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Research Article

Optimal Control of Pseudoparabolic Variational Inequalities Involving State Constraint

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We establish the necessary condition of optimality for optimal control problem governed by some pseudoparabolic differential equations involving monotone graphs. Some approximating control process and examples are given.

1. Introduction

We will study the following optimal control problem governed by nonlinear pseudoparabolic variational inequalities of the following form:

$$\frac{dMy}{dt} + Ay + \beta(y) \ni Bu \quad \text{a.e in } (0,T), \qquad y(0) = y_0, \tag{1}$$

with the state constraint

$$F(y(\cdot)) \subset S. \tag{2}$$

The pay-off function is given by

$$L(y(\cdot), u(\cdot)) = \int_0^T [g(t, y(\cdot)) + h(u(\cdot))]\, dt, \tag{3}$$

where $Q = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

For the problem (1)–(3), we have the following assumptions.

(H1) M is a selfadjoint operator in $H = Ł^2(\Omega)$ with $D(M) \subset D(A+\beta)$ such that for every $y \in D(M)$,

$$(My, y) \geq a|y|^2, \quad a > 0. \tag{4}$$

Throughout in the sequel, we will denote by $|\cdot|$ and $(\cdot,\cdot)$ the norm and the scalar product of H, respectively. The norm of the control set U will be denoted by $|\cdot|_U$ and the scalar product $\langle\cdot,\cdot\rangle$, respectively. $D(M), D(A+\beta)$ denote the domain of operator $M, A+\beta$, respectively.

(H2) $V \subset H$ is a real Hilbert space such that V is dense in H and $V \subset H \subset V'$ algebraically and topologically, where V' is the dual of V. Further, the injection of V into H is compact.

$A : V \to V'$ is a linear continuous and symmetric operator from V to V' satisfying the coercivity condition

$$(Ay, y) \geq w\|y\|_V^2 + \alpha|y|_H^2 \quad \forall y \in V, \tag{5}$$

where $w > 0$ and $\alpha \geq 0$.

(H3) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. Let $\phi(y) : H \to \mathbb{R} = (-\infty, +\infty]$ be the lower semicontinuous convex function defined by $\phi(y) = \int_\Omega j(y)dx$, where $j : \mathbb{R} \to \overline{\mathbb{R}}$ is such that $\partial j = \beta$. Moreover,

$$(Ay, \beta_\epsilon(y)) \geq 0 \quad \forall y \in D(A), \epsilon > 0, \tag{6}$$

where $\beta_\epsilon(r) = \epsilon^{-1}(r - (1+\epsilon\beta)^{-1}r)$ for all $\epsilon > 0$, $r \in \mathbb{R}$. For every $\xi \in \beta$, there exists a constant c such that

$$|\xi(s)| \le c\left(1 + |s|^{p+1}\right), \tag{7}$$

where $0 \le p \le 2/(N-2)$ if $N > 2$ and $0 \le p < +\infty$ if $N = 1, 2$. ∂j denotes the generalized Clarke subdifferential of the function j.

(H4) B is a linear continuous operator from a real Hilbert space U to H.

(H5) Let $\mathbb{Z}$ be a Banach space with the dual $\mathbb{Z}^*$ strictly convex. $S \subset \mathbb{Z}$ is a closed convex subset with finite codimensionality [1–3]. $F : L^2(0,T;V) \to \mathbb{Z}$ is of class C^1.

(H6) The functional $h : U \to \overline{\mathbb{R}}$ is convex and lower semicontinuous (l. s. c), such that

$$h(u) \ge c_1|u|_U^2 + c_2, \tag{8}$$

where $c_1 > 0$, $c_2 \in \mathbb{R}$, for all $u \in U$.

(H7) $g : [0,T]\times H \to \mathbb{R}^+$ is measurable in t, and for every $\delta > 0$, there exists $L_\delta > 0$ independent of t such that $g(t,0) \in L^\infty(0,T)$ and

$$\begin{gathered}|g(t,y_1) - g(t,y_2)| \le L_\delta|y_1 - y_2|_H \\ \forall t \in [0,T], \\ |y_1|_H + |y_2|_H \le \delta.\end{gathered} \tag{9}$$

Remark 1. Note that, by (H3), system (1) is equivalent to

$$\begin{gathered}\frac{dMy}{dt} + Ay + \partial\phi(y(t)) \ni Bu \quad \text{a.e } t \in (0,T), \\ y(0) = y_0.\end{gathered} \tag{10}$$

As we know, by Barbu [4] (see Chapter 4) and Theorem 1.1 of [5], we have the following.

Lemma 2. *Let (H1)–(H4) hold. Then, for any $y_0 \in D(M)\cap V$, $u \in L^2(0,T;U)$, (1) admits a unique solution $y(x,t)$ satisfying*

$$y \in W^{1,2}([0,T];H) \cap L^2(0,T;D(M)\cap V) \cap C([0,T];H). \tag{11}$$

Now we formulate the optimal control problems as follows.

Let $A_{ad} = \{(y,u) \in W^{1,2}([0,T];H) \cap C([0,T];H) \cap L^2(0,T;D(M)) \times L^2(0,T;U) \mid y$ is the solution of (10) with (2)$\}$.

We will find

$$\min \quad L(y,u) \text{ over } (y,u) \in A_{ad}. \tag{P}$$

Recently, some optimal control problems governed by pseudoparabolic equations have already been discussed. Linear optimal control problems for pseudoparabolic equations were considered by many authors (cf. [6–12]). However, these problems studied in [7–12] do not involve state constraints and maximal monotone graph. On the other hand, optimal control problems governed by some parabolic variational inequalities (cf. [4, 13–19]) have already been discussed. Li and Yong [1] studied the maximal principle for optimal control governed by some nonlinear parabolic equations with two point boundary (time variable) state constraints. In Cases' work [20], the state constraint was considered, but the state equation did not involve monotone graph. He [21] studied the optimal control problems involving some special maximal monotone graph (Lipschitz continuous) with state constraint. Wang [2, 3] also discussed the optimal control problem governed by the state equation involving some maximal monotone graph.

The present work in this paper considers the optimal control problem governed by the pseudoparabolic equations which is different from what they discussed in [7–9, 12], with the state constraints which is similar to those in [3, 4, 21].

The plan of this paper is as follows. Section 2 gives an approximating control process. In Section 3, we state and prove the necessary conditions on optimality for the problem (P). In Section 4, some examples are given.

2. The Approximating Control Process

Let (y^*, u^*) be optimal for the problem (P). Then

$$\begin{gathered}\frac{dMy^*}{dt} + Ay^* + \partial\phi(y^*(t)) \ni Bu^* \quad \text{a.e } t \in (0,T), \\ y^*(0) = y_0,\end{gathered} \tag{12}$$

with

$$\begin{gathered}F(y^*) \in S, \\ L(y^*, u^*) = \inf\ L(y,u) \quad \text{over } (y,u) \in A_{ad}.\end{gathered} \tag{13}$$

From a perturbation theorem for m-accretive operators ([22], Lemma 5) and (H2), (H3), we easily know that $C(= A + \beta)$ is m-accretive in H.

Now consider the following approximating equation:

$$\begin{gathered}\frac{dMy}{dt} + C_\epsilon J_\epsilon^M y = Bu \quad \text{a.e in } (0,T), \\ y(0) = y_0,\end{gathered} \tag{14}$$

where $C_\epsilon = \epsilon^{-1}(I - J_\epsilon^C)$ and $J_\epsilon^C = (I+\epsilon C)^{-1}$. By Lemma 2, for any $y_0 \in D(M)\cap V$, $u \in L^2(0,T;U)$, (14) has a unique solution in $W^{1,2}((0,T];H) \cap C([0,T];H) \cap L^2(0,T;V)$.

Besides, we have the following result on (14).

Lemma 3. *For $\epsilon > 0$ given, let $u_n \in L^2(0,T;U)$, $u_n \to \tilde{u}$ weakly in $L^2(0,T;U)$, and $\tilde{y}, y_n$ the solutions of (14) corresponding to $\tilde{u}$ and u_n, respectively. Then, there exists some subsequence of $\{y_n\}$, still denoted by itself, such that $y_n \to \tilde{y}$ strongly in $C([0,T];H)\cap L^2(0,T;V)$.*

Proof. Multiplying (14) by $J_\epsilon^M y_n(t)$ and using the self-adjointness of M, we see the following:

$$\frac{d}{dt}\left|M_\epsilon^{1/2} y_n(t)\right|^2 + 2\left(C_\epsilon J_\epsilon^M y_n(t), J_\epsilon^M y_n(t)\right) = 2\left(Bu_n, J_\epsilon^M y_n(t)\right). \tag{15}$$

Then (H1)–(H3) yield

$$\frac{d}{dt}\left|M_\epsilon^{1/2} y_n(t)\right|^2 \le c\left|M_\epsilon^{1/2} y_n(t)\right|^2 + c\left|Bu_n\right|^2, \tag{16}$$

where $M_\epsilon = \epsilon^{-1}(I - J_\epsilon^M)$. Integrating the above inequality from 0 to $t (t \in (0,T])$ and using Gronwall's inequality, we see the following:

$$\left|M_\epsilon^{1/2} y_n(t)\right|^2 \le c \quad \forall t \in [0,T]. \tag{17}$$

Note that from (H1), M has a bounded inverse operator on H and

$$a\left|J_\epsilon^M y_n(t)\right|^2 \le \left|M^{1/2} J_\epsilon^M y_n(t)\right|^2 \le \left|M_\epsilon^{1/2} y_n(t)\right|^2. \tag{18}$$

Together (17) and (18), we have the following:

$$\left|J_\epsilon^M y_n(t)\right|^2 \le c \quad \forall t \in [0,T]. \tag{19}$$

Since $|v|^2 = \epsilon(M_\epsilon v, v) + (J_\epsilon^M v, v)$ for every $v \in H$, taking into account (17) and (19), we have the following:

$$\left|y_n(t)\right|^2 \le c \quad \forall t \in [0,T]. \tag{20}$$

Multiplying (14) by $M_\epsilon J_\epsilon^M y_n(t)$, we see

$$\frac{d}{dt}\left|M_\epsilon y_n(t)\right|^2 + 2\left(C_\epsilon J_\epsilon^M y_n(t), M_\epsilon J_\epsilon^M y_n(t)\right) = 2\left(Bu_n, M_\epsilon J_\epsilon^M y_n(t)\right). \tag{21}$$

Then we get the following:

$$\frac{d}{dt}\left|M_\epsilon y_n(t)\right|^2 \le c\left|M_\epsilon y_n(t)\right|^2 + c\left|u_n\right|_U^2. \tag{22}$$

Applying Gronwall's inequality to the above inequality and noting that $\{u_n\}$ is bounded, we have the following:

$$\left|M_\epsilon y_n(t)\right| \le c \quad \forall t \in [0,T]. \tag{23}$$

From (H2), (H3) and (18), we see

$$\left|C_\epsilon J_\epsilon^M y_n(t)\right| \le c \quad \forall t \in [0,T]. \tag{24}$$

Then in view of (14), (24) gives

$$\left|M\frac{d}{dt} y_n(t)\right|^2 \le c + c\left|u_n\right|_U^2; \tag{25}$$

thus we see

$$\left|M\frac{d}{dt} y_n(t)\right|_T \le c, \tag{26}$$

which implies

$$\left|My_n(t)\right| \le c \quad \forall t \in [0,T], \qquad \left|\frac{d}{dt} y_n(t)\right|_T \le c. \tag{27}$$

Here, $|\cdot|_T$ is the norm in $L^2(0,T;H)$. For every m, $n > 0$

$$\frac{d}{dt}\left|M^{1/2}(y_m - y_n)\right|^2 + 2\left(C_\epsilon J_\epsilon^M (y_m - y_n), (y_m - y_n)\right) \le \left|u_m - u_n\right|_U^2 + \left|y_m - y_n\right|^2. \tag{28}$$

By some calculation, we see

$$\left|M^{1/2}(y_m - y_n)\right|^2 \le c\left|u_m - u_n\right|_T^2 \quad \forall t \in [0,T]. \tag{29}$$

Hence $\{M^{1/2} y_n\}$ and $\{y_n\}$ are Cauchy sequences in $C([0,T];H)$. Note that (H2); then there exists a function $\tilde{y} \in C([0,T]; D(M^{1/2}))$ such that as $n \to \infty$

$$y_n \longrightarrow \tilde{y} \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V),$$
$$M^{1/2} y_n \longrightarrow M^{1/2}\tilde{y} \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V). \tag{30}$$

This completes the proof. □

Next, we define the approximation g^ϵ of g and h^ϵ of h as follows. For the details, we refer to [2–4]. Let

$$g^\epsilon(t,y) = \int_{R^N} g\left(t, P_N y(s) - \epsilon\Lambda_N s\right)\rho(s)\,ds, \quad \epsilon > 0. \tag{31}$$

Here, ρ is a mollifier in R^N, $N = [\epsilon^{-1}]$. $P_N : L^2 \to X_N$ is the projection of $L^2(\Omega)$ on X_N, which is the finite dimensional space generated by $\{e_i\}_{i=1}^N$, where $\{e_i\}_{i=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$. $\Lambda_N : R^N \to X_N$ is the operator defined by $\Lambda_N(s) = \sum_{i=1}^N s_i e_i$, $s = (s_1, \ldots, s_N)$.

We define $h_\epsilon : U \to \mathbb{R}$:

$$h_\epsilon(y) = \inf\left\{\frac{\|y - x\|_U^2}{2\epsilon} + h(x) : x \in L^2(0,T;U)\right\}, \quad \epsilon > 0. \tag{32}$$

Now we define the penalty $L_\epsilon : L^2(0,T;U) \to \mathbb{R}$ by

$$L_\epsilon(u) = \int_0^T \left[g^\epsilon(t, y_\epsilon(t)) + h_\epsilon(u)\right] dt + \frac{1}{2}\left|u - u^*\right|_{L^2(0,T;U)}^2 + \frac{1}{2\epsilon^{1/2}}\left[\epsilon^{1/2} + d_S\left(F\left(y_\epsilon(t)\right)\right)\right]^2, \tag{33}$$

where y_ϵ is the solution of (14). $d_S(F(y_\epsilon))$ denotes the distance of $F(y_\epsilon)$ to S.

The approximating optimal control problems are as follows:

$$\text{Minimize} \quad L_\epsilon(u) \quad \text{over } u \in L^2(0,T;U). \tag{P$^\epsilon$}$$

From Lemma 3, we easily show the following existence of the optimal solutions for (P^ϵ) (see [2, 3]).

Theorem 4. (P^{ϵ}) *has at least one optimal solution.*

The following results are useful in discussing the approximating control problems.

Lemma 5. *Let* $u_{\epsilon} \to u$ *weakly in* $L^2(0,T;U)$ *as* $\epsilon \to 0$. *Then there exists a subsequence* $\{y_{\epsilon}\}$, *still denoted itself*

$$y_{\epsilon} \longrightarrow y \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V), \tag{34}$$

as $\epsilon \to 0$, *where* y_{ϵ} *is the solutions of* (14) *corresponding to* u_{ϵ} *and* y *is the solutions of* (10) *corresponding to* u.

Proof. Rewrite (14) as follows:

$$\begin{gathered}\frac{dMy_{\epsilon}(t)}{dt} + C_{\epsilon}J_{\epsilon}^{M}y_{\epsilon}(t) = Bu_{\epsilon}(t) \quad \text{a.e in } (0,T), \\ y_{\epsilon}(0) = y_0.\end{gathered} \tag{35}$$

Multiplying (35) by $J_{\epsilon}^{M}y_{\epsilon}(t)$, we see

$$\begin{aligned}\frac{d}{dt}\left|M_{\epsilon}^{1/2}y_{\epsilon}(t)\right|^2 &+ 2\left(C_{\epsilon}J_{\epsilon}^{M}y_{\epsilon}(t), J_{\epsilon}^{M}y_{\epsilon}(t)\right) \\ &= 2\left(Bu_{\epsilon}, J_{\epsilon}^{M}y_{\epsilon}(t)\right).\end{aligned} \tag{36}$$

Then, (H1)–(H3) yield

$$\frac{d}{dt}\left|M_{\epsilon}^{1/2}y_{\epsilon}(t)\right|^2 \le c\left|M_{\epsilon}^{1/2}y_{\epsilon}(t)\right|^2 + c\left|Bu_{\epsilon}\right|^2. \tag{37}$$

Integrating the above inequality from 0 to $t (t \in (0,T])$ and using Gronwall's inequality, we have the following:

$$\left|M_{\epsilon}^{1/2}y_{\epsilon}(t)\right|^2 \le c \quad \forall t \in [0,T], \tag{38}$$

together with (18) implies

$$\left|J_{\epsilon}^{M}y_{\epsilon}(t)\right|^2 \le c \quad \forall t \in [0,T]. \tag{39}$$

Since $|v|^2 = \epsilon(M_{\epsilon}v, v) + (J_{\epsilon}^{M}v, v)$ for every $v \in H$, taking into account (36) (39), we see

$$\left|y_{\epsilon}(t)\right|^2 \le c \quad \forall t \in [0,T]. \tag{40}$$

Multiplying (35) by $M_{\epsilon}J_{\epsilon}^{M}y_{\epsilon}(t)$, we see

$$\begin{aligned}\frac{d}{dt}\left|M_{\epsilon}y_{\epsilon}(t)\right|^2 &+ 2\left(C_{\epsilon}J_{\epsilon}^{M}y_{\epsilon}(t), M_{\epsilon}J_{\epsilon}^{M}y_{\epsilon}(t)\right) \\ &= 2\left(Bu_{\epsilon}, M_{\epsilon}J_{\epsilon}^{M}y_{\epsilon}(t)\right).\end{aligned} \tag{41}$$

Then we get the following:

$$\frac{d}{dt}\left|M_{\epsilon}y_{\epsilon}(t)\right|^2 \le c\left|M_{\epsilon}y_{\epsilon}(t)\right|^2 + c\left|u_{\epsilon}\right|_U^2, \tag{42}$$

from which it follows that

$$\left|M_{\epsilon}y_{\epsilon}(t)\right| \le c \quad \forall t \in [0,T]. \tag{43}$$

From (H2), (H3), and (18), we see

$$\left|C_{\epsilon}J_{\epsilon}^{M}y_{\epsilon}(t)\right| \le c. \tag{44}$$

Then in view of (14) and (24) give

$$\left|M\frac{d}{dt}y_{\epsilon}(t)\right|^2 \le c + c\left|u_n\right|_U^2, \tag{45}$$

Thus, we see

$$\left|M\frac{d}{dt}y_{\epsilon}(t)\right|_T \le c, \tag{46}$$

which implies

$$\left|My_{\epsilon}(t)\right| \le c \quad \forall t \in [0,T], \tag{47}$$

$$\left|\frac{d}{dt}y_{\epsilon}(t)\right|_T \le c. \tag{48}$$

For every $m, n > 0$,

$$\begin{aligned}&\frac{d}{dt}\left|M^{1/2}\left(y_{\epsilon_m} - y_{\epsilon_n}\right)\right|^2 \\ &\qquad + 2\left(C_{\epsilon_m}J_{\epsilon_m}^{M}y_{\epsilon_m} - C_{\epsilon_n}J_{\epsilon_n}^{M}y_{\epsilon_n}, y_{\epsilon_m} - y_{\epsilon_n}\right) \\ &\quad \le \left|u_{\epsilon_m} - u_{\epsilon_n}\right|_U^2 + \left|y_{\epsilon_m} - y_{\epsilon_n}\right|^2,\end{aligned} \tag{49}$$

Using the identities $w = J_{\epsilon_m}^{M}w + \epsilon_m M_{\epsilon_m}w$ for every $w \in H$, and so forth, we see

$$\begin{aligned}&\left(C_{\epsilon_m}J_{\epsilon_m}^{M}y_{\epsilon_m} - C_{\epsilon_n}J_{\epsilon_n}^{M}y_{\epsilon_n}, y_{\epsilon_m} - y_{\epsilon_n}\right) \\ &\quad = \left(C_{\epsilon_m}J_{\epsilon_m}^{M}y_{\epsilon_m} - C_{\epsilon_n}J_{\epsilon_n}^{M}y_{\epsilon_n}, J_{\epsilon_m}^{M}y_{\epsilon_m} - J_{\epsilon_n}^{M}y_{\epsilon_n}\right) \\ &\qquad + \left(C_{\epsilon_m}J_{\epsilon_m}^{M}y_{\epsilon_m} - C_{\epsilon_n}J_{\epsilon_n}^{M}y_{\epsilon_n}, \epsilon_m M_{\epsilon_m}y_{\epsilon_m} - \epsilon_n M_{\epsilon_n}y_{\epsilon_n}\right) \\ &\quad \ge \left(C_{\epsilon_m}J_{\epsilon_m}^{M}y_{\epsilon_m} - C_{\epsilon_n}J_{\epsilon_n}^{M}y_{\epsilon_n}, \epsilon_m C_{\epsilon_m}J_{\epsilon_m}^{M}y_{\epsilon_m} - \epsilon_n C_{\epsilon_n}J_{\epsilon_n}^{M}y_{\epsilon_n}\right) \\ &\qquad + \left(C_{\epsilon_m}J_{\epsilon_m}^{M}y_{\epsilon_m} - C_{\epsilon_n}J_{\epsilon_n}^{M}y_{\epsilon_n}, \epsilon_m M_{\epsilon_m}y_{\epsilon_m} - \epsilon_n M_{\epsilon_n}y_{\epsilon_n}\right) \\ &\quad \ge -c\left(\epsilon_m + \epsilon_n\right).\end{aligned} \tag{50}$$

Because of (43) and (44), we obtain the following:

$$\begin{aligned}&\left|\frac{d}{dt}M^{1/2}\left(y_{\epsilon_m} - y_{\epsilon_n}\right)\right|^2 \\ &\qquad \le c\left|M^{1/2}\left(y_{\epsilon_m} - y_{\epsilon_n}\right)\right|^2 + c\left|u_{\epsilon_m} - u_{\epsilon_n}\right|_U^2 \\ &\qquad\quad + c\left(\epsilon_m + \epsilon_n\right),\end{aligned} \tag{51}$$

where c is a constant independent of m and n. Then Gronwall's inequality yields

$$\begin{aligned}\left|M^{1/2}\left(y_{\epsilon_m} - y_{\epsilon_n}\right)\right|^2 \le c\left\{\left|u_{\epsilon_m} - u_{\epsilon_n}\right|_T^2 + \left(\epsilon_m + \epsilon_n\right)\right\} \\ \forall t \in [0,T].\end{aligned} \tag{52}$$

Hence, $\{M^{1/2}y_{\epsilon_n}\}$ and $\{y_{\epsilon_n}\}$ are Cauchy sequences in $C([0,T];H)$. Note that (H2); then there exists a function $y \in C([0,T];D(M^{1/2})) \cap L^2(0,T;V)$ such that as $n \to \infty$, $\epsilon_n \to 0$,

$$y_{\epsilon_n} \longrightarrow y \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V), \qquad M^{1/2}y_{\epsilon_n} \longrightarrow M^{1/2}y \quad \text{strongly in } C([0,T];H). \tag{53}$$

Thus, we deduce that as $\epsilon_n \to 0$,

$$J^M_{\epsilon_n} y_{\epsilon_n} \longrightarrow y \quad \text{strongly in } C([0,T];H), \tag{54}$$

Note that

$$M^{1/2}J^M_{\epsilon_n} y_{\epsilon_n} \longrightarrow M^{1/2}y \quad \text{strongly in } C([0,T];H). \tag{55}$$

Indeed, we see

$$\begin{aligned}\left|M^{1/2}J^M_{\epsilon_n}y_{\epsilon_n} - M^{1/2}y\right|^2 &\le 2\left|M^{1/2}\left(J^M_{\epsilon_n}y_{\epsilon_n} - y_{\epsilon_n}\right)\right|^2 + \left|M^{1/2}\left(y_{\epsilon_n} - y\right)\right|^2 \\ &= -2\epsilon_n\left(M_{\epsilon_n}y_{\epsilon_n}, M_{\epsilon_n}y_{\epsilon_n} - My_{\epsilon_n}\right) + \left|M^{1/2}\left(y_{\epsilon_n} - y\right)\right|^2 \\ &\le c\epsilon_n + \left|M^{1/2}\left(y_{\epsilon_n} - y\right)\right|^2 \longrightarrow 0,\end{aligned} \tag{56}$$

for all $t \in [0,T]$. From (43) and (46), $\{M_{\epsilon_n}y_{\epsilon_n}\}$ is uniformly bounded and equicontinuous in $C([0,T];H)$. Hence the Ascoli-Arzela theorem gives that as $\epsilon_n \to 0$, for every $v \in H$, $(M_{\epsilon_n}y_{\epsilon_n}, v) \to (My, v)$ strongly in $C([0,T])$. In virtue of (46) and (48), weak closedness of d/dt, and M, it is shown that

$$\frac{dy_{\epsilon_n}}{dt} \longrightarrow \frac{dy}{dt} \quad \text{weakly in } L^2(0,T;H), \qquad \frac{Mdy_{\epsilon_n}}{dt} \longrightarrow \frac{Mdy}{dt} \quad \text{weakly in } L^2(0,T;H). \tag{57}$$

Therefore, $y \in AC([0,T];D(M))$ and $dy/dt \in L^2(0,T;D(M))$. By $AC([0,T];H])$, we denote the space of all H-valued strongly absolutely continuous functions on $[0,T]$. We easily get that $y(t) \in D(C)$ a.e. $t \in (0,T)$ and there exists a function $\xi \in L^\infty(0,T;H)$ such that as $\epsilon_n \to 0$,

$$C_{\epsilon_n}J^M_{\epsilon_n}y_{\epsilon_n} \longrightarrow \xi \quad \text{weakly star in } L^\infty(0,T;H), \tag{58}$$

and $\xi(t) \in Cy = Ay + \beta(y)$ a.e. $t \in (0,T)$. Thus, letting $\epsilon_n \to 0$ in (35), we see

$$\frac{dMy(t)}{dt} + Ay + \xi(t) = Bu(t) \quad \text{a.e in } (0,T), \qquad y(0) = y_0. \tag{59}$$

□

Lemma 6. *Let $y_0 \in D(M) \cap V$, $u \in L^2(0,T;U)$; then $y_\epsilon \to y$ strongly in $C([0,T];H)$ as $\epsilon \to 0$, where y_ϵ is the solutions of* (14) *corresponding to u and y is the solutions of* (1) *corresponding to u with the initial condition $y(0) = y_0$. Furthermore,*

$$|y_\epsilon - y|_{C([0,T];H)} \le c\epsilon^{1/2}. \tag{60}$$

Proof. By the same argument in the proof of Lemma 5, we have the following:

$$y_\epsilon \longrightarrow y \quad \text{strongly in } C([0,T];H) \cap L^2(0,T;V). \tag{61}$$

We have for all ϵ and λ,

$$\frac{dM(y_\epsilon(t) - y_\lambda(t))}{dt} + C_\epsilon J^M_\epsilon y_\epsilon(t) - C_\lambda J^M_\lambda y_\lambda(t) = 0 \quad \text{a.e in } (0,T), \qquad y_\epsilon(0) - y_\lambda(0) = 0. \tag{62}$$

Multiplying (62) by $y_\epsilon(t) - y_\lambda(t)$, we have

$$\frac{d\left|M^{1/2}(y_\epsilon(t) - y_\lambda(t))\right|^2}{dt} + 2\left(C_\epsilon J^M_\epsilon y_\epsilon(t) - C_\lambda J^M_\lambda y_\lambda(t), y_\epsilon(t) - y_\lambda(t)\right) = 0. \tag{63}$$

Using the identities $w = J^M_\epsilon w + \epsilon M_\epsilon w$ for every $w \in H$, and so forth, we get the following:

$$\begin{aligned}&\left(C_\epsilon J^M_\epsilon y_\epsilon - C_\lambda J^M_\lambda y_\lambda, y_\epsilon - y_\lambda\right) \\ &\quad = \left(C_\epsilon J^M_\epsilon y_\epsilon - C_\lambda J^M_\lambda y_\lambda, J^M_\epsilon y_\epsilon - J^M_\lambda y_\lambda\right) \\ &\qquad + \left(C_\epsilon J^M_\epsilon y_\epsilon - C_\lambda J^M_\lambda y_\lambda, \epsilon M_\epsilon y_\epsilon - \lambda M_\lambda y_\lambda\right) \\ &\quad \ge -c(\epsilon + \lambda).\end{aligned} \tag{64}$$

Thus, we see

$$\frac{d\left|M^{1/2}(y_\epsilon(t) - y_\lambda(t))\right|^2}{dt} \le c(\epsilon + \lambda); \tag{65}$$

then

$$\left|M^{1/2}(y_\epsilon(t) - y_\lambda(t))\right|^2_{C([0,T];H)} \le c(\epsilon + \lambda). \tag{66}$$

Because of (61), letting $\lambda \to 0$ in (66), we get (60). □

Lemma 7. *Let u_ϵ be optimal for the problem (P^ϵ) and y_ϵ be the solution of* (14) *corresponding to u_ϵ. For $\epsilon \to 0$, then*

$$y_\epsilon \longrightarrow y_* \quad \textit{strongly in } C([0,T];H) \cap L^2(0,T;V), \qquad u_\epsilon \longrightarrow u_* \quad \textit{strongly in } L^2(0,T;U). \tag{67}$$

Proof. For any $\epsilon > 0$, we have the following:

$$\begin{aligned}L_\epsilon(u_\epsilon) &\le L_\epsilon(u_*) \\ &= \int_0^T \left[g^\epsilon(t, y_\epsilon(t)) + h_\epsilon(u_*(t))\right] dt \\ &\quad + \frac{1}{2\epsilon^{1/2}}\left[\epsilon^{1/2} + d_S(F(y_\epsilon(t)))\right]^2.\end{aligned} \tag{68}$$

By Lemma 5, we know $y_\epsilon \to y_*$ strongly in $C([0,T];H)$. So we have the following:

$$g^\epsilon(t,y_\epsilon) \longrightarrow g(t,y_*) \quad \forall t\in[0,T], \qquad h_\epsilon(u_\epsilon)\longrightarrow h(u_*). \tag{69}$$

So

$$\lim_{\epsilon\to 0}\int_0^T g^\epsilon(t,y_\epsilon(t))\,dt = \int_0^T g(t,y_*(t))\,dt, \qquad \lim_{\epsilon\to 0}\int_0^T h_\epsilon(u_*(t))\,dt = \int_0^T h(u_*(t))\,dt. \tag{70}$$

Similarly, by (60) and (H5), we obtain the following:

$$\frac{1}{2\epsilon^{1/2}}\left[\epsilon^{1/2}+d_S(F(y_\epsilon))\right]^2 \le \frac{1}{2\epsilon^{1/2}}\left[\epsilon^{1/2}+\|F(y_\epsilon)-F(y_*)\|_Z\right]^2 \le c\epsilon \longrightarrow 0 \quad \text{as } \epsilon\longrightarrow 0. \tag{71}$$

Then, we get the following:

$$\limsup_{\epsilon\to 0} L_\epsilon(u_\epsilon)\le L(u_*). \tag{72}$$

On the other hand, since $\{u_\epsilon\}$ is bounded in $L^2(0,T;U)$, there exists $u_1\in L^2(0,T;U)$ such that, on some subsequence ϵ, still denoted by itself, as $\epsilon\to 0$,

$$u_\epsilon \longrightarrow u_1 \quad \text{weakly in } L^2(0,T;U), \tag{73}$$

and so, by Lemma 5,

$$y_\epsilon\longrightarrow y_1 = y(u_1) \quad \text{strongly in } C(0,T;H)\cap L^2(0,T;V). \tag{74}$$

By (66), one can check easily that

$$\frac{1}{2\epsilon^{1/2}}\left[\epsilon^{1/2}+d_S(F(y_\epsilon))\right]^2\le c. \tag{75}$$

Thus, $d_S(F(y_\epsilon))\to 0$ as $\epsilon\to 0$. Since S is closed and convex, $F(y_1)=\lim_{\epsilon\to 0}F(y_\epsilon)\in S$. Since the function $u\to\int_0^T h(u)dt$ is weakly lower semicontinuous on $L^2(0,T;U)$, we see

$$\liminf_{\epsilon\to 0} L_\epsilon(u_\epsilon)\ge L(u_1)\ge L(u_*). \tag{76}$$

Together with (72), we obtain

$$\lim_{\epsilon\to 0} L_\epsilon(u_\epsilon) = L(u_*). \tag{77}$$

Therefore,

$$\lim_{\epsilon\to 0}\int_0^T |u_\epsilon-u_*|_U^2\,dt = 0. \tag{78}$$

Hence, $y_1=y_*$, $u_1=u_*$. This completes the proof. □

3. Necessary Condition on Optimality

Let ∂g the generalized gradient of $y\to g(t,y)$. Let $Y^*=(H^s(\Omega))'+V'$ which is the dual of $Y=H^s(\Omega)\cap V$ with $s>N/2$.

Firstly, we consider the following Cauchy problem:

$$\frac{dMp_\epsilon}{dt}-Ap_\epsilon-\dot\beta^\epsilon(y_\epsilon)p_\epsilon-\left[F'(y_\epsilon)\right]^*\xi_\epsilon = \lambda_\epsilon\nabla g^\epsilon(t,y_\epsilon) \quad \text{in } (0.T), \qquad p_\epsilon(T)=0, \tag{79}$$

where $\dot\beta^\epsilon=(\beta^\epsilon)'$, $\beta_\epsilon=\epsilon^{-1}(I-(I+\epsilon\beta)^{-1})$, $\beta^\epsilon=\int_{-\infty}^{\infty}[\beta_\epsilon(r-\epsilon^2\theta)-\beta_\epsilon(-\epsilon^2\theta)]\rho(\theta)d\theta+\beta_\epsilon(0)$, and ρ is a C_0^∞-mollifier on $\mathbb{R}$.

Lemma 8. *Problem (79) has a unique absolutely continuous function $p_\epsilon\in L^2(0,T;V)\cap C([0,T];H)$ with $p_\epsilon'\in L^2(0,T;V')$, such that*

$$|p_\epsilon(t)|_2^2+\int_0^T\|p_\epsilon(t)\|_V^2dt\le c \quad \forall\epsilon>0,\ t\in[0,T], \tag{80}$$

$$\int_Q\left|p_\epsilon\dot\beta^\epsilon(y_\epsilon)\right|dx\,dt\le c\quad\forall\epsilon>0. \tag{81}$$

Proof. From (H1)–(H3) and $\dot\beta^\epsilon(y_\epsilon)\ge 0$, it is seen that $C=M^{-1}(A+\dot\beta^\epsilon(y_\epsilon)):V\to V'$ is demicontinuous monotone operator that satisfies

$$(C\omega,\omega)\ge w\|\omega\|^p+c\quad\forall\omega\in V, \qquad \|C\omega\|_*\le c\left(1+\|\omega\|^{p-1}\right), \tag{82}$$

where $w>0$ and $p\ge 2$. It follows by Theorem 1.9′ of [4] that (79) has a unique solution $p_\epsilon\in L^2(0,T;V)\cap C([0,T];H)$ with $p_\epsilon'\in L^2(0,T;V')$. Multiplying (79) by $J_\epsilon^M p_\epsilon(t)$ and using the self-adjointness of M and integrating over $[t,T]$, we see

$$\left|M_\epsilon^{1/2}p_\epsilon(t)\right|^2+w\int_t^T\|p_\epsilon(s)\|_V^2\,ds\le c\int_t^T\left|M_\epsilon^{1/2}p_\epsilon(s)\right|^2ds+c, \tag{83}$$

Because of $a|J_\epsilon^M p_\epsilon(t)|^2\le|M^{1/2}J_\epsilon^M p_\epsilon(t)|^2\le|M_\epsilon^{1/2}p_\epsilon(t)|^2$, $|\lambda_\epsilon\nabla g^\epsilon(t,\tilde y_n)|_{L^\infty(0,T;H)}\le c$ and $|[F'(y_\epsilon)]^*\xi_\epsilon|_{L^2(0,T;V')}\le c$. And so by Gronwall's lemma we obtain the following:

$$\left|M_\epsilon^{1/2}p_\epsilon(t)\right|_2^2+\int_0^T\|p_\epsilon(s)\|_V^2ds\le c\quad\forall t\in[0,T]. \tag{84}$$

Combining the above equalities, we see

$$\left|J_\epsilon^M p_\epsilon(t)\right|_2^2\le c\quad\forall t\in[0,T]. \tag{85}$$

Since $w=J_{\epsilon_m}^M w+\epsilon_m(M_{\epsilon_m}w,w)$ for every $w\in H$, taking into account the above equalities, we have the following:

$$|p_\epsilon(t)|_2^2\le c\quad\forall t\in[0,T]. \tag{86}$$

Thus, we obtain (80).

Multiplying (79) by $\zeta(p_\epsilon)$ and integrate on Q, where ζ is a smooth monotonically increasing approximation of the sign function such that $\zeta(0) = 0$. For instance

$$\zeta = \zeta_\lambda(r) = \int_{-\infty}^{\infty} (\zeta_\lambda(r - \lambda\theta) - \zeta_\lambda(-\lambda\theta))\,\rho(\theta)\,d\theta, \quad (87)$$

where $\zeta_\lambda(r) = r|r|^{-1}$ for $|r| \ge \lambda, \zeta_\lambda(r) = \lambda^{-1}r$ for $|r| < \lambda$, and ρ is a C_0^∞-mollifier. Then $(Ap_\epsilon(t), \zeta(p_\epsilon(t))) \ge 0$; therefore,

$$\int_Q \dot{\beta}^\epsilon(y_\epsilon)\,\zeta(p_\epsilon)\,p_\epsilon\,dx\,dt \le \int_Q \left|\nabla_y g^\epsilon(t, y_\epsilon)\,\zeta(p_\epsilon)\right| dx\,dt, \quad \forall\epsilon > 0. \quad (88)$$

Then, letting ζ tend to the sign function, we get (81). □

We state the main results of the necessary conditions on optimality as follows.

Theorem 9. *Suppose that* (*H*1)–(*H*7) *hold. Let* (y_*, u_*) *be an optimal pair of problem* (*P*). *Then, there exists function* $p \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap BV([0, T]; Y^*)$, *a measure* $\mu \in (L^\infty(Q))'$, $\lambda_0 \in \mathbb{R}, \xi_0 \in \mathbb{Z}^*$ *satisfying*

$$\begin{gathered}\frac{d}{dt}Mp - Ap - \mu - \left[F'(y_*)\right]^*\xi_0 \in L^\infty(0, T; H), \\ \frac{d}{dt}Mp(t) - Ap(t) - \mu - \left[F'(y_*)\right]^* \\ \times \xi_0 \in \lambda_0 \partial g(t, y_*) \quad a.e.in\ (0, T), \\ p(T) = 0, \\ \langle \xi_0, w - F(y_*)\rangle \le 0 \quad \forall w \in S, \\ B^* p \in \lambda_0 \partial h(u_*(t)), \quad a.e.\, t \in (0, T), \\ (\lambda_0, \xi_0) \ne 0.\end{gathered} \quad (89)$$

Proof. Since (y_ϵ, u_ϵ) is optimal for problem (P^ϵ), we see

$$L_\epsilon(u_\epsilon^\rho) \ge L_\epsilon(u_\epsilon) \quad \text{for any } \rho > 0, v \in L^2(0, T; V). \quad (90)$$

Here $u_\epsilon^\rho = u_\epsilon + \rho v$. Thus,

$$\frac{L_\epsilon(u_\epsilon^\rho) - L_\epsilon(u_\epsilon)}{\rho} \ge 0. \quad (91)$$

By some calculation, we have the following:

$$\lim_{\rho\to 0}\int_0^T \frac{g^\epsilon(t, y_\epsilon^\rho(t)) - g^\epsilon(t, y_\epsilon(t))}{\rho} = \int_0^T (\nabla g^\epsilon(t, y_\epsilon(t)), z_\epsilon)\,dt, \quad (92)$$

where $z_\epsilon \in C([0, T]; H) \cap L^2(0, T; V) \cap W^{1,2}([0, T]; H)$ is the following solution to the linear equation

$$\frac{dMz}{dt} + Az + \dot{\beta}^\epsilon(y_\epsilon)\,z = Bv \quad \text{in } (0, T), \qquad z(0) = 0. \quad (93)$$

Hence, we also have the following:

$$\lambda_\epsilon\left[\int_0^T \langle \nabla g^\epsilon(t, y_\epsilon), z_\epsilon\rangle\,dt + \int_0^T \langle \nabla h_\epsilon(u_\epsilon), v\rangle\,dt\right] + \langle \xi_\epsilon, F'(y_\epsilon)\,z_\epsilon\rangle \ge \int_0^T \langle u_* - u_\epsilon, v\rangle dt, \quad (94)$$

where

$$\lambda_\epsilon = \frac{\epsilon^{1/2}}{d_S(F(y_\epsilon)) + \epsilon^{1/2}}, \qquad \xi_\epsilon = \begin{cases}\nabla d_S(F(y^\epsilon)), & \text{if } F(y^\epsilon) \notin S, \\ 0 & \text{otherwise,}\end{cases} \quad (95)$$

and $\xi_\epsilon \in \partial d_S(F(y^\epsilon))$. Since S is convex and closed, we see

$$\|\xi_\epsilon\|_{Z^*} = \begin{cases}1, & \text{if } F(y^\epsilon) \notin S, \\ 0 & \text{otherwise,}\end{cases} \qquad 1 \le \varphi_\epsilon^2 + \|\xi_\epsilon\|_{Z^*}^2 \le 2. \quad (96)$$

So, we see

$$\lambda_\epsilon \longrightarrow \lambda_0, \quad \xi_\epsilon \longrightarrow \xi_0 \quad \text{weakly in } Z^*. \quad (97)$$

It follows from Lemma 7 that $y_\epsilon \to y_*$ strongly in $C([0, T]; H) \cap L^2(0, T; V)$. By the same arguments as those in [2–4], there exists $p \in C([0, T]; H) \cap L^2(0, T; V) \cap BV([0, T]; Y^*)$ and $\mu \in (L^\infty(Q))^*$ such that, on some subsequence ϵ, still denoted itself

$$p_\epsilon(t) \longrightarrow p \quad \text{strongly in } Y^*, \quad \forall t \in [0, T], \quad (98)$$

where $BV([0, T]; Y^*)$ is the space of all Y^*-valued functions $p : [0, T] \to Y^*$ with bounded variation on $[0, T]$. On the other hand, by (80), we see

$$p_\epsilon \longrightarrow p \quad \text{weakly star in } L^\infty(0, T; H), \quad \text{weakly in } L^2(0, T; V). \quad (99)$$

Note that $V \hookrightarrow H$ is compact, for every $\lambda > 0$, there is $\delta(\lambda) > 0$ such that

$$|p_\epsilon(t) - p(t)|_2 \le \|p_\epsilon(t) - p(t)\|_V + \delta(\lambda)\,\|p_\epsilon(t) - p(t)\|_{Y^*} \quad \forall t \in [0, T]. \quad (100)$$

This yields

$$\begin{gathered} p_\epsilon \longrightarrow p \quad \text{strongly in } L^2(0, T; H), \\ p_\epsilon(t) \longrightarrow p(t) \quad \text{weakly in } H\ \forall t \in [0, T]. \end{gathered} \quad (101)$$

Moreover, by (81) we infer that there is $\mu \in (L^\infty(Q))^*$ such that, on some generalized subsequence ϵ,

$$\begin{gathered} \dot{\beta}^\epsilon(y_\epsilon)\,p_\epsilon \longrightarrow \mu \quad \text{weakly star in } (L^\infty(Q))^*, \\ \nabla g^\epsilon(t, y_\epsilon) \longrightarrow \eta \quad \text{weakly star in } L^\infty(0, T; H))^*, \\ \eta(t) \in \partial g(t, y_*) \quad \text{a.e. } t \in (0, T). \end{gathered} \quad (102)$$

Since F is continuously differentiable from $L^2(0,T;V)$ to Z,

$$[F'(y_\epsilon)]^* \xi_\epsilon \longrightarrow [F'(y_*)]^* \xi_0 \quad \text{weakly } L^2(0,T;V'). \tag{103}$$

Now letting $\epsilon \to 0$ in (79), it follows that

$$\frac{d}{dt}Mp - Ap - \mu - [F'(y_*)]^* \xi_0 \in L^\infty(0,T;H), \tag{104}$$

$$\begin{gathered}\frac{d}{dt}Mp(t) - Ap(t) - \mu - [F'(y_*)]^* \\ \times \xi_0 \in \lambda_0 \partial g(t, y_*) \quad \text{a.e. in } (0,T), \\ p(T) = 0.\end{gathered} \tag{105}$$

It follows from (93), (94), and (79) that

$$\begin{gathered}-\int_0^T \langle B^* p_\epsilon, v\rangle dt + \lambda_\epsilon \int_0^T \langle \nabla h_\epsilon(u_\epsilon), v\rangle dt \\ \geq \int_0^T \langle u_* - u_\epsilon, v\rangle dt, \quad \forall v \in L^2(0,T;V).\end{gathered} \tag{106}$$

By Lemma 7, $u_\epsilon \to u_*$ strongly in $L^2(0,T;U)$, it follows

$$\begin{gathered}\int_0^T \langle \nabla h_\epsilon(u_\epsilon), v\rangle dt \longrightarrow \int_0^T \langle \nabla\zeta(t), v\rangle dt, \\ \zeta(t) \in \partial h(u_*) \quad \text{a.e. in } (0,T), \ \forall v \in L^2(0,T;V).\end{gathered} \tag{107}$$

Thus,

$$\begin{gathered}-\int_0^T \langle B^* p, v\rangle dt + \lambda_0 \int_0^T \langle \zeta(t), v\rangle dt \geq 0, \\ \forall v \in L^2(0,T;V).\end{gathered} \tag{108}$$

Since $\xi_\epsilon \in d_S(F(y_\epsilon))$, we get $\langle \xi_\epsilon, w - F(y_\epsilon)\rangle \leq 0$ for all $w \in S$. Now we claim that $(\lambda_0, \xi_0) \neq 0$. Indeed, if $\lambda_0 = 0$, we have that $\{\xi_\epsilon\}$ is bounded in Z^*. By (H3), S has finite codimentionality, so dose $S - F(y^*)$. Thus, it follows that $\xi_\epsilon \to \xi_0$ weakly in Z^* and

$$\langle \xi_0, w - F(y^*)\rangle \leq 0 \quad \forall w \in S. \tag{109}$$

Finally, if $(\lambda_0, p) = 0$, it follows from (105) that $\mu + [F'(y^*)]^* \xi_0 = 0$. So in the case that $\mu \notin R([F'(y^*)]^*)$, we must have $(\lambda_0, p) \neq 0$. Together with (104), (105), and (109), we completes the proof. □

4. Some Examples

In this section, we present two examples.

Example 1. Consider the initial value controlled system

$$\begin{gathered}y_t - y_{xxt} + yy_x + \beta(y) \ni Bu \quad \text{in } (0,1) \times [0,T], \\ y(0,t) = y(1,t) = 0 \quad t \in [0,T], \\ y(x,0) = y_0 \quad \text{in } (0,1),\end{gathered} \tag{110}$$

where 1 $y = y(x,t)$ is a function on $\mathbb{R} \times [0,T]$ and $\beta(\cdot)$ is a multivalued function on $\mathbb{R}$.

If $\beta(y) = 0$, rewrite (110) in the form

$$\begin{gathered}y_t - y_{xxt} + yy_x = Bu \quad \text{in } (0,1) \times [0,T], \\ y(0,t) = y(1,t) = 0 \quad t \in [0,T], \\ y(x,0) = y_0 \quad \text{in } (0,1).\end{gathered} \tag{111}$$

(111) was introduced by Benjamin et al. [23] as an approximate equation of the propagation of one-dimensional waves of small amplitude in water. If $y_x \geq 0$, $\beta(\cdot)$ satisfies (H3). $y_0 \in H^2((0,1)) \cap H_0^1((0,1))$. Since β_ϵ is a Lipschitz continuous and monotone increasing function, integration by parts yields

$$\int_{\mathbb{R}} \beta_\epsilon(y)\left(I - \frac{d^2}{dx^2}\right) y\,dx \geq 0 \quad \text{for every } y \in H^2(\mathbb{R}). \tag{112}$$

Thus, $C(= A + \beta)$ is m-accretive in H. We easily proof the following result.

Theorem 10. *Suppose that (H1)–(H7) hold. Let (y_*, u_*) be an optimal pair of problem (P). Then there exists function $p \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap BV([0,T];Y^*)$, a measure $\mu \in (L^\infty(Q))'$ and R with λ_0, ξ_0 satisfying*

$$\begin{gathered}\left(I - \frac{d^2}{dx^2}\right)\frac{d}{dt}p - yp_x - \mu - [F'(y_*)]^* \\ \times \xi_0 \in L^\infty(0,T;H), \\ \left(I - \frac{d^2}{dx^2}\right)\frac{d}{dt}p - yp_x - \mu - [F'(y_*)]^* \\ \times \xi_0 \in \lambda_0 \partial g(t, y_*) \quad \textit{a.e.in } (0,1) \times [0,T], \\ p(0,t) = p(1,t) = 0 \quad \textit{in } [0,T], \\ p(x,T) = 0 \quad \textit{in } (0,1). \\ \langle \xi_0, w - F(y_*)\rangle \leq 0 \quad \forall w \in S, \\ B^* p \in \lambda_0 \partial h(u_*)(t), \quad \textit{a.e. } t \in (0,T), \\ (\lambda_0, \xi_0) \neq 0.\end{gathered} \tag{113}$$

Example 2. Consider the initial boundary value controlled system

$$\begin{gathered}(I - \Delta)\frac{dy(x,t)}{dt} - \Delta y(x,t) + \beta(y(x,t)) \ni Bu(x,t) \\ \text{in } \Omega \times [0,T], \\ y(x,t) = 0 \quad \text{on } \partial\Omega \times [0,T], \\ y(x,0) = y_0 \quad \text{in } \Omega,\end{gathered} \tag{114}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\beta(\cdot)$ satisfies (H3). $My = (I - \Delta)y$ with

$D(M) = H_0^1(\Omega) \cap H^2(\Omega)$, $Ay = -\Delta y$. Since β_ϵ is a monotone function,

$$\int_\Omega \beta_\epsilon(y)(-\Delta y)\,dx \geq 0, \quad \text{for every } y \in H_0^2(\Omega) \cap H^2(\Omega). \tag{115}$$

Then, $C(= A + \beta)$ is m-accretive in H. We easily obtain similar necessary condition of optimality of problem (P).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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