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Research Article

On Higher-Order Sequential Fractional Differential Inclusions with Nonlocal Three-Point Boundary Conditions

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We study a nonlinear three-point boundary value problem of sequential fractional differential inclusions of order $\xi+1$ with $n-1<\xi\leq n$, $n\geq 2$. Some new existence results for convex as well as nonconvex multivalued maps are obtained by using standard fixed point theorems. The paper concludes with an example.

1. Introduction

The topic of fractional differential equations has attracted a great attention in the recent years. It is mainly due to the intensive development of the theory and applications of fractional calculus. In fact, the tools of fractional calculus have considerably improved the modeling of several real world phenomena in physics, chemistry, bioengineering, etc. The systematic development of theory, methods, and applications of fractional differential equations can be found in [1–6]. For some recent results on fractional differential equations and inclusions, see [7–23] and the references cited therein.

In this paper, we study the following boundary value problem:

$$\begin{gathered}{}^cD^{\xi}(D+\lambda)x(t)\in F(t,x(t)),\\ 0<t<1,\quad n-1<\xi\leq n,\\ x(0)=0,\qquad x'(0)=0,\\ x''(0)=0,\dots,x^{(n-1)}(0)=0,\qquad x(1)=\alpha x(\sigma),\end{gathered}\tag{1}$$

where ${}^cD$ is the Caputo fractional derivative, D is the ordinary derivative, $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $0<\sigma<1$, λ is a positive real number, and α is a real number.

The present work is motivated by a recent paper of the authors [14], where the problem (1) was considered for a single-valued case. The existence of solutions for the given multivalued problem is discussed for three cases: (a) convex-valued maps; (b) not necessarily convex-valued maps; (c) nonconvex-valued maps. To establish the existence results, we make use of nonlinear alternative for Kakutani maps, nonlinear alternative of Leray-Schauder type for single-valued maps, selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, and a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. The tools employed in this paper are standard; however, their exposition in the framework of the problem at hand is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we used in the sequel. Section 3 contains the main results and an example. In Section 4, we summarize the work obtained in this paper and discuss some special cases.

2. Preliminaries

Let us recall some basic definitions of fractional calculus [2, 4, 6].

Definition 1. For $(n-1)$-times absolutely continuous function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^{c}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\int_{0}^{t}(t-s)^{n-q-1}g^{(n)}(s)\,ds, \quad n-1<q<n,\quad n=[q]+1, \tag{2}$$

where $[q]$ denotes the integer part of the real number q.

Definition 2. The Riemann-Liouville fractional integral of order q is defined as

$$I^{q}g(t)=\frac{1}{\Gamma(q)}\int_{0}^{t}\frac{g(s)}{(t-s)^{1-q}}ds,\quad q>0, \tag{3}$$

provided the integral exists.

Definition 3. A function $x\in AC^{n-1}([0,1],\mathbb{R})$ is called a solution of problem (1) if there exists a function $v\in L^{1}([0,1],\mathbb{R})$ with $v(t)\in F(t,x(t))$, a.e. $[0,1]$, such that ${}^{c}D^{\xi}(D+\lambda)x(t)=v(t)$, a.e. $[0,1]$, and $x(0)=0$, $x'(0)=0$, $x''(0)=0,\ldots,x^{(n-1)}(0)=0$, and $x(1)=\alpha x(\sigma)$.

For the forthcoming analysis, we define

$$\begin{aligned}P(t)&=P_{o}(t)\\&=\frac{t^{n-1}}{\lambda}-\frac{(n-1)t^{n-2}}{\lambda^{2}}+\frac{(n-1)(n-2)t^{n-3}}{\lambda^{3}}\\&\quad-\cdots-\frac{(n-1)!t}{\lambda^{n-1}}+\frac{(n-1)!}{\lambda^{n}}\left(1-e^{-\lambda t}\right),\\&\qquad n \text{ is odd},\end{aligned} \tag{4}$$

$$\begin{aligned}P(t)&=P_{e}(t)\\&=\frac{t^{n-1}}{\lambda}-\frac{(n-1)t^{n-2}}{\lambda^{2}}+\frac{(n-1)(n-2)t^{n-3}}{\lambda^{3}}\\&\quad-\cdots+\frac{(n-1)!t}{\lambda^{n-1}}-\frac{(n-1)!}{\lambda^{n}}\left(1-e^{-\lambda t}\right),\\&\qquad n \text{ is even}.\end{aligned} \tag{5}$$

Furthermore, we assume the nonresonance condition, that is, for $P=P_{o}$ and $P=P_{e}$, we choose α such that

$$P(1)-\alpha P(\sigma)\neq 0,\quad \text{for } 0<\sigma<1. \tag{6}$$

Lemma 4 (see [14]). *Assume that the nonresonance condition* (6) *holds. Given $y\in C([0,1],\mathbb{R})$, the unique solution of the problem*

$$\begin{gathered}{}^{c}D^{\xi}(D+\lambda)x(t)=y(t),\quad 0<t<1,\\ x(0)=0,\qquad x'(0)=0,\\ x''(0)=0,\ldots,x^{(n-1)}(0)=0,\qquad x(1)=\alpha x(\sigma)\end{gathered} \tag{7}$$

is given by

$$\begin{aligned}x(t)&=\int_{0}^{t}e^{-\lambda(t-s)}\left(\int_{0}^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}y(u)\,du\right)ds\\&\quad+\frac{P(t)}{P(1)-\alpha P(\sigma)}\\&\quad\times\left[\alpha\int_{0}^{\sigma}e^{-\lambda(\sigma-s)}\left(\int_{0}^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}y(u)\,du\right)ds\right.\\&\quad\left.-\int_{0}^{1}e^{-\lambda(1-s)}\left(\int_{0}^{s}\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}y(u)\,du\right)ds\right],\end{aligned} \tag{8}$$

where $P(t)=P_{o}(t)$ and $P(t)=P_{e}(t)$ are given by (4) *and* (5), *respectively.*

3. Existence Results

We begin this section with some preliminary material on multivalued maps [24, 25] that we need in the sequel.

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\|=\sup_{t\in[0,1]}|x(t)|$. Let $L^{1}([0,1],\mathbb{R})$ be the Banach space of measurable functions $x:[0,1]\to\mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^{1}}=\int_{0}^{1}|x(t)|dt$.

Let $(X,\|\cdot\|)$ denote a normed space. Then we define

$$\begin{aligned}\mathscr{P}_{\mathrm{cl}}(X)&=\{\mathscr{Z}\in\mathscr{P}(X):\mathscr{Z}\text{ is closed}\},\\ \mathscr{P}_{b}(X)&=\{\mathscr{Z}\in\mathscr{P}(X):\mathscr{Z}\text{ is bounded}\},\\ \mathscr{P}_{\mathrm{cp}}(X)&=\{\mathscr{Z}\in\mathscr{P}(X):\mathscr{Z}\text{ is compact}\},\\ \mathscr{P}_{\mathrm{cp},c}(X)&=\{\mathscr{Z}\in\mathscr{P}(X):\mathscr{Z}\text{ is compact and convex}\}.\end{aligned} \tag{9}$$

Definition 5. A multivalued map $\mathfrak{G}:X\to\mathscr{P}(X)$ is convex- (closed-) valued if $\mathfrak{G}(x)$ is convex (closed) for all $x\in X$.

Definition 6. The map $\mathfrak{G}$ is bounded on bounded sets if $\mathfrak{G}(\mathbb{B})=\cup_{x\in\mathbb{B}}\mathfrak{G}(x)$ is bounded in X for all $\mathbb{B}\in\mathscr{P}_{b}(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y|:y\in\mathfrak{G}(x)\}\}<\infty$).

Definition 7. $\mathfrak{G}$ is called upper semicontinuous (u.s.c.) on X if for each $x_{0}\in X$, the set $\mathfrak{G}(x_{0})$ is a nonempty closed subset of X, and if, for each open set N of X containing $\mathfrak{G}(x_{0})$, there exists an open neighborhood $\mathscr{N}_{0}$ of x_{0} such that $\mathfrak{G}(\mathscr{N}_{0})\subseteq N$.

Definition 8. $\mathfrak{G}$ is said to be completely continuous if $\mathfrak{G}(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in\mathscr{P}_{b}(X)$.

If the multivalued map $\mathfrak{G}$ is completely continuous with nonempty compact values, then $\mathfrak{G}$ is u.s.c. if and only if $\mathfrak{G}$ has a closed graph; that is, $x_{n}\to x_{*}$, $y_{n}\to y_{*}$, and $y_{n}\in\mathfrak{G}(x_{n})$ imply that $y_{*}\in\mathfrak{G}(x_{*})$. $\mathfrak{G}$ has a fixed point if there is $x\in X$ such that $x\in\mathfrak{G}(x)$. The fixed point set of the multivalued operator $\mathfrak{G}$ will be denoted by Fix $\mathfrak{G}$.

Definition 9. A multivalued map $\mathfrak{G} : [0;1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, \mathfrak{G}(t)) = \inf\{|y - z| : z \in \mathfrak{G}(t)\} \tag{10}$$

is measurable.

3.1. The Carathéodory Case

Definition 10. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\rho(t) \tag{11}$$

for all $\|x\| \le \rho$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1], \mathbb{R})$, define the set of selections of F by

$$\begin{aligned} S_{F,y} := \Big\{ & v \in L^1([0,1], R) : v(t) \in F(t, y(t)) \\ & \text{for a.e. } t \in [0,1] \Big\}. \end{aligned} \tag{12}$$

For the forthcoming analysis, we need the following lemmas.

Lemma 11 (nonlinear alternative for Kakutanimaps [26]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Lemma 12 (see [27]). *Let X be a Banach space. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator*

$$\begin{aligned} \Theta \circ S_F : \ & C([0,1], X) \longrightarrow \mathcal{P}_{cp,c}(C([0,1], X)), \\ & x \longmapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x}) \end{aligned} \tag{13}$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

Now we are in a position to prove the existence of the solutions for the boundary value problem (1) when the right-hand side is convex-valued.

Theorem 13. *Assume that the nonresonance condition (6) holds. In addition, we suppose that*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values;*

(H_2) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1], \mathbb{R}^+)$ such that*

$$\begin{aligned} \|F(t,x)\|_{\mathcal{P}} &:= \sup\{|y| : y \in F(t,x)\} \\ &\le p(t)\,\psi(\|x\|) \quad \text{for each } (t,x) \in [0,1] \times \mathbb{R}; \end{aligned} \tag{14}$$

(H_3) *there exists a constant $M > 0$ such that*

$$\begin{aligned} M \bigg(\frac{\psi(M)}{\Gamma(\xi)} \Big\{ & (1 + P_1) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \\ & + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma - s)} p(s)\, ds \Big\} \bigg)^{-1} > 1, \end{aligned} \tag{15}$$

where $P_1 = \max_{t \in [0,1]} |P(t)/(P(1) - \alpha P(\sigma))|$ ($P(t)$ is defined in (4) and (5)).

Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. Define the operator $\Omega_F : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ by

$$\begin{aligned} \Omega_F(x) &= \Big\{ h \in C([0,1], \mathbb{R}) : h(t) \\ &= \Big\{ \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ &\quad + \frac{P(t)}{P(1) - \alpha P(\sigma)} \\ &\quad \times \Big[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ &\quad - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \Big] \Big\} \Big\}, \end{aligned} \tag{16}$$

for $v \in S_{F,x}$. We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that *Ω_F is convex for each $x \in C([0,1], \mathbb{R})$*. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega_F(x)$, $x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = {} & \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ & + \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ & \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right]. \end{aligned} \tag{17}$$

Then for $t \in [0,1]$, we have

$$\begin{aligned} & |h(t)| \\ & \le \left| \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad + \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad \left.\left. + \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right] \right| \\ & \le \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} p(s) \psi(\|x\|)\, du \right) ds \\ & \quad + \left| \frac{P(t)}{P(1)-\alpha P(\sigma)} \right| \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} p(s) \psi(\|x\|)\, du \right) ds \right. \\ & \quad \left. + \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} p(s) \psi(\|x\|)\, du \right) ds \right] \\ & \le \frac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ \int_0^1 e^{-\lambda(1-s)} p(s)\, ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right. \\ & \quad \left. + P_1 \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \right\} \\ & = \frac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ (1+P_1) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \right. \\ & \quad \left. + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right\}. \end{aligned} \tag{18}$$

Consequently,

$$\begin{aligned} \|h\| \le \frac{\psi(r)}{\Gamma(\xi)} & \left\{ (1+P_1) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \right. \\ & \left. + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right\}. \end{aligned} \tag{19}$$

Now we show that Ω_F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ and $x \in B_r$. For each $h \in \Omega_F(x)$, we obtain

$$\begin{aligned} & |h(t_1) - h(t_2)| \\ & = \left| \int_0^{t_1} e^{-\lambda(t_1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad - \int_0^{t_2} e^{-\lambda(t_2-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ & \quad + \frac{[P(t_1) - P(t_2)]}{P(1)-\alpha P(\sigma)} \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad \left.\left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right] \right| \\ & \le \int_0^{t_1} e^{-\lambda(t_1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \\ & \quad - \int_0^{t_2} e^{-\lambda(t_2-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \\ & \quad + \left| \frac{P(t_1) - P(t_2)}{P(1)-\alpha P(\sigma)} \right| \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \right. \\ & \quad \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \right]. \end{aligned} \tag{20}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, therefore it follows from the Ascoli-Arzelá theorem that $\Omega_F : C([0,1],\mathbb{R}) \to \mathscr{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F *has a closed graph.* Let $x_n \to x_*$, $h_n \in \Omega_F(x_n)$, and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(x_*)$. Associated with $h_n \in \Omega_F(x_n)$, there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$h_n(t) = \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\, du \right) ds + \frac{P(t)}{P(1) - \alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\, du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\, du \right) ds \right]. \tag{21}$$

Thus, it suffices to show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in [0,1]$,

$$h_*(t) = \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\, du \right) ds + \frac{P(t)}{P(1) - \alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\, du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\, du \right) ds \right]. \tag{22}$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$f \longmapsto \Theta(v)(t) = \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds + \frac{P(t)}{P(1) - \alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right]. \tag{23}$$

Observe that

$$\|h_n(t) - h_*(t)\| = \Bigg\| \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} (v_n(u) - v_*(u))\, du \right) ds + \frac{P(t)}{P(1) - \alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} (v_n(u) - v_*(u))\, du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} (v_n(u) - v_*(u))\, du \right) ds \right] \Bigg\| \longrightarrow 0, \tag{24}$$

as $n \to \infty$.

Thus, it follows from Lemma 12 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\, du \right) ds + \frac{P(t)}{P(1) - \alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\, du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\, du \right) ds \right], \tag{25}$$

for some $v_* \in S_{F,x_*}$.

Finally, we show that there exists an open set $U \subseteq C([0,1],\mathbb{R})$ with $x \notin \Omega_F(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega_F(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$x(t) = \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds + \frac{P(t)}{P(1) - \alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right]. \tag{26}$$

Using the computations of the second step above we have

$$|x(t)| \le \frac{\psi(\|x\|)}{\Gamma(\xi)} \times \left\{ \left(1 + P_1\right) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds + \alpha P_1 \int_0^{\sigma} e^{-\lambda(\sigma-s)} p(s)\, ds \right\}. \tag{27}$$

Consequently, we have

$$\|x\| \left(\frac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ \left(1 + P_1\right) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds + \alpha P_1 \int_0^{\sigma} e^{-\lambda(\sigma-s)} p(s)\, ds \right\} \right)^{-1} \le 1. \tag{28}$$

In view of (H_3), there exists M such that $\|x\| \ne M$. Let us set

$$U = \{x \in C([0,1], \mathbb{R}) : \|x\| < M\}. \tag{29}$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(C([0,1], \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda \Omega_F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 11), we deduce that Ω_F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This completes the proof. □

Remark 14. The condition (H_3) in the statement of Theorem 13 may be replaced with the following one.

$(H_3)'$ There exists a constant $M > 0$ such that

$$\frac{M}{(\psi(M)/\Gamma(\xi)) \left\{ \left(1 + (1+\alpha) P_1\right) \|p\|_{L^1} \right\}} > 1, \tag{30}$$

where P_1 is the same as defined in (H_3).

3.2. The Lower Semicontinuous Case. As a next result, we study the case when F is not necessarily convex-valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [28] for lower semicontinuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and let $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semicontinuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \ne \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1], \mathbb{R})$ is decomposable if, for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 15. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semicontinuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \left\{ w \in L^1([0,1], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1] \right\}, \tag{31}$$

which is called the Nemytskii operator associated with F.

Definition 16. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semicontinuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Lemma 17 (see [29]). *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection; that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 18. *Assume that (H_2), (H_3), and the following condition hold:*

(H_4) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t, x) \mapsto F(t, x)$ *is $\mathcal{L} \otimes \mathcal{B}$ measurable,*

(b) $x \mapsto F(t, x)$ *is lower semicontinuous for each $t \in [0,1]$.*

Further the nonresonance condition (6) holds. Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 17, there exists a continuous function $f : AC^1([0,1], \mathbb{R}) \to L^1([0,1], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1], \mathbb{R})$.

Consider the problem

$$\begin{gathered} {}^cD^{\xi}(D + \lambda) x(t) = f(x(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(0) = 0, \\ x''(0) = 0, \ldots, x^{(n-1)}(0) = 0, \qquad x(1) = \alpha x(\sigma). \end{gathered} \tag{32}$$

Observe that if $x \in AC^1([0,1], \mathbb{R})$ is a solution of (32), then x is a solution to the problem (1). In order to transform the problem (32) into a fixed point problem, we define the operator $\overline{\Omega_F}$ as

$$\overline{\Omega_F}x(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(x(u))\,du\right)ds + \frac{P(t)}{P(1)-\alpha P(\sigma)} \times \left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(x(u))\,du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(x(u))\,du\right)ds\right]. \tag{33}$$

It can easily be shown that $\overline{\Omega_F}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 13. So we omit it. This completes the proof. □

3.3. The Lipschitz Case. Now we prove the existence of solutions for the problem (1) with a nonconvex-valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [30].

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathscr{P}(X)\times\mathscr{P}(X) \to \mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{34}$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathscr{P}_{b,\mathrm{cl}}(X), H_d)$ is a metric space and $(\mathscr{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [31]).

Definition 19. A multivalued operator $N : X \to \mathscr{P}_{\mathrm{cl}}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \quad \text{for each } x, y \in X; \tag{35}$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 20 (see [30]). *Let (X, d) be a complete metric space. If $N : X \to \mathscr{P}_{cl}(X)$ is a contraction, then* Fix $N \neq \emptyset$.

Theorem 21. *Assume that the nonresonance condition* (6) *holds. In addition, suppose that the following conditions hold:*

(H_5) *$F : [0,1]\times\mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0,1] \to \mathscr{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(H_6) *$H_d(F(t,x), F(t,\overline{x})) \le m(t)|x-\overline{x}|$ for almost all $t \in [0,1]$ and $x, \overline{x} \in \mathbb{R}$ with $m \in L^1([0,1],\mathbb{R}^+)$ and $d(0, F(t,0)) \le m(t)$ for almost all $t \in [0,1]$.*

Then the boundary value problem (1) *has at least one solution on* $[0,1]$ *if*

$$\frac{1}{\Gamma(\xi)}\left\{(1+P_1)\int_0^1 e^{-\lambda(1-s)}m(s)\,ds + \alpha P_1\int_0^\sigma e^{-\lambda(\sigma-s)}m(s)\,ds\right\} < 1. \tag{36}$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$ by the assumption (H_5), so F has a measurable selection (see Theorem III.6 [32]). Now we show that the operator Ω_F, defined in the beginning of proof of Theorem 13, satisfies the assumptions of Lemma 20. To show that $\Omega_F(x) \in \mathscr{P}_{\mathrm{cl}}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\ge0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n\to\infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds + \frac{P(t)}{P(1)-\alpha P(\sigma)} \times \left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds\right]. \tag{37}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and, for each $t \in [0,1]$, we have

$$v_n(t) \longrightarrow v(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds + \frac{P(t)}{P(1)-\alpha P(\sigma)} \times \left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\right]. \tag{38}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\delta < 1$ such that

$$H_d(\Omega_F(x), \Omega_F(\overline{x})) \le \delta\|x-\overline{x}\| \quad \text{for each } x, \overline{x} \in AC^1([0,1],\mathbb{R}). \tag{39}$$

Let $x, \overline{x} \in AC^1([0,1],\mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_1(u)\, du \right) ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_1(u)\, du \right) ds \right. \\ &\left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_1(u)\, du \right) ds \right]. \end{aligned} \tag{40}$$

By (H_6), we have

$$H_d(F(t,x), F(t,\overline{x})) \le m(t)\,|x(t) - \overline{x}(t)|. \tag{41}$$

So, there exists $w \in F(t, \overline{x}(t))$ such that

$$|v_1(t) - w| \le m(t)\,|x(t) - \overline{x}(t)|, \quad t \in [0,1]. \tag{42}$$

Define $U : [0,1] \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{ w \in \mathbb{R} : |v_1(t) - w| \le m(t)\,|x(t) - \overline{x}(t)| \}. \tag{43}$$

Since the multivalued operator $U(t) \cap F(t, \overline{x}(t))$ is measurable (Proposition III.4 [32]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \overline{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \overline{x}(t)|$.

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) &= \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_2(u)\, du \right) ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_2(u)\, du \right) ds \right. \\ &\left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_2(u)\, du \right) ds \right]. \end{aligned} \tag{44}$$

Thus,

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ &\le \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} |v_1(u) - v_2(u)|\, du \right) ds \\ &+ \left| \frac{P(t)}{P(1)-\alpha P(\sigma)} \right| \\ &\times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} |v_1(u) - v_2(u)|\, du \right) ds \right. \\ &\left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} |v_1(u) - v_2(u)|\, du \right) ds \right]. \end{aligned} \tag{45}$$

Hence,

$$\begin{aligned} &\|h_1 - h_2\| \\ &\le \frac{1}{\Gamma(\xi)} \left\{ (1 + P_1) \int_0^1 e^{-\lambda(1-s)} m(s)\, ds \right. \\ &\left. + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} m(s)\, ds \right\} \\ &\times \|x - \overline{x}\|. \end{aligned} \tag{46}$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} &H_d(\Omega_F(x), \Omega_F(\overline{x})) \\ &\le \delta \|x - \overline{x}\| \\ &\le \frac{1}{\Gamma(\xi)} \left\{ (1 + P_1) \int_0^1 e^{-\lambda(1-s)} m(s)\, ds \right. \\ &\left. + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} m(s)\, ds \right\} \|x - \overline{x}\|. \end{aligned} \tag{47}$$

Since Ω_F is a contraction, it follows from Lemma 20 that Ω_F has a fixed point x which is a solution of (1). This completes the proof. □

Remark 22. An alternative to the condition (36) in the statement of Theorem 21 may be the following one:

$$\frac{1}{\Gamma(\xi)} \{ (1 + (1+\alpha) P_1) \|m\|_{L^1} \} < 1. \tag{48}$$

Example 23. Consider the problem

$$\begin{gathered} {}^cD^{7/2}(D+2)\,x(t) \in F(t, x(t)), \quad 0 \le t \le 1, \\ x(0) = 0, \qquad x'(0) = 0, \\ x''(0) = 0, \qquad x'''(0) = 0, \qquad x(1) = x\left(\frac{1}{2}\right). \end{gathered} \tag{49}$$

Here, $\xi = 7/2$, $n = 4$, $\lambda = 2$, $\alpha = 1$, $\sigma = 1/2$, and $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$\begin{aligned} x &\longrightarrow F(t,x) \\ &= \left[\frac{|x|^5}{|x|^5 + 3} + t^3 + t^2 + 4, \frac{|x|^3}{|x|^3 + 1} + t + 2 \right]. \end{aligned} \tag{50}$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{|x|^5}{|x|^5 + 3} + t^3 + t^2 + 4, \frac{|x|^3}{|x|^3 + 1} + t + 2 \right) \le 9, \quad x \in \mathbb{R}. \tag{51}$$

Thus,

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leq 7$$
$$= p(t)\,\psi(\|x\|), \quad x \in \mathbb{R}, \tag{52}$$

with $p(t) = 1, \psi(\|x\|) = 7$. In this case

$$P(t) = P_e(t) = \frac{t^3}{2} - \frac{3t(t-1)}{4} - \frac{3\left(1-e^{-2t}\right)}{8},$$
$$P_1 \approx 6.214821. \tag{53}$$

By the condition (H_3), that is,

$$M\left(\frac{\psi(M)}{\Gamma(\xi)}\left\{\left(1+P_1\right)\int_0^1 e^{-\lambda(1-s)}p(s)\,ds\right.\right.$$
$$\left.\left. +\alpha P_1\int_0^\sigma e^{-\lambda(\sigma-s)}p(s)\,ds\right\}\right)^{-1} > 1, \tag{54}$$

we find that $M > M_1$ with $M_1 \approx 10.707326$. Therefore, it follows from Theorem 13 that problem (49) has at least one solution.

4. Conclusions

In this paper, we have solved a three-point boundary value problem of Caputo-type sequential fractional differential inclusions of an arbitrary order $\xi \in (n-1,n)$. The existence of solutions for the given problem with the convex-valued map is obtained by means of nonlinear alternative for Kakutani maps, while the existence result for not necessarily convex-valued map is established by combining nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with decomposable values. The nonconvex-valued case relies on a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. Some new existence results follow by fixing the parameters involved in the given problem. For instance, by taking $\alpha = 0$, our results correspond to a two-point Caputo-type multivalued problem of an arbitrary order $\xi \in (n-1,n)$, while the results for sequential differential inclusions of order $(n+1)$ can be obtained by fixing $\xi = n$ in the results of this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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