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Research Article

The Exponential Diophantine Equation $\left(4m^2+1\right)^x+\left(5m^2-1\right)^y=(3m)^z$

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Let m be a positive integer. In this paper, using some properties of exponential diophantine equations and some results on the existence of primitive divisors of Lucas numbers, we prove that if $m > 90$ and $3|m$, then the equation $\left(4m^2+1\right)^x + \left(5m^2-1\right)^y = (3m)^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers, respectively. Given a triple (a, b, c) of coprime positive integers with $\min\{a, b, c\} > 1$, there are many papers that investigated the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N} \tag{1}$$

(see [1–8]). Recently, Terai [8] proved that if a, b, c satisfy

$$a = 4m^2 + 1, \quad b = 5m^2 - 1, \quad c = 3m, \quad m \in \mathbb{N}, \tag{2}$$

then (1) has only the solution $(x, y, z) = (1, 1, 2)$, provided that $m \not\equiv 3 (\bmod 6)$ or $m \leq 20$. The proof of this result is based on elementary methods and Baker's method. In this paper, using some properties of exponential diophantine equations and some results on the existence of primitive divisors of Lucas numbers, we prove a general result as follows.

Theorem 1. *Let a, b, c be positive integers satisfying (2). If $m > 90$ and $3 \mid m$, then (1) has only the solution $(x, y, z) = (1, 1, 2)$.*

Some Note. Combining Terai [8] and our Theorem, we know that only the values $20 < m \leq 90$, $3 \mid m$ are left to investigate (2). In this case, the equation $(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z$ has only finitely many solutions in (x, y, z). Moreover, they are not only effectively but also practically solvable. With some computer assistance, we would be able to solve completely the equation.

2. Preliminaries

In this section, we assume that a, b, c are positive integers satisfying (2). Then, (1) can be written as

$$\left(4m^2+1\right)^x + \left(5m^2-1\right)^y = (3m)^z, \quad x, y, z \in \mathbb{N}. \tag{3}$$

Further, let (x, y, z) be a solution of (3) with $(x, y, z) \neq (1, 1, 2)$.

Lemma 2. *If $m > 90$, then $2 \nmid m$, $2 \nmid y$ and*

$$z > \frac{1}{5}m^2. \tag{4}$$

Proof. Since $\min\{x, y\} \geq 1$, by (3), we have $z \geq 2$ and $1 + (-1)^y \equiv (4m^2 + 1)^x + (5m^2 - 1)^y \equiv (3m)^z \equiv 0 (\bmod m^2)$. Since $m > 1$, we get

$$2 \nmid y. \tag{5}$$

Since $(x, y, z) \neq (1, 1, 2)$, we have $\max\{x, y\} \geq 2$ and $(3m)^z > \min\{(4m^2 + 1)^2, (5m^2 - 1)^2\} > 16m^4$. It implies that

$$z \geq 4. \tag{6}$$

Hence, by (3), (5), and (6), we get

$$4x + 5y \equiv 0 \left(\operatorname{mod} m^2\right). \tag{7}$$

If $2 \mid m$, then from (7) we get $2 \mid y$, which contradicts (5). So we have

$$2 \nmid m. \tag{8}$$

Since $4x + 5y$ is a positive integer, by (7), we have

$$4x + 5y \geq m^2. \tag{9}$$

On the other hand, by (3), we have $(3m)^z > (4m^2+1)^x$ and $(3m)^z > (5m^2-1)^y$. It implies that

$$\frac{z}{2} > x\frac{\log\left(4m^2+1\right)}{\log\left(9m^2\right)}, \qquad \frac{z}{2} > y\frac{\log\left(5m^2-1\right)}{\log\left(9m^2\right)}; \tag{10}$$

whence we obtain

$$z > \frac{4x+5y}{2\log\left(9m^2\right)/\log\left(4m^2+1\right) + 5\log\left(9m^2\right)/2\log\left(5m^2-1\right)}. \tag{11}$$

Since $m > 90$, we have $(\log(9m^2))/\log(5m^2-1) < (\log(9m^2))/\log(4m^2+1) < 1.08$. Thus, by (9) and (11), we get (4). The lemma is proved.

Let D_1, D_2, k be positive integers such that $\min\{D_1, D_2, k\} > 1$ and $\gcd(D_1, D_2) = \gcd(k, 2D_1D_2) = 1$. □

Lemma 3 ([9, Lemmas 1 and 6]). *If (X, Y, Z) is a fixed solution of the equation*

$$D_1X^2 + D_2Y^2 = k^Z, \quad X, Y, Z \in \mathbb{Z}, \qquad \gcd(X, Y) = 1, \quad Z > 0, \tag{12}$$

then there exists a unique positive integer l such that

$$l \equiv -\frac{D_1X}{Y} \ (\operatorname{mod} k), \quad l < k. \tag{13}$$

The positive integer l is called the characteristic number of the solution (X, Y, Z) and is denoted by the symbol $\langle X, Y, Z\rangle$.

Lemma 4 ([9, Theorems 1 and 3]). *For a fixed characteristic number l, let $S(l)$ be the set of all solutions (X, Y, Z) of (13) with $\langle X, Y, Z\rangle \equiv \pm l (\operatorname{mod} k)$. Then one has the following.*

(i) *$S(l)$ has a unique solution (X_1, Y_1, Z_1) satisfying $X_1 > 0$, $Y_1 > 0$ and $Z_1 \leq Z$, where Z through all solutions (X, Y, Z) of $S(l)$. Such (X_1, Y_1, Z_1) is called the least solution of $S(l)$.*

(ii) *Every element (X, Y, Z) of $S(l)$ can be expressed as*

$$Z = Z_1 t, \quad t \in \mathbb{N}, 2 \nmid t,$$

$$X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1\left(X_1\sqrt{D_1} + \lambda_2 Y_1\sqrt{-D_2}\right)^t, \tag{14}$$

$$\lambda_1, \lambda_2 \in \{\pm 1\}.$$

Lemma 5. *If $\min\{D_1, D_2\} \geq 4$ and $3 \nmid D_1D_2$, then the equality*

$$D_1^r\sqrt{D_1} + D_2^s\sqrt{-D_2} = \lambda_1\left(\sqrt{D_1} + \lambda_2\sqrt{-D_2}\right)^t, \qquad \lambda_1, \lambda_2 \in \{\pm 1\}, \tag{15}$$

$$r, s, t \in \mathbb{Z}, \quad r \geq 0, \quad s \geq 0, \quad t > 1, \quad 2 \nmid t$$

cannot hold.

Proof. If (15) holds, then

$$D_1^r = \lambda_1 \sum_{i=0}^{(t-1)/2} (-i)^i \binom{t}{2i} D_1^{(t-1)/2-i} D_2^i,$$
$$D_2^s = \lambda_1\lambda_2 \sum_{i=0}^{(t-1)/2} (-i)^i \binom{t}{2i+1} D_1^{(t-1)/2-i} D_2^i, \tag{16}$$

$$D_1^r\sqrt{D_1} - D_2^s\sqrt{-D_2} = \lambda_1\left(\sqrt{D_1} - \lambda_2\sqrt{-D_2}\right)^t. \tag{17}$$

By (15) and (17), we have

$$D_1^{2r+1} + D_2^{2s+1} = \left(D_1 + D_2\right)^t. \tag{18}$$

We may assume that $D_1^{2r+1} < D_2^{2s+1}$; then, by (18), we have

$$2D_2^{2s+1} > \left(D_1 + D_2\right)^t > D_2^t. \tag{19}$$

Since $D_2 \geq 4$ and $t \geq 3$, we see from (19) that $s \geq 1$. Further, since $\gcd(D_1, D_2) = 1$, by (16), we get

$$D_2 \mid t. \tag{20}$$

It implies that $2 \nmid D_2$.

Let p be an odd prime divisor of D_2. Since $3 \nmid D_2$, we have $p \geq 5$. Further let

$$p^\alpha \parallel D_2, p^\beta \parallel t, p^{\gamma_j} \parallel 2j+1, \quad j = 1, \ldots, \frac{t-1}{2}. \tag{21}$$

Since $p \geq 5$, by (21), we have

$$\gamma_j \leq \frac{\log(2j+1)}{\log p} < j, \quad j = 1, \ldots, \frac{t-1}{2}. \tag{22}$$

Further, by (21) and (22), we get

$$(-1)^j \binom{t}{2j+1} D_1^{(t-1)/2-j} D_2^j \equiv (-1)^j t \binom{t-1}{2j} D_1^{(t-1)/2-j}$$
$$\frac{D_2^j}{2j+1} \equiv 0 \left(\operatorname{mod} p^{\beta+1}\right), \quad j = 1, \ldots, \frac{t-1}{2}. \tag{23}$$

Therefore, by (16), (21), and (23), we obtain

$$\alpha s \geq \beta. \tag{24}$$

Putting p through all odd prime divisors of D_2, we get from (24) that $D_2^s \mid t$ and

$$t \geq D_2^s. \tag{25}$$

By (19) and (25), we have $D_2^{2s+2} > 2D_2^{2s+1} > D_2^t \geq D_2^{D_2^s}$ and

$$2s + 2 > D_2^s \geq 4^s. \tag{26}$$

But (26) is impossible for any positive integer s. Thus, (15) is false. The lemma is proved. □

Let α, β be algebraic integers. If $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers and α/β is not a root of unity, then (α, β) is called a Lucas pair. Further, let $A = \alpha + \beta$ and $C = \alpha\beta$. Then we have

$$\alpha = \frac{1}{2}\left(A + \lambda\sqrt{B}\right), \quad \beta = \frac{1}{2}\left(A - \lambda\sqrt{B}\right), \quad \lambda \in \{\pm 1\}, \tag{27}$$

where $B = A^2 - 4C$. We call (A, B) the parameters of the Lucas pair (α, β). Two Lucas pairs (α_1, β_1) and (α_2, β_2) are equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 = \pm 1$. Given a Lucas pair (α, β), one defines the corresponding sequence of Lucas numbers by

$$L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n = 0, 1, 2, \ldots. \tag{28}$$

For equivalent Lucas pairs (α_1, β_1) and (α_2, β_2), we have $L_n(\alpha_1, \beta_1) = \pm L_n(\alpha_2, \beta_2)$ for any n. A prime p is called a primitive divisor of $L_n(\alpha, \beta)(n > 1)$ if $p \mid L_n(\alpha, \beta)$ and $p \nmid BL_1(\alpha, \beta), \ldots, L_{n-1}(\alpha, \beta)$. A Lucas pair (α, β) such that $L_n(\alpha, \beta)$ has no primitive divisors will be called an n-defective Lucas pair. Further, a positive integer n is called totally nondefective if no Lucas pair is n-defective.

Lemma 6 (see [10]). *Let n satisfy $4 < n \leq 30$ and $n \neq 6$. Then, up to equivalence, all parameters of n-defective Lucas pair are given as follows:*

(i) $n = 5$, $(A, B) = (1, 5), (1, -7), (2, -40), (1, -11), (1, -15), (12, -76), (12, -1364)$,

(ii) $n = 7$, $(A, B) = (1, -7), (1, -19)$,

(iii) $n = 8$, $(A, B) = (2, -24), (1, -7)$,

(iv) $n = 10$, $(A, B) = (2, -8), (5, -3), (5, -47)$,

(v) $n = 12$, $(A, B) = (1, 5), (1, -7), (1, -11), (2, -56), (1, -15), (1, -19)$,

(vi) $n \in \{13, 18, 30\}$, $(A, B) = (1, -7)$.

Lemma 7 (see [11]). *If $n > 30$, then n is totally nondefective.*

Let D, k be positive integers such that $\min\{D, k\} > 1$ and $\gcd(k, 2D) = 1$.

Lemma 8 (see [9, Theorems 1 and 3]). *Every solution (X, Y, Z) of the equation*

$$X^2 + DY^2 = k^Z, \quad X, Y, Z \in \mathbb{Z}, \qquad \gcd(X, Y) = 1, \quad Z > 0 \tag{29}$$

can be expressed as

$$Z = Z_1 t, \quad t \in \mathbb{N}, \tag{30}$$

$$X + Y\sqrt{-D} = \lambda_1\left(X_1 + \lambda_2 Y_1\sqrt{-D}\right)^t, \quad \lambda_1, \lambda_2 \in \{\pm 1\}, \tag{31}$$

where X_1, Y_1, Z_1 are positive integers satisfying

$$X_1^2 + DY_1^2 = k^{Z_1}, \qquad \gcd(X_1, Y_1) = 1, \qquad h(-4D) \equiv 0 \pmod{Z_1}, \tag{32}$$

where $h(-4D)$ is the class number of positive binary quadratic primitive forms of discriminant $-4D$.

For any positive integer a, let $P(a)$ denote the set of distinct prime divisors of a.

Lemma 9. *If (X, Y, Z) is a solution of (29) with $P(|Y|) \subseteq P(D)$, then $h(-4D) \equiv 0 \pmod{Z}$, except the possibility of the following cases:*

(i) $t \in \{2, 3, 4, 6\}$, *where t is defined as in* (30),

(ii) $(D, k, X, Y, Z) = (10, 11, \pm 401, \pm 5, 5), (19, 55, \pm 22434, \pm 1, 5), (341, 377, \pm 2759646, \pm 1, 5)$.

Proof. Let (X, Y, Z) be a solution of (29) with $P(|Y|) \subseteq P(D)$. By Lemma 8, X, Y, and Z satisfy (30) and (31), where X_1, Y_1, and Z_1 satisfy (32). Let

$$\alpha = X_1 + Y_1\sqrt{-D}, \qquad \beta = X_1 - Y_1\sqrt{-D}. \tag{33}$$

By (32) and (33), we have that $\alpha + \beta = 2X_1$, $\alpha - \beta = 2Y_1\sqrt{-D}$, $\alpha\beta = k^{Z_1}$, and α/β satisfies $k^{Z_1}(\alpha/\beta)^2 - 2(X_1^2 - DY_1^2)(\alpha/\beta) + k^{Z_1} = 0$. It implies that (α, β) is a Lucas pair with parameters $(2X_1, -4DY_1^2)$. Let $L_n(\alpha, \beta)(n = 0, 1, 2, \ldots)$ denote the corresponding Lucas numbers. By (28), (31), and (33), we get

$$Y = Y_1\left|\frac{\alpha^t - \beta^t}{\alpha - \beta}\right| = Y_1\left|L_t(\alpha, \beta)\right|. \tag{34}$$

Since $P(|Y|) \subseteq P(D)$, by the definition of primitive divisors, we see from (34) that either $t = 1$ or $t > 1$ and the Lucas number $L_t(\alpha, \beta)$ has no primitive divisor.

If $t = 1$, then from (30) and (32) we get $h(-4D) \equiv 0 (\mathrm{mod}\, Z)$. If $t > 1$, by Lemmas 6 and 7, using an easy computation, the solution (X, Y, Z) satisfies the case (i) or (ii). Thus, the lemma is proved. □

Lemma 10 ([12, Theorems 12.10.1 and 12.14.3]). *For any positive integer D, one has*

$$h(-4D) < \frac{4}{\pi}\sqrt{D}\log\left(2e\sqrt{D}\right). \tag{35}$$

3. Proof of Theorem

We now assume that (x, y, z) is a solution of (3) with $(x, y, z) \neq (1, 1, 2)$. By Lemma 2, we have

$$2 \nmid m, \qquad 2 \nmid y. \tag{36}$$

We first consider the case that $2 \nmid x$. Then, by (3) and (36), the equation

$$\left(4m^2+1\right)X^2+\left(5m^2-1\right)Y^2=(3m)^Z, \quad X,Y,Z\in\mathbb{Z}, \qquad \gcd(X,Y)=1, \quad Z>0 \tag{37}$$

has a solution

$$(X,Y,Z)=\left(\left(4m^2+1\right)^{(x-1)/2},\left(5m^2-1\right)^{(y-1)/2},z\right). \tag{38}$$

Let $l=\langle(4m^2+1)^{(x-1)/2},(5m^2-1)^{(y-1)/2},z\rangle$. Since $3\mid m$, we have $3m \mid m^2$. Hence, by Lemma 3, we get

$$l\equiv-\frac{\left(4m^2+1\right)^{(x-1)/2}}{\left(5m^2-1\right)^{(y-1)/2}}\equiv(-1)^{(y+1)/2}\,(\operatorname{mod} 3m). \tag{39}$$

In addition, (37) has another solution

$$(X,Y,Z)=(1,1,2). \tag{40}$$

Let $l'=\langle 1,1,2\rangle$. We have

$$l'\equiv-\left(4m^2+1\right)\equiv-1\,(\operatorname{mod} 3m). \tag{41}$$

By (39) and (41), we get $l'\equiv\pm l(\operatorname{mod} 3m)$. It implies that the solutions (38) and (40) belong to a same class $S(l)$ of solutions of (37). Further, since $(4m^2+1)X^2+(5m^2-1)Y^2\geq(4m^2+1)+(5m^2-1)=(3m)^2$, (40) is the least solution of $S(l)$. Therefore, applying Lemma 4 to (38), we get $2\mid z$, $2\nmid z/2$, and

$$\begin{aligned}&\left(4m^2+1\right)^{(x-1)/2}\sqrt{4m^2-1}\\&\quad+\left(5m^2-1\right)^{(y-1)/2}\sqrt{-\left(5m^2-1\right)}\\&\quad=\lambda_1\left(\sqrt{4m^2+1}+\lambda_2\sqrt{-\left(5m^2-1\right)}\right)^{z/2},\\&\qquad\qquad\lambda_1,\lambda_2\in\{\pm1\}.\end{aligned} \tag{42}$$

However, since $(x,y,z)\neq(1,1,2)$ and $3\mid m$, we have $z/2>1$ and $3\nmid(4m^2+1)(5m^2-1)$. By Lemma 5, (42) is false.

We finally consider the case that $2\mid x$. Then the equation

$$X^2+\left(5m^2-1\right)Y^2=(3m)^Z, \quad X,Y,Z\in\mathbb{Z}, \qquad \gcd(X,Y)=1, \quad Z>0 \tag{43}$$

has a solution

$$(X,Y,Z)=\left(\left(4m^2+1\right)^{x/2},\left(5m^2-1\right)^{(y-1)/2},z\right). \tag{44}$$

Since $2\nmid 3m$, applying Lemma 8 to (44), we have

$$z=Z_1t, \quad t\in\mathbb{N}, \tag{45}$$

$$\begin{aligned}&\left(4m^2+1\right)^{x/2}+\left(5m^2-1\right)^{(y-1)/2}\sqrt{-\left(5m^2-1\right)}\\&\quad=\lambda_1\left(X_1+\lambda_2Y_1\sqrt{-\left(5m^2-1\right)}\right)^t,\end{aligned} \tag{46}$$

where $\lambda_1,\lambda_2\in\{\pm1\}$, X_1,Y_1,Z_1 are positive integers satisfying

$$\begin{aligned}&X_1^2+\left(5m^2-1\right)Y_1^2=(3m)^{Z_1}, \qquad \gcd(X_1,Y_1)=1,\\&h\left(-4\left(5m^2-1\right)\right)\equiv0\,(\operatorname{mod} Z_1).\end{aligned} \tag{47}$$

If $2\mid t$, then from (46) we get

$$\begin{aligned}&\left(4m^2+1\right)^{x/2}+\left(5m^2-1\right)^{(y-1)/2}\sqrt{-\left(5m^2-1\right)}\\&\quad=\lambda_1\left(X_2+Y_2\sqrt{-\left(5m^2-1\right)}\right)^2,\end{aligned} \tag{48}$$

where X_2,Y_2 are integers satisfying

$$X_2^2+\left(5m^2-1\right)Y_2^2=(3m)^{Z/2}, \qquad \gcd(X_2,Y_2)=1. \tag{49}$$

By (49), we have

$$\begin{aligned}\left(4m^2+1\right)^{x/2}&=\lambda_1\left(X_2^2-\left(5m^2-1\right){Y_2}^2\right),\\\left(5m^2-1\right)^{(y-1)/2}&=2\lambda_1X_2Y_2.\end{aligned} \tag{50}$$

Further, since $\gcd(4m^2+1,5m^2-1)=1$, we see from (50) that $|X_2|=\pm1$, $|Y_2|=(5m^2-1)^{(y-1)/2}/2$ and

$$\left(4m^2+1\right)^{x/2}=\frac{1}{4}\left(5m^2-1\right)^y-1. \tag{51}$$

Furthermore, by (51), we get $1\equiv(4m^2+1)^{x/2}\equiv(1/4)(5m^2-1)^y-1\equiv-1/4-1(\operatorname{mod} m^2)$; whence we obtain $m^2\mid 9$. But, since $m>90$, it is impossible. So we have $2\nmid t$.

If $t=3$, then from (46) we get

$$\left(4m^2+1\right)^{x/2}=\lambda_1X_1\left(X_1^2-3\left(5m^2-1\right)Y_1^2\right), \tag{52}$$

$$\left(5m^2-1\right)^{(y-1)/2}=\lambda_1\lambda_2Y_1\left(3X_1^2-\left(5m^2-1\right)Y_1^2\right). \tag{53}$$

Since $\gcd(3X_1,5m^2-1)=1$, by (47), we see from (53) that $Y_1=(5m^2-1)^{(y-1)/2}$ and

$$3X_1^2-\left(5m^2-1\right)^y=\pm1. \tag{54}$$

Further, since $3\mid m$ and $2\nmid y$, by (54), we have

$$3X_1^2-\left(5m^2-1\right)^y=1. \tag{55}$$

But, since $2\nmid m$, we get from (55) that $4\mid 5m^2-1$, $2\nmid X_1$, and $3\equiv 3X_1^2-(5m^2-1)^y\equiv 1(\operatorname{mod} 4)$, a contradiction. So we have $t\neq3$.

Notice that the solution (44) satisfies $P(|Y|)\subseteq P(5m^2-1)$. Therefore, since $2\nmid t$ and $t\neq3$, by Lemma 9, we have $h(-4(5m^2-1))\equiv 0(\operatorname{mod} z)$ and

$$z\leq h\left(-4\left(5m^2-1\right)\right). \tag{56}$$

Further, applying Lemma 10 to (56), we get

$$z<\frac{4}{\pi}\sqrt{5m^2-1}\log\left(2e\sqrt{5m^2-1}\right). \tag{57}$$

On the other hand, by Lemma 2, z satisfies (4). The combination of (4) and (57) yields

$$m^2 < \frac{20}{\pi}\sqrt{5m^2-1}\log\left(2e\sqrt{5m^2-1}\right). \quad (58)$$

But (58) is false for $m > 90$. Thus, the solution (x, y, z) with $(x, y, z) \neq (1, 1, 2)$ does not exist. The theorem is proved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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