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Research Article

Parabolic Equations of Infinite Order with L^1 Data

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We prove an existence result of a nonlinear parabolic equation under Dirichlet null boundary conditions in Sobolev spaces of infinite order, where the second member belongs to $L^1(Q_T)$.

1. Introduction

This paper is devoted to the study of the following strongly nonlinear parabolic problem of Dirichlet type in the cylinder Q_T:

$$\begin{gathered} \frac{\partial u}{\partial t} + Au + g(t,x,u) = f(t,x) \quad \text{in } Q_T, \\ u(0,x) = 0, \\ D^{\omega} u = 0 \ \text{ on } \ S_T \quad \forall |\omega| = 0, 1 \ldots, \end{gathered} \tag{P}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $Q_T = (0,T)\times\Omega$ is a cylinder with lateral surface $S_T = [0,T]\times\Gamma$, with Γ is the boundary of Ω. A is a nonlinear elliptic operator of infinite order defined by

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha}\left(A_{\alpha}\left(t,x,D^{\gamma}u\right)\right), \quad |\gamma| \le |\alpha|. \tag{1}$$

Such operators include as a special case Leray-Lions types in the usual sense.

The real functions $A_{\alpha}(t,x,\xi)$ are assumed to satisfy some growth and coerciveness conditions without supposing a monotonicity condition in ξ, for all multi-indices α.

The nonlinear term g satisfies natural growth on $|u|$ and has to fulfil a sign condition.

The data f is assumed to satisfy

$$f \in L^1(Q_T). \tag{2}$$

In the case of infinite order, Dubinskiĭ [1] has proved, under some growth hypothesis and certain monotonicity conditions, the existence of solutions for the Dirichlet problem associated with the equation $Au = f$ in some general functional Sobolev spaces of infinite order $W_0^{\infty}(a_{\alpha}, p_{\alpha})$ of variables exponents p_{α}, with α being a multi-indice. The same author has investigated the existence result for parabolic elliptic problems governed by operators of infinite orders. In fact, also in [1], Dubinskiĭ has proved by considering, further, the monotonicity of the operator A that the problem $\partial u/\partial t + Au = f$ has a solution in $L^p(0,T,W_0^{\infty}(a_{\alpha},p))$, $p>1$, in the variational case (i.e., where f belongs to the dual space).

Another work has been shown, in the variational case in [2], the existence of solutions for strongly parabolic nonlinear equations of infinite order related to the problem $\partial u/\partial t + Au + g(t,x,u) = f$.

Our purpose in this paper is to prove the existence of solutions for parabolic equations, in Sobolev spaces of infinite order with L^1 data, associated with the problem $\partial u/\partial t + Au + g(t,x,u) = f$.

More precisely, we will assume more less restrictions on the operator A (no monotonicity condition) and deal with a different approach by involving a truncation of the perturbations g. Next, we use the monotonicity of a part of approximate operator which contains a linear term of higher order of derivation that satisfies the monotonicity condition and prove the existence of solutions in the framework of function space $L^p(0,T,W_0^\infty(a_\alpha,p))$, $p>1$.

Let us mention that an interesting result concerning the stationary counterpart of the problem (P) has been proved in [3, 4].

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$, $a_\alpha \geq 0$, $p > 1$ real numbers for all multi-index α, and $\|\cdot\|_p$ the usual Lebesgue norm in the space $L^p(\Omega)$. The Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha,p)(\Omega) = \left\{u \in C^\infty(\Omega) : \|u\|_\infty^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \left\|D^\alpha u\right\|_p^p < \infty\right\}. \tag{3}$$

Here $D^\alpha = \partial^{|\alpha|}/(\partial x_1)^{\alpha_1}\cdots(\partial x_N)^{\alpha_N}$.

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.

Since we will deal with the Dirichlet problem, we will use the functional space $W_0^\infty(a_\alpha,p)(\Omega)$ defined by

$$W_0^\infty(a_\alpha,p)(\Omega) = \left\{u \in C_0^\infty(\Omega) : \|u\|_\infty^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \left\|D^\alpha u\right\|_p^p < \infty\right\}. \tag{4}$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of the spaces $W^\infty(a_\alpha,p)(\Omega)$, is the question of their nontriviality (or nonemptiness), that is, the question of the existence of a function u such that $\|u\|_\infty < \infty$.

Definition 1 (see [1]). The space $W_0^\infty(a_\alpha,p)(\Omega)$ is called nontrivial space if it contains at least one function which is not identically equal to zero; that is, there is a function $u \in C_0^\infty(\Omega)$ such that $\|u\|_\infty < \infty$.

It turns out that the answer of this question depends not only on the given parameters a_α and p of the spaces $W^\infty(a_\alpha,p)(\Omega)$, but also on the domain Ω.

The dual space of $W_0^\infty(a_\alpha,p)(\Omega)$ is defined as follows:

$$W^{-\infty}(a_\alpha,p')(\Omega) = \left\{f : f = \sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}D^\alpha f_\alpha,\quad \|f\|_{-\infty}^{p'} = \sum_{|\alpha|=0}^{\infty} a_\alpha \left\|f_\alpha\right\|_{p'}^{p'} < \infty\right\}, \tag{5}$$

where $f_\alpha \in L^{p'}(\Omega)$ for all multi-indices α and p' is the conjugate of p; that is, $p' = p/(p-1)$ (for more details about these spaces, see [1, 5]).

By the definition, the duality of the space $W^{-\infty}(a_\alpha,p')(\Omega)$ and $W_0^\infty(a_\alpha,p)(\Omega)$ is given by relation

$$\langle f,v\rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega f_\alpha(x)\,D^\alpha v(x)\,dx, \tag{6}$$

which, as it is not difficult to verify, is correct.

Let us denote by $L^p(0,T,W_0^\infty(a_\alpha,p))$ the space of functions $u(t,x)$ which has finite norm

$$\|u\|_{p,\infty}^p = \int_0^T \|u\|_\infty^p\,dt \tag{7}$$

and is equal to zero together with all derivatives $D^\omega u$ on the lateral surface S. In other words one has

$$L^p(0,T,W_0^\infty(a_\alpha,p)) = \left\{u \text{ measurable} : \|u\|_{p,\infty}^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \left\|D^\alpha u\right\|_p^p dt < \infty,\quad D^\omega u\big|_S = 0,\ |\omega| = 0,1,\ldots\right\}. \tag{8}$$

Further, let $L^{p'}(0,T,W^{-\infty}(a_\alpha,p'))$ be the dual space of the space $L^p(0,T,W_0^\infty(a_\alpha,p))$, that is, the space of generalized functions $f(t,x)$ having a form

$$f(t,x) = \sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^\alpha f_\alpha(t,x), \tag{9}$$

where $f_\alpha(t,x) \in L^{p'}(Q_T)$ and

$$\rho'(f) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \left\|f_\alpha(t,x)\right\|_{p'}^{p'} dt < \infty. \tag{10}$$

The value of $f(t,x) \in L^{p'}(0,T,W^{-\infty}(a_\alpha,p'))$ on an element $v(t,x) \in L^p(0,T,W_0^\infty(a_\alpha,p))$ is defined by the formula

$$\langle f,v\rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \int_\Omega f_\alpha(t,x)\,D^\alpha v(t,x)\,dx\,dt, \tag{11}$$

which, as easy to see, is correct.

Sobolev spaces of infinite order have extensive applications to the theory of partial differential equations and, among their number, in mathematical physics. The basis of these applications is the nonformal algebra of differential operators of infinite orders as the operators, acting in the corresponding Sobolev spaces of infinite order. This makes it possible, by considering $\partial/\partial x$ as a parameter, to solve a partial equation as ordinary differential equation, to which are adjoined the initial or boundary conditions.

More explicitly, we cite the following examples of operators of infinite order which are closely inspired from the ones used in Dubinskiĭ [1].

Example 2. Consider the following operator:

$$Au = \left(-\sqrt{I+\triangle}\right)u. \tag{12}$$

Our technique here consists in exploiting certain results in the setting of functional spaces of infinite order. Thus as in [1], we can write the operator A as follows:

$$Au = \left(-\sqrt{I+\triangle}\right)u = \sum_{k=0}^{\infty} a_k (-\triangle)^k u, \tag{13}$$

where $a_k > 0$, $k = 0, 1, \ldots$, are real numbers which guarantee the nontriviality of the corresponding functional space defined by

$$W_0^{\infty}(a_k, 2)(\Omega) = \left\{ u \in C_0^{\infty}(\Omega) : \rho(u) = \sum_{k=0}^{\infty} a_k \|\nabla u\|_2^2 < \infty \right\}. \tag{14}$$

Moreover for any $f \in L^2(0, T, W^{-\infty}(a_k, 2)(\Omega))$, the parabolic problem

$$\frac{\partial u}{\partial t} + \sum_{k=0}^{\infty} a_k (-\triangle)^k u = f, \quad x \in \Omega \tag{15}$$

has a solution $u \in L^2(0; T; W_0^{\infty}(a_k; 2)(\Omega))$, in the variational sense.

By using the recent work of authors (see Theorem 3.1. in [2]), the strongly nonlinear parabolic problem

$$\frac{\partial u}{\partial t} + \sum_{k=0}^{\infty} a_k (-\triangle)^k u + g(t, x, u) = f \quad x \in \Omega, \tag{16}$$

has also a solution $u \in L^2(0; T; W_0^{\infty}(a_k; 2)(\Omega))$, in the variational sense, for any $f \in L^2(0, T, W^{-\infty}(a_k, 2)(\Omega))$. Here g is a nonlinear term which has to fulfil a sign condition (see Section 3).

Remark 3. For examples of the nontriviality of Sobolev spaces of infinite order, we refer the reader to [1, 5–7] for details.

3. Main Result

In this section we formulate and prove the main result. We denote by λ_α the number of multi-indices γ such that $|\gamma| \le |\alpha|$. Let A be the nonlinear operator of infinite order defined as in (1), with $A_\alpha : (0, T) \times \Omega \times \mathbb{R}^{\lambda_\alpha} \to \mathbb{R}$ being a real function.

Let us now formulate the following assumptions.

(A_1) $A_\alpha(t, x, \xi_\gamma)$ is a Carathéodory function for all α, $|\gamma| \le |\alpha|$.

(A_2) For a.e. $(t, x) \in Q_T$, all $m \in \mathbb{N}^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \le |\alpha|$, and some constant $c_0 > 0$, we assume that

$$\left| \sum_{|\alpha|=0}^{m} A_\alpha(t, x, \xi_\gamma) \eta_\alpha \right| \le c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p-1} |\eta_\alpha|, \tag{17}$$

where $p > 1$, $a_\alpha \ge 0$ are real numbers for all multi-indices α.

(A_3) There exist constants $c_1 > 0$, $c_2 \ge 0$ such that

$$\sum_{|\alpha|=0}^{m} A_\alpha(t, x, \xi_\gamma) \xi_\alpha \ge c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^p - c_2 \tag{18}$$

for all $m \in \mathbb{N}^*$, for all ξ_γ, ξ_α; $|\gamma| \le |\alpha|$ and for a.e. $(t, x) \in Q_T$.

(A_4) The space $W_0^{\infty}(a_\alpha, p)(\Omega)$ is nontrivial.

As regards the nonlinear term g, we assume that g satisfies the following natural growth on $|u|$ and the classical sign condition.

(G) $g : Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|g(t, x, s)| \le b_1 |s|^{p-1} + b_2, \quad g(t, x, s)\, s \ge 0, \tag{19}$$

for a.e. $(t, x) \in Q_T$, $s \in \mathbb{R}$, and some constants b_1 and b_2.

Concerning the second member f, we assume that

$$f \in L^1(Q_T). \tag{20}$$

We will prove the following existence theorem.

Theorem 4. *Under assumptions (A_1)–(A_4) and (G), for any right side $f \in L^1(Q_T)$ there exists at least a function u such that*

(1) *$u(t, x) \in L^p(0, T, W_0^{\infty}(a_\alpha, p))$, $\partial u/\partial t \in L^{p'}(0, T, W^{-\infty}(a_\alpha, p'))$;*

(2) *$u(0, x) = 0$;*

(3) *for any function $v(t, x) \in L^{\infty}(0, T, W_0^{\infty}(a_\alpha, p))$, the following identity*

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, v \right\rangle dt + \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \int_\Omega A_\alpha(t, x, D^\gamma u) D^\alpha v\, dx\, dt + \int_0^T \int_\Omega g(t, x, u)\, v\, dx\, dt = \int_0^T \langle f, v \rangle dt \tag{21}$$

is valid.

Proof. we proceed by steps in order to prove our result.

Step 1. The approximate problem.

Set for a.e. $(t, x) \in Q_T$

$$f_k(t, x) = T_k f(t, x), \qquad g_k(t, x, u) = T_k g(t, x, u), \tag{22}$$

where T_k is the usual truncation given by

$$T_k \eta = \begin{cases} \eta & \text{if } |\eta| < k \\ \dfrac{k\eta}{|\eta|} & \text{if } |\eta| \ge k. \end{cases} \tag{23}$$

It is clear that $|f_k| \le k$ for a.e. $(t,x) \in Q_T$. Thus, it follows that $f_k \in L^\infty(Q_T)$.

Further, we have

$$f_k \longrightarrow f \quad \text{for a.e. } (t,x) \in Q_T, \qquad |f_k| \le |f| \in L^1(Q_T), \tag{24}$$

and from Lebesgue's dominated convergence theorem, see [8], we conclude that

$$f_k \longrightarrow f \quad \text{in } L^1(Q_T). \tag{25}$$

Let $k \in \mathbb{N}^*$ sufficiently large. Define the operator A_{2k+2} of order $2k+2$ by

$$A_{2k+2}u = \sum_{|\alpha|=k+1} (-1)^{k+1} c_\alpha D^{2\alpha} u + \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha \left(A_\alpha\left(t,x,D^\gamma u\right)\right). \tag{26}$$

Note that c_α are constants small enough such that they fulfil the conditions of the following lemma introduced in [1].

In fact, such a condition imposed on each c_α is required to ensure the nontriviality of the space $W_0^\infty(c_\alpha, 2)$.

Lemma 5 (cf. [1]). *For any nontrivial space $W_0^\infty(a_\alpha, p_\alpha)$ there exists a nontrivial space $W_0^\infty(c_\alpha, 2)$ such that $W_0^\infty(a_\alpha, p_\alpha) \subset W_0^\infty(c_\alpha, 2)$.*

The operator A_{2k+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition (see [1, 3]). Moreover, thanks to the truncation T_k as in [9] and from assumptions (A_1), (A_2), and (A_3), we deduce that the operator $A_{2k+2} + g_k$ is bounded, coercive, and pseudo-monotone. Then, it is well known (see Lions [10]) that there exists $u_k \in L^p(0,T,W_0^{k+1,p}(\Omega))$ such that

$$\frac{\partial u_k}{\partial t} + A_{2k+2}u_k + g_k(t,x,u_k) = f_k(t,x), \qquad u_k(0,x) = 0. \tag{P_k}$$

In the variational formulation, we get

$$\int_0^T \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle dt + \int_0^T \langle A_{2k+2}u_k, v\rangle\, dt + \int_0^T \int_\Omega g_k(t,x,u_k)\, v\, dx\, dt = \int_0^T \langle f_k, v\rangle dt, \tag{27}$$

for any $v \in L^\infty(0,T,W_0^{k+1}(\Omega))$.

Step 2 (a priori estimates). Let us choose $v = u_k$ as a test function in (P_k). Then using the sign condition in (G), one has the estimates

$$\sum_{|\alpha|=k+1} c_\alpha \int_0^T \|D^\alpha u_k\|_2^2\, dt + \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|D^\alpha u_k\|_p^p\, dt \le c_2, \tag{28}$$

$$\int_Q g_k(t,x,u_k)\, u_k dx\, dt \le c_2. \tag{29}$$

In the sequel $c_2, c_3, c_4, \ldots$ designate arbitrary constants not depending on k.

From the first equality in (P_k) and estimates (28) and (29), we remark that $\partial u_k/\partial t \in L^{p'}(0,T,W_0^{-k-1,p'}(\Omega))$. In addition, for any $v \in L^\infty(0,T,W_0^\infty(a_\alpha,p))$ the following equality is valid:

$$\int_0^T \left| \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle \right| dt \le Q_1 + Q_2 + Q_3, \tag{30}$$

where

$$\begin{aligned} Q_1 &= \int_0^T \int_\Omega |f_k(t,x)\, v|\, dx\, dt, \\ Q_2 &= \int_0^T \int_\Omega |g_k(t,x,u_k)\, v|\, dx\, dt, \\ Q_3 &= \sum_{|\alpha|=k+1} c_\alpha \int_0^T \int_\Omega |D^\alpha u_k|\, |D^\alpha v|\, dx\, dt \\ &\quad + \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \int_\Omega |D^\alpha u_k|^{p-1} |D^\alpha v|\, dx\, dt. \end{aligned} \tag{31}$$

Regarding the quantity Q_1, one has

$$\begin{aligned} Q_1 &= \int_0^T \int_\Omega |f_k(t,x)\, v|\, dx\, dt \\ &\le \int_0^T \int_\Omega |f(t,x)\, v|\, dx\, dt \\ &\le \int_0^T \int_\Omega |f| \cdot |v|\, dx\, dt \\ &\le \|f\|_{L^1(Q_T)} \|v\|_{L^\infty(Q_T)}, \end{aligned} \tag{32}$$

and so

$$Q_1 \le c_3 \|v\|_{L^\infty(Q_T)}. \tag{33}$$

We also have

$$\begin{aligned} Q_2 &\le \int_0^T \int_\Omega \left(|u_k|^{p-1} |v| + |v|\right) dx\, dt \\ &\le \int_0^T \|u_k\|_p^{p-1} \|v\|_p\, dt + c_4 \left(\int_0^T \|v\|_p^p\, dt\right)^{1/p} \\ &\le \left(\int_0^T \|u_k\|_p^p\, dt\right)^{1/p'} \left(\int_0^T \|v\|_p^p\, dt\right)^{1/p} \\ &\quad + c_4 \left(\int_0^T \|v\|_p^p\, dt\right)^{1/p} \\ &\le (c_2 + c_4) \|v\|_{p,\infty}, \end{aligned} \tag{34}$$

where c_2 is the constant of the estimate (28). Then one gets

$$Q_2 \le c_5 \|v\|_{p,\infty}. \tag{35}$$

Moreover, for the last term Q_3, one has

$$Q_3 = J_1 + J_2, \tag{36}$$

where

$$\begin{aligned} J_1 &= \sum_{|\alpha|=k+1} c_\alpha \int_0^T \int_\Omega |D^\alpha u_k| |D^\alpha v| \, dx \, dt \\ &\le \left(\sum_{|\alpha|=k+1} c_\alpha \int_0^T \|D^\alpha u_k\|_2^2 \, dt \right)^{1/2} \\ &\quad \times \left(\sum_{|\alpha|=k+1} c_\alpha \int_0^T \|D^\alpha v\|_2^2 \, dt \right)^{1/2} \le (c_2)^{1/2} \|v\|_{p,\infty}, \\ J_2 &= \sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \int_\Omega |D^\alpha u_k|^{p-1} |D^\alpha v| \, dx \, dt \\ &\le \left(\sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|D^\alpha u_k\|_p^p \, dt \right)^{1/p'} \\ &\quad \times \left(\sum_{|\alpha|=0}^{k} a_\alpha \int_0^T \|D^\alpha v\|_p^p \, dt \right)^{1/p} \\ &\le (c_2)^{1/p'} \|v\|_{p,\infty}. \end{aligned} \tag{37}$$

Then, one deduces that

$$Q_3 \le c_6 \|v\|_{p,\infty}. \tag{38}$$

Combining (30), (33), (35), and (38), it follows that

$$\int_0^T \left| \left\langle \frac{\partial u_k}{\partial t}, v \right\rangle \right| dt \le c_7 \|v\|_{p,\infty} + c_3 \|v\|_{L^\infty(Q_T)}. \tag{39}$$

This implies that

$$\left\| \frac{\partial u_k}{\partial t} \right\|_{p',-\infty} \le c_8; \tag{40}$$

that is, the derivatives $\partial u_k/\partial t$ form a bounded set in the space $L^{p'}(0,T,W^{-\infty}(a_\alpha,p'))$.

Now, estimates (28) and (40) permit us to apply the well known lemma of compactness (see Lions [11]).

Let B_0, B, and B_1 be Banach spaces. Let us set

$$Y = \left\{ u : u \in L^{p_0}(0,T,B_0),\ u' \in L^{p_1}(0,T,B_1) \right\}, \tag{41}$$

where $p_0 > 1$, $p_1 > 1$ are real numbers.

Lemma 6 (cf. [1]). *Let the imbeddings*

$$B_0 \subset B \subset B_1 \tag{42}$$

hold; moreover, let the imbedding $B_0 \subset B$ be compact. Then

$$Y \subset L^{p_0}(0,T,B) \tag{43}$$

and this imbedding is compact.

In order to apply this lemma, define

$$\begin{gathered} B_0 = W^{S+1}(a_\alpha, p) = \left\{ u(x) : \sum_{|\alpha|=0}^{S} a_\alpha \|D^\alpha u\|_p^p < \infty \right\}, \\ B = W^S(a_\alpha, p), \qquad B_1 = W^{-\infty}(a_\alpha, p'); \\ p_0 = p, \qquad p_1 = p', \end{gathered} \tag{44}$$

where $S \ge 0$ is arbitrary and $p' = p/(p-1)$.

Step 3 (convergence of the approximate problem (P_k)). In view of (28) and (40), we deduce that the family u_k of solutions of problems (P_k) is compact in the space $L^p(0,T,W^S(a_\alpha,p))$, where S is arbitrary. Consequently, by similar argument as in the elliptic case (using the diagonal process), see [3] or [1], one gets that the sequence u_k converges strongly together with all derivatives $D^\omega u_k$ to a function $u \in L^p(0,T,W_0^\infty(a_\alpha,p))$.

Letting now $m > 0$ be fixed, E a measurable subset of Q_T, and $\varepsilon > 0$, we have

$$\begin{aligned} &\int_E |g_k(t,x,u_k)| \, dx \, dt \\ &\quad \le \int_{E\cap\{|u_k|\le m\}} |g_k(t,x,u_k)| \, dx \, dt \\ &\qquad + \frac{1}{m} \int_{E\cap\{|u_k|>m\}} |g_k(t,x,u_k) u_k| \, dx \, dt \\ &\quad \le \int_{E\cap\{|u_k|\le m\}} \left(b_1 |u_k|^{p-1} + b_2 \right) dx \, dt \\ &\qquad + \frac{1}{m} \int_{Q_T} g_k(t,x,u_k) u_k dx \, dt \\ &\quad \le \left(|m|^{p-1} + 1 \right) |E| + \frac{c_2}{m}, \end{aligned} \tag{45}$$

where c_2 is the constant of (29) which is independent of k.

For $|E|$ sufficiently small and $c_2/m < \varepsilon/2$, we obtain

$$\int_E g_k(t,x,u_k) \le \varepsilon. \tag{46}$$

Using Vitali's theorem, we get

$$g_k(x,t,u_k) \longrightarrow g(x,t,u) \quad \text{in } L^1(Q_T). \tag{47}$$

On the other hand, in view of Fatou's lemma and (29), we obtain

$$\int_{Q_T} g(x,t,u) u \, ds \le \lim_{k\to+\infty} \int_{Q_T} g_k(x,t,u_k) u_k ds \le c_2; \tag{48}$$

this implies that

$$g(x,t,u) u \in L^1(Q_T). \tag{49}$$

Now, we will prove that

$$\lim_{k\to+\infty} \int_0^T \langle A_{2k+2}(u_k), v \rangle \, dt = \int_0^T \langle A(u), v \rangle \, dt \tag{50}$$

for all $v \in L^\infty(0,T,W_0^\infty(a_\alpha,p))$.

In fact, let k_0 be a fixed number sufficiently large ($k > k_0$) and let $v \in L^\infty(0,T,W_0^\infty(a_\alpha,p))$. Set

$$\int_0^T \langle A(u) - A_{2k+2}(u_k), v\rangle\, dt = I_1 + I_2 + I_3, \tag{51}$$

where

$$\begin{aligned} I_1 &= \sum_{|\alpha|=0}^{k_0} \int_0^T \langle A_\alpha(t,x,D^\gamma u) - A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle\, dt, \\ I_2 &= \sum_{|\alpha|=k_0+1}^{\infty} \int_0^T \langle A_\alpha(t,x,D^\gamma u), D^\alpha v\rangle\, dt, \\ I_3 &= -\sum_{|\alpha|=k_0+1}^{k} \int_0^T \langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle \\ &\quad - \sum_{|\alpha|=k+1} c_\alpha \langle D^\alpha u, D^\alpha v\rangle\, dt, \end{aligned} \tag{52}$$

or, in another form,

$$I_3 = -\sum_{|\alpha|=k_0+1}^{k+1} \int_0^T \langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle\, dt, \tag{53}$$

with $A_\alpha(t,x,\xi_\gamma) = c_\alpha \xi_\alpha$ and $c_\alpha \geq 0$ for $|\alpha| = k+1$ (c_α are constants given in Lemma 5).

We will go to limit as $k \to +\infty$ to prove that I_1, I_2, and I_3 tend to 0. Starting by I_1, we have $I_1 \to 0$ since $A(t,x,\xi_\gamma)$ is of Carathéodory type.

The term I_2 is the remainder of a convergence series; hence $I_2 \to 0$.

For what concerns I_3, for all $\varepsilon > 0$, there holds $k(\varepsilon) > 0$ (see [8, page 56]) such that

$$\begin{aligned} &\left| \sum_{|\alpha|=k_0+1}^{k+1} \int_0^T \langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle\, dt \right| \\ &\quad \leq \sum_{|\alpha|=k_0+1}^{k+1} \int_0^T \left| \langle A_\alpha(t,x,D^\gamma u_k), D^\alpha v\rangle \right| dt \\ &\quad \leq c_0 \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \int_\Omega \left|D^\alpha u_k\right|^{p-1} \left|D^\alpha v\right| dx\, dt \\ &\quad \leq c_0 \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \left\|D^\alpha u_k\right\|_p^{p-1} \left\|D^\alpha v\right\|_p dt \\ &\quad \leq \varepsilon c_0 \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \left\|D^\alpha u_k\right\|_p^p dt \\ &\qquad + c_0 k(\varepsilon) \sum_{|\alpha|=k_0+1}^{k+1} a_\alpha \int_0^T \left\|D^\alpha v\right\|_p^p dt \\ &\quad \leq \varepsilon c_0 c_2 + c_0 k(\varepsilon) \sum_{|\alpha|=k_0+1}^{\infty} a_\alpha \int_0^T \left\|D^\alpha v\right\|_p^p dt, \end{aligned} \tag{54}$$

where c_2 is the constant given in the estimate (28). Moreover the term

$$\sum_{|\alpha|=k_0+1}^{\infty} a_\alpha \int_0^T \left\|D^\alpha v\right\|_p^p dt \tag{55}$$

is the remainder of a convergent series; therefore $I_3 \to 0$ holds.

Finally, we conclude that

$$\int_0^T \langle A_{2k+2}(u_k), v\rangle\, dt \longrightarrow \int_0^T \langle A(u), v\rangle\, dt \quad \text{as } k \longrightarrow +\infty \tag{56}$$

for all $v \in L^\infty(0,T,W_0^\infty(a_\alpha,p))$.

Moreover, it is clear that

$$\int_0^T \langle f_k, v\rangle\, dt \longrightarrow \int_0^T \langle f, v\rangle\, dt \tag{57}$$

as $k \to +\infty$ since $f_k \to f$ in $L^1(Q_T)$.

Consequently, by passing to the limit in (P_k), we obtain

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial u}{\partial t}, v\right\rangle dt + \int_0^T \langle A(u), v\rangle\, dt \\ &\quad + \int_{Q_T} g(t,x,u)\, v\, dx\, dt = \int_0^T \langle f, v\rangle\, dt, \end{aligned} \tag{58}$$

for all $v \in L^\infty(0,T,W_0^\infty(a_\alpha,p))$.

That is,

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial u}{\partial t}, v\right\rangle dt \\ &\quad + \sum_{|\alpha|=0}^{\infty} a_\alpha \int_0^T \int_\Omega A_\alpha(t,x,D^\gamma u)\, D^\alpha v\, dx\, dt \\ &\quad + \int_0^T \int_\Omega g(t,x,u)\, v\, dx\, dt = \int_0^T \langle f, v\rangle dt, \end{aligned} \tag{59}$$

for all $v \in L^\infty(0,T,W_0^\infty(a_\alpha,p))$.

This completes the proof. □

4. Example

The following example of an operator of infinite order is closely related to the one used in [12].

Let us consider the operator:

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^\alpha D^\alpha \left(a_\alpha \left|D^\alpha u\right|^{p-2} D^\alpha u \right), \tag{60}$$

where $a_\alpha \geq 0$ is a sequence of numbers, $p > 1$ is a number such that the space $W^\infty(a_\alpha,p)(\Omega)$ is not trivial (e.g., if $a_\alpha = [(2\alpha)!]^{-p}$, $p > 1$ and $\dim\Omega = 1$); then the conditions A_1, A_2, and A_3 are satisfied.

As regards a function g that satisfies the condition (G), let us consider

$$g(t,x,s) = s\,|s|^r\, h(x), \quad \text{with } r > 0, \tag{61}$$

where $h \in L^1(\Omega)$, $h(x) \geq 0$, a.e.

Consequently, for the described above operator A and the nonlinear term g the existence result follows immediately from Theorem 4.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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