_2)(|y_1-y_2|)\\
&\le\frac{1}{10}[2\max(\alpha_1,\alpha_2)\times\max(|x_1-x_2|,|y_1-y_2|)]\\
&=\frac{1}{5}\max(\alpha_1,\alpha_2)(\max(|x_1-x_2|,|y_1-y_2|)).
\end{aligned} \tag{66}$$

Therefore, $\varphi_1(t) = (1/5)\max(\alpha_1(t),\alpha_2(t))$ and $\varphi_1 \in \mathcal{A}$.

On the other hand,

$$\begin{aligned}
&|g(t,x_1,y_1)-g(t,x_2,y_2)|\\
&\le\frac{1}{9}|x_1-x_2|+\frac{1}{10}|x_2-y_2|\\
&\le\frac{1}{9}(|x_1-y_1|+|y_2-y_2|)\\
&\le\frac{1}{9}(2\max(|x_1-y_1|,|y_2-y_2|))\\
&=\frac{2}{9}\max(|x_1-y_1|,|y_2-y_2|),
\end{aligned} \tag{67}$$

and $\varphi_2(t) = (2/9)t$. It is clear that $\varphi_2 \in \mathcal{A}$. Therefore, assumption (H_2) of Theorem 15 is satisfied.

In our case, the inequality appearing in assumption (H_3) of Theorem 15 has the expression

$$\left[\frac{1}{5}\max\left(\arctan r,\frac{r}{1+r}\right)+\frac{1}{4}\right]\left[\frac{2}{9}r+\frac{1}{7}\right]\le r\Gamma\left(\frac{3}{2}\right). \tag{68}$$

It is easily seen that $r_0 = 1$ satisfies the last inequality. Moreover,

$$\frac{2}{9}r_0+\frac{1}{7}=\frac{2}{9}+\frac{1}{7}\le\Gamma\left(\frac{3}{2}\right)\cong 0.88623. \tag{69}$$

Finally, Theorem 15 says that problem (17) has at least one solution $(x,y)\in C[0,1]$ such that $\max(\|x\|,\|y\|)\le 1$.

Conflict of Interests

The authors declare that there is no conflict of interests in the submitted paper.

References

[1] W. Deng, "Numerical algorithm for the time fractional Fokker-Planck equation," *Journal of Computational Physics*, vol. 227, no. 2, pp. 1510–1522, 2007.

[2] S. Z. Rida, H. M. El-Sherbiny, and A. A. M. Arafa, "On the solution of the fractional nonlinear Schrödinger equation," *Physics Letters A*, vol. 372, no. 5, pp. 553–558, 2008.

[3] K. Diethelm and A. D. Freed, "On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity," in *Scientific Computing in Chemical Engineering II*, pp. 217–224, Springer, Berlin, Germany, 1999.

[4] F. Mainardi, "Fractional calculus: some basic proble ms in continuum and statistical mechanics," in *Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996)*, vol. 378 of *CISM Courses and Lectures*, pp. 291–348, Springer, Vienna, Austria, 1997.

[5] R. Metzler, W. Schick, H. Kilian, and T. F. Nonnenmacher, "Relaxation in filled polymers: a fractional calculus approach," *The Journal of Chemical Physics*, vol. 103, no. 16, pp. 7180–7186, 1995.

[6] B. C. Dhage and V. Lakshmikantham, "Basic results on hybrid differential equations," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 3, pp. 414–424, 2010.

[7] Y. Zhao, S. Sun, Z. Han, and Q. Li, "Theory of fractional hybrid differential equations," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1312–1324, 2011.

[8] J. Caballero, M. A. Darwish, and K. Sadarangani, "Solvability of a fractional hybrid initial value problem with supremum by using measures of noncompactness in Banach algebras," *Applied Mathematics and Computation*, vol. 224, pp. 553–563, 2013.

[9] C. Bai and J. Fang, "The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations," *Applied Mathematics and Computation*, vol. 150, no. 3, pp. 611–621, 2004.

[10] Y. Chen and H. An, "Numerical solutions of coupled Burgers equations with time- and space-fractional derivatives," *Applied Mathematics and Computation*, vol. 200, no. 1, pp. 87–95, 2008.

[11] M. P. Lazarević, "Finite time stability analysis of PD^{α} fractional control of robotic time-delay systems," *Mechanics Research Communications*, vol. 33, no. 2, pp. 269–279, 2006.

[12] V. Gafiychuk, B. Datsko, V. Meleshko, and D. Blackmore, "Analysis of the solutions of coupled nonlinear fractional reaction-diffusion equations," *Chaos, Solitons and Fractals*, vol. 41, no. 3, pp. 1095–1104, 2009.

[13] B. Ahmad and J. J. Nieto, "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1838–1843, 2009.

[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[15] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1980.

[16] B. N. Sadovskii, "On a fixed point principle," *Functional Analysis and Its Applications*, vol. 1, pp. 151–153, 1967.

[17] J. Banaś and L. Olszowy, "On a class of measures of noncompactness in Banach algebras and their application to nonlinear integral equations," *Journal of Analysis and its Applications*, vol. 28, no. 4, pp. 475–498, 2009.

[18] A. Aghajani, J. Banas, and N. Sabzali, "Some generalizations of Darbo fixed point theorem and applications," *Bulletin of the Belgian Mathematical Society*, vol. 20, no. 2, pp. 345–358, 2013.

[19] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, vol. 55 of *Operator Theory: Advances and Applications*, Birkhäauser, Basel, Switzerland, 1992.