Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 634974, 6 pages
http://dx.doi.org/10.1155/2014/634974

Research Article

Comparison Analysis Based on the Cubic Spline Wavelet and Daubechies Wavelet of Harmonic Balance Method

Jing Gao

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

Correspondence should be addressed to Jing Gao; jgao@mail.xjtu.edu.cn

Received 25 January 2014; Revised 25 March 2014; Accepted 5 April 2014; Published 22 April 2014

Academic Editor: Eugene B. Postnikov

This paper develops a theoretical analysis of harmonic balance method, based on the cubic spline wavelet and Daubechies wavelet, for steady state analysis of nonlinear circuits under periodic excitation. The properties of the resulting Jacobian matrix for harmonic balance are analyzed. Numerical experiments illustrate the theoretical analysis.

1. Introduction

The rapid growth in integrated circuits has placed new demands on the simulation tools. Many quantities properties of circuits are of interest to circuits designer. Especially, the steady-state analysis of nonlinear circuits represents one of the most computationally challenging problems in microwave design.

Harmonic balance (HB) [1–3] is very favorable for periodic or quasiperiodic steady-state analysis of mildly nonlinear circuits using Fourier series expansion. However, the density of the resulting Jacobian matrix seriously affects the efficiency of HB based on Fourier series expansion. More effective simulators are required to study steady-state analysis. Soveiko and Nakhla in [4] have provided the elaborate formulation for HB approach applying Daubechies wavelet series instead of Fourier series and obtain the sparser Jacobian matrix to reduce the whole computational cost. And Steer and Christoffersen in [5] have given the possibility of wavelet expansion for steady-state analysis. One advantage of wavelet bases is a sparse representation matrix of operators or functions which is favorable for solving the nonlinear system by Newton iterative method. But the main disadvantage is the waste of much time in storing Jacobian matrix due to the complex computation of Daubechies wavelet functions. And few studies have been reported on the efficient wavelet matrix transform which is very important in the wavelet HB approach. The cubic spline wavelet in [6, 7] has the explicit form and sparse transform matrix and derivative matrix. In this paper, we provide the theoretical analysis for HB method by using the cubic spline wavelet and Daubechies wavelets.

The remainder of this paper is organized as follows. In Section 2, we develop the HB method based on the cubic spline wavelet and Daubechies wavelets in [4] for nonlinear circuits simulations, respectively. Section 3 provides the theoretical comparison analysis in the sparsity and computation of Jacobian matrix obtained by the transform. And it is shown that the cubic spline wavelet HB method has sparser Jacobian matrix. Numerical experiments are provided in Section 4. It is concluded in Section 5.

2. HB Formulation Based on the Cubic Spline Wavelets and Daubechies Wavelets

2.1. Generalized HB Formulation. The harmonic balance (HB) method is a powerful technique for the analysis of high-frequency nonlinear circuits such as mixers, power amplifiers, and oscillators. The basic idea of HB is to expand the unknown state variable $x(t)$ in electrical circuit equations by some series $x(t) = \sum X_k v_k(t)$. Then the problem is transformed into the frequency domain focusing on the coefficients X_k.

Let us consider the general approach of HB which assumes obtaining the solution $x(t)$ of the nonlinear modified nodal analysis (MNA) equation in [8]

$$C\dot{x} + Gx + f(x) + u = 0, \quad (1)$$

which satisfies the following periodical boundary condition:

$$x(t+L) = x(t), \tag{2}$$

where C and G are $N_x \times N_x$ matrices, x is a N_x dimensional column vector of unknown circuit variables, and u is a N_x dimensional column vector of independent sources. Let $\{v_k\}$ be the basis; then the unknown function $x(t)$ can be expanded $x(t) = \sum X_k v_k(t)$. To solve (1) with periodic boundary condition (2), assume that the expansion basis is periodic with period τ and $[x_l]$ is a discrete vector containing values of $x(t)$ sampled in the time domain at time points $[t_l]$, $l = 1, \ldots, N_t$. Then (1) can be written in the transform domain as a nonlinear algebraic equation system:

$$\Phi(X) = \left(\widehat{C}D + \widehat{G}\right) X + F(X) + U = 0, \tag{3}$$

where

$$X = Tx, \qquad x = T^{-1}X, \qquad U = Tu, \tag{4}$$

$\widehat{C}$, D, and $\widehat{G}$ are $N_t N_x \times N_t N_x$ matrices, especially, the matrix D is a representation matrix of the derivative operator d/dt in expansion basis $\{v_i\}$

$$\left[D_{i,j}\right] = \left\langle \frac{d}{dt} v_i, v_j \right\rangle, \tag{5}$$

and, finally, T and T^{-1} are the matrices associated with the forward and inverse transform arising from the chosen expansion basis. The nonlinear matrix system (3) can be solved by Newton iterative method

$$J\left(X^{(i)}\right)\left(X^{(i+1)} - X^{(i)}\right) = -\Phi\left(X^{(i)}\right), \tag{6}$$

where $X^{(i)}$ is the solution of the ith iteration and $J(X)$ is the Jacobian matrix of $\Phi(X)$

$$\begin{aligned} J(X) = \left[J_{kl}(X)\right] &= \left[\frac{\partial \Phi_k}{\partial X_l}\right] = \frac{\partial \Phi}{\partial X} \\ &= \widehat{C}D + \widehat{G} + T\left[\frac{\partial f_k}{\partial x_l}\right]T^{-1}, \\ &k, l = 1, \ldots, \left(N_t N_x\right). \end{aligned} \tag{7}$$

Hence, the sparsity of this Jacobian matrix $J(X)$ affects the computational cost of iterative method. Because these matrices $\widehat{C}$ and $\widehat{G}$ have a rather sparse structure due to the MNA formulation and $[(\partial f_k)/(\partial x_l)]$ for time-invariant systems is just a block matrix consisting of diagonal blocks, the sparsity of the Jacobian matrix $J(X)$ is determined by three matrices T, T^{-1}, and the representation matrix D of the differential operator d/dt.

Given the base $\{v_k\}_{k=1}^{N}$, the matrices D, T, and T^{-1} are constructed before those iterative methods are used. So the sparsity of the Jacobian matrix based on these different basis functions indicates how to solve the nonlinear algebraic system. Next, we give the formulation for two kinds of wavelet bases.

2.2. Description of the Periodic Daubechies Wavelets. Two functions ψ and ϕ are the wavelet function and its corresponding scaling function described by Daubechies [9]. They are defined in the frame of the wavelet theory and can be constructed with finite spatial support under the following conditions:

$$\begin{aligned} \psi(t) &= \sqrt{2}\sum_{k=0}^{M-1} g_{k+1}\phi(2t-k), \\ \phi(t) &= \sqrt{2}\sum_{k=0}^{M-1} h_{k+1}\phi(2t-k), \\ &\int_{-\infty}^{+\infty} \phi(t)\,dt = 1, \end{aligned} \tag{8}$$

where the coefficients $\{h_k\}_{k=0}^{M-1}$ and $\{g_k\}_{k=0}^{M-1}$ are the quadrature mirror filters (QMFs) of length L_M. The quadrature mirror filters $\{h_k\}$ and $\{g_k\}$ are defined

$$g_k = (-1)^k h_{M-k-1}, \quad k = 0, 1, \ldots, M-1. \tag{9}$$

The function ψ has p vanishing moments; that is,

$$\int_{-\infty}^{\infty} \psi(t)\, t^m dt = 0, \quad 0 \le m \le p-1. \tag{10}$$

The number M of the filter coefficients is related to the number of vanishing moments p, and $M = 2p$ for the wavelets constructed in [9].

We observe that once the filter $\{h_k\}$ has been chosen, the functions ϕ and ψ can be confirmed. Moreover, due to the recursive definition of the wavelet bases, via the two-scale equation, all of the manipulations are performed with the quadrature mirror filters $\{h_k\}$ and $\{g_k\}$. Especially, the wavelet transform matrix T and the derivative matrix D for the differential operator d/dt can be obtained by the filters.

In HB method the wavelets on the interval $[0, L]$ are required. Hence, periodic Daubechies wavelets on the interval $[0, L]$ are constructed by periodization. Here, we describe the discrete wavelet transform matrix by the periodic Daubechies wavelets. The discrete wavelet transform with the period $L = 2^n$ can be considered as a linear transformation taking the vector $\mathbf{f}_J \in V_J$ determined by its sampling data into the vector

$$\mathbf{d} = \left(\mathbf{c}_0, \mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \ldots, \mathbf{d}_{J-1}\right)^T, \tag{11}$$

where $\mathbf{c}_j$ stands for the scaling coefficients of the function $x(t)$ and $\mathbf{d}_j$ for the wavelet coefficients.

This linear transform can be represented by the $N = 2^n$ dimensional matrix T_{Daube} such that

$$T_{\text{Daube}}\mathbf{f}_J = \mathbf{d}. \tag{12}$$

If the level of the DWT is $J \le n$, then the DWT of the sequence has exactly 2^n coefficients. The transform matrix

T_{Daube} is composed of QMFs coefficients $\{h_k\}$ and $\{g_k\}$ as follows:

$$T_{\text{Daube}} = \begin{pmatrix} h_0 & h_1 & \cdots & h_{M-1} & 0 & \cdots & 0 & \cdots & 0 \\ g_0 & g_1 & \cdots & g_{M-1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & h_0 & h_1 & \cdots & h_{M-1} & 0 & \cdots & 0 \\ 0 & 0 & g_0 & g_1 & \cdots & g_{M-1} & 0 & \cdots & 0 \\ \cdots \\ h_2 & h_3 & \cdots & h_{M-1} & \cdots & \cdots & 0 & h_1 & h_2 \\ g_2 & g_3 & \cdots & g_{M-1} & \cdots & \cdots & 0 & g_1 & g_2 \end{pmatrix}, \tag{13}$$

where M is the length of the filters.

The periodized Daubechies wavelet HB formulation has been formulated in [4], so we have $D_{\text{Daube}} = T_{\text{Daube}} R T_{\text{Daube}}^{-1}$, where R is a band limited circulant matrix with its diagonals filled by r_m in [10], where with the following properties:

$$\begin{gathered} r_m \neq 0, \quad \text{for } -M+2 \leq m \leq M-2, \\ r_0 = 0, \qquad r_{-m} = -r_m, \qquad \sum_m m r_m = -1, \\ r_m = 2\left[r_{2m} + \frac{1}{2}\sum_{k=1}^{M/2} a_{2k-1}\left(r_{2m-2k+1} + r_{2m+2k-1}\right)\right], \end{gathered} \tag{14}$$

in which a_i are autocorrelation coefficients of the QMFs

$$a_i = 2\sum_{m=0}^{M-i-1} \widetilde{h}_m h_{m+1}, \quad i = 1, \ldots, M-1. \tag{15}$$

And the matrix T_{Daube}^{-1} is the inverse matrix of the forward transform matrix T_{Daube} which satisfies $T_{\text{Daube}}^{-1} = T_{\text{Daube}}^{T}$ due to the orthogonality of the matrix T_{Daube}.

2.3. The Cubic Spline Wavelet Basis. Consider the cubic spline wavelets as the expansion base in HB technique. The cubic spline wavelets are constructed in [7], which are semiorthogonal wavelets. The high approximation rate and the interpolation property can be inherited from spline functions. Therefore, the cubic spline wavelet transform matrix T_{cubic} and the differential operator representation matrix D_{cubic} have the following properties which are suitable for HB method.

Due to the periodic condition $x(t+L) = x(t)$, we must use the periodization functions of the cubic spline wavelets on the interval $[0, L]$, $L > 4$. For convenience, we still denote by $v_i(t)$ the periodic function. Let us assume that expansion bases are

$$\begin{aligned} \{v_i\}_{i=1}^{N_s} = \{&\phi_{0,-1}, \phi_{0,k}\,(0 \leq k \leq L-4), \phi_{0,L-3}, \\ &\psi_{j,k}\left(0 \leq j \leq J-1, -1 \leq k \leq n_j - 2\right)\}, \end{aligned} \tag{16}$$

where $n_j = 2^j L$, $N_s = 2^J L - 1$. Correspondingly, the unknown state variable $x(t)$ is approximated by the bases of these spaces

$$\begin{aligned} V_J &= V_{J-1} \oplus W_{J-1} \\ &\vdots \\ &= V_0 \oplus W_0 \oplus \cdots \oplus W_{J-1}, \end{aligned} \tag{17}$$

where

$$\begin{gathered} V_0 = \operatorname{span}\{\phi_{-1,-1}(t), \ldots, \phi_{-1,L-4}(t), \phi_{-1,L-3}(L-t)\}, \\ W_i = \operatorname{span}\left\{\psi_{i,-1}(t), \psi_{i,0}(t), \ldots, \psi_{i,n_i-2}(t)\right\}, \\ 0 \leq i \leq J-1. \end{gathered} \tag{18}$$

Based on the interpolation property of the cubic spline wavelets, we have

$$\begin{aligned} P_{V_J} x(t) &= I_{V_b} x(t) + \sum_{j=0}^{J-1} I_{W_j} x(t) \\ &= \widehat{x}_{-1,-3}\eta_1(t) + \widehat{x}_{-1,-2}\eta_2(t) + \widehat{x}_{-1,-1}\phi_b(t) \\ &\quad + \sum_{k=0}^{L-4} \widehat{x}_{-1,k}\phi_k(t) + \widehat{x}_{-1,L-3}\phi_b(L-t) \\ &\quad + \widehat{x}_{-1,L-2}\eta_2(L-t) + \widehat{x}_{-1,L-1}\eta_1(L-t) \\ &\quad + \sum_{j=0}^{J-1}\left[\sum_{k=-1}^{n_j-2} \widehat{x}_{j,k}\psi_{j,k}(t)\right]. \end{aligned} \tag{19}$$

Denote the expansion coefficients by a $N_s \times 1$ dimensional vector $\widehat{x}_J$,

$$\begin{aligned} \widehat{x}_J = \big(&\widehat{x}_{-1,-3}, \ldots, \widehat{x}_{-1,L-1}, \widehat{x}_{0,-1}, \ldots, \widehat{x}_{0,n_0-2}, \ldots, \widehat{x}_{J-1,-1}, \\ &\ldots, \widehat{x}_{J-1,k}, \ldots, \widehat{x}_{J-1,n_J-2}\big)^T, \end{aligned} \tag{20}$$

that will be determined by satisfying the collocation conditions, $N_s = 2^J L + 3$. Interpolate P_{V_J} at the collocation points

$$\begin{gathered} \Big\{t_1^{(-1)} = 0, \quad t_2^{(-1)} = \frac{1}{2}, \\ t_k^{(-1)} = k-2, \quad k = 3, \ldots, L+1; \\ t_{L+2}^{(-1)} = L - \frac{1}{2}, \qquad t_{L+3}^{(-1)} = L\Big\}, \\ \Big\{t_{-1}^{(j)} = \frac{1}{2^{j+2}}, \qquad t_k^{(j)} = \frac{k+1.5}{2^j}, \quad 0 \leq k \leq n_j - 3, \\ t_{n_j-2}^{(j)} = L - \frac{1}{2^{j+2}}\Big\}, \end{gathered} \tag{21}$$

as follows:

$$P_{V_J} x\left(t_k^{-1}\right) = x\left(t_k^{-1}\right), \quad 1 \le k \le L+3,$$
$$P_{V_J} x\left(t_k^{j}\right) = x\left(t_k^{j}\right), \quad j \ge 0,\ -1 \le k \le n_j - 2,\ 0 \le j \le J-1. \tag{22}$$

Substituting the expressions into (1), we obtain nonlinear discrete algebraic systems.

Denote by T_{cubic} the cubic spline wavelet transform matrix. We introduce an inverse wavelet transform (IWT) T_{cubic}^{-1} which maps its wavelet coefficients $\hat{x}_J$ to discrete sample values $\mathbf{f}_J$ with length N_s; that is $T_{\text{cubic}}^{-1}\hat{x}_J = \mathbf{f}_J$. The inverse transform matrix T_{cubic}^{-1} is

$$T_{\text{cubic}}^{-1} = \begin{pmatrix} B & & & & \\ & M_0 & & & \\ & & M_1 & & \\ & & & \vdots & \\ & & & & M_{J-1} \end{pmatrix}, \tag{23}$$

where B denotes a tridiagonal matrix with dimension $L+2$ and M_j is a tridiagonal matrix with dimension $2^j L$.

We obtain the derivative matrix D_{cubic} in [11] as follows:

$$D_{\text{cubic}} = H_1^{-1} H_2, \tag{24}$$

where

$$H_1 = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ \lambda_1 & 2 & \mu_1 & & & & \\ & \lambda_2 & 2 & \mu_2 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \lambda_i & 2 & \mu_i & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \lambda_{N_s-1} & 2 & \mu_{N_s-1} \\ & & & & & 1 & \mu_{N_s-1} \end{bmatrix}_{(N_s+1)\times(N_s+1)},$$

$$H_2 = \begin{bmatrix} a_1 & a_2 & a_3 & & & \\ c_1 & d_1 & e_1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & c_i & d_i & e_i & \\ & & \cdot & \cdot & \cdot & \\ & & & c_{N_s-1} & d_{N_s-1} & e_{N_s-1} \\ & & & b_3 & b_2 & b_1 \end{bmatrix}_{(N_s+1)\times(N_s+1)}, \tag{25}$$

and these constants in these matrices H_1 and H_2 can be referenced from the formulae (2.20a)–(2.20d) in [11].

For the whole nonlinear equation system

$$\left(\widehat{C}H_1^{-1}H_2 + \widehat{G} + T_{\text{cubic}}\left[\frac{\partial f_k}{\partial x_l}\right]T_{\text{cubic}}^{-1}\right)\left(X^{(i+1)} - X^{(i)}\right) = -\Phi\left(X^{(i)}\right), \tag{26}$$

where H_1, H_2, and T_{cubic}^{-1} are tridiagonal matrices, the triangular decomposition of the tridiagonal matrix can be used to decompose the Jacobian iterative matrix.

3. Comparison Analysis

Using HB method to solve nonlinear ODEs, the Newton iterative form is obtained. Here we want to analyze the sparsity of derivative matrix D and wavelet transform matrix T of the Jacobian matrix based on two wavelets.

3.1. The Comparison of the Wavelet Transform Matrixes T_{Daube} and T_{cubic}. Now we analyze the sparsity of the matrixes T_{Daube} and T_{cubic}. The computation cost $T[\partial f_k/\partial x_l]T^{-1}$ of Jacobian matrix $J(X)$ results from the number of nonzero elements of the wavelet transform matrix T. We analyze the nonzero elements (NZ) of matrices T_{Daube} and T_{cubic}. Let the maximum level of wavelet decomposition be J, $N = 2^n$, $J \le n$. From [12], we have the nonzero numbers $\text{NZ}_{T_{\text{Daube}}}$ of the matrix T_{Daube} are

$$\text{NZ}_{T_{\text{Daube}}} \le J2^{J-1}M + \left(2^J - 1\right)M + 2^J - 1 \sim O\left(N\log(N)\right). \tag{27}$$

For the cubic spline interpolation wavelets, the transform matrix T_{cubic}^{-1} has the following property:

$$\begin{aligned}\text{NZ}_{T_{\text{cubic}}^{-1}} &= [3(L-1)-2] + \sum_{j=0}^{J-1}\left(2^j \times 3 \times L - 2\right) \\ &= 3L \times 2^J - 2J - 5 \sim O\left(N_s\right).\end{aligned} \tag{28}$$

3.2. Comparison of the Derivative Matrices D_{Daube} and D_{cubic}. The sparsity of the derivative matrix D is an important property of Jacobian matrix $J(X)$. According to the approach in [4], matrix D_{Daube} is composed of $T_{\text{Daube}}RT_{\text{Daube}}^{-1}$, where T_{Daube}^{-1} is the transpose of the matrix T_{Daube}. Since both T_{Daube} and T_{Daube}^{-1} in this case are band-limited matrices, as well as R, the resulting matrix D_{Daube} is also a band-limited matrix. Especially, the nonzero element numbers of matrix R are $(M-5)\times N + 2\sum_{i=1}^{M-2} i \sim O(N)$, so we have

$$\text{NZ}_{D_{\text{Daube}}} \sim O\left(N\log(N)\right). \tag{29}$$

By the formulation in Section 2.3, the number of nonzero element of matrix $D_{\text{cubic}} = H_1^{-1}H_2$ is $O(N_s)$. It follows that cubic spline wavelets yield a sparser derivative matrix than that of Daubechies wavelets. Thus, the Jacobian matrix of the cubic spline wavelets is much sparser than the periodic Daubechies wavelet.

4. Numerical Experiments

In this section, we will give the sparsity figures of the transform matrix and the derivative matrix based on two kinds of wavelets. For simplicity, assume the matrices $\widehat{C}$, $\widehat{G}$,

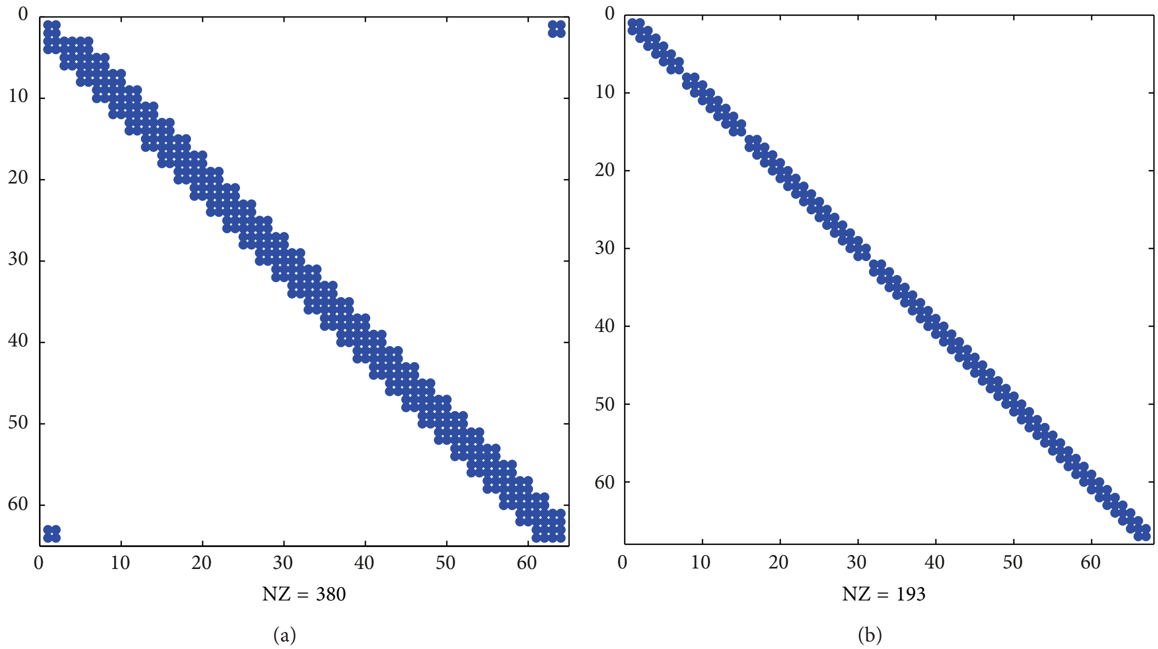

FIGURE 1: (a): Sparsity pattern of the periodized transform matrix T_{Daube} by the periodized D4 wavelets; (b): sparsity pattern of the inverse transform matrix T_{cubic}^{-1} of the cubic spline wavelet.

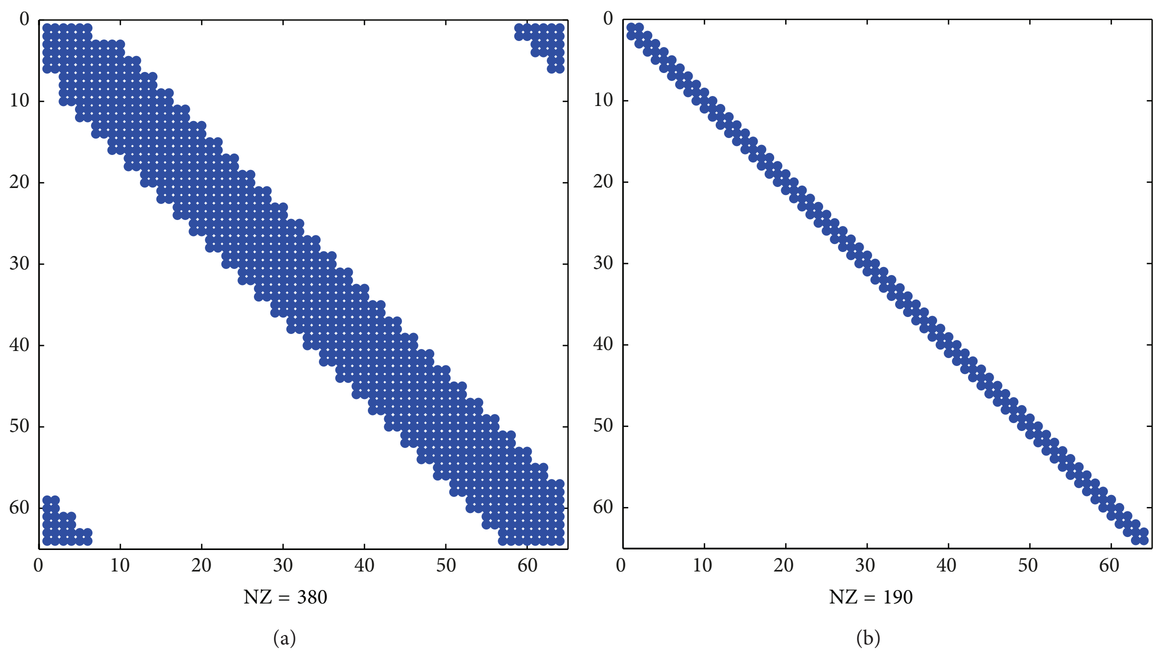

FIGURE 2: (a): Sparsity pattern of the derivative matrix D_{Daube} by the periodized D4 wavelets; (b): sparsity pattern of H_1 or H_2 of the derivative matrix of the cubic spline wavelet, where $D_{\text{cubic}} = H_1^{-1}H_2$.

and $[\partial f/\partial x_l]$ are diagonal. Figure 1 is the sparsity of T_{Daube} using the periodized D4 Daubechies wavelets.

In Figure 2 we plot the sparsity of the derivative matrix of periodic Daubechies wavelets and the matrix H_2 or H_1 of the derivative matrix D_{cubic}.

5. Conclusions

In this paper, we formulate the comparison analysis of harmonic balance method based on the cubic spline wavelets and periodic Daubechies wavelets. It is shown that the cubic spline wavelet HB method has the special structure for

Jacobian matrix compared to the Daubechies wavelet HB method to solve steady-state analysis of nonlinear circuits.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work was supported by the Natural Science Foundation of China (NSFC) (Grant nos. 11201370 and 11171270) and the Fundamental Research Funds for the Central Universities.

References

[1] D. Long, R. Melville, K. Ashby, and B. Horton, "Full-chip harmonic balance," in *Proceedings of the IEEE Custom Integrated Circuits Conference*, pp. 379–382, Santa Clara, Calif, USA, May 1997.

[2] M. S. Nakhla and J. Vlach, "A piecewise harmonic balance technique for determination of periodic response of nonlinear systems," *IEEE Transactions on Circuits and Systems*, vol. 23, no. 2, pp. 85–91, 1976.

[3] V. Rizzoli, F. Mastri, F. Sgallari, and G. Spaletta, "Harmonic-balance simulation of strongly nonlinear very large-size microwave circuits by inexact Newton methods," in *Proceedings of the IEEE MTT-S International Microwave Symposium Digest*, pp. 1357–1360, June 1996.

[4] N. Soveiko and M. S. Nakhla, "Steady-state analysis of multitone nonlinear circuits in wavelet domain," *IEEE Transactions on Microwave Theory and Techniques*, vol. 52, no. 3, pp. 785–797, 2004.

[5] M. Steer and C. Christoffersen, "Generalized circuit formulation for the transient simulation of circuits using wavelet convolution and time marching techniques," in *Proceedings of the 15th European Conference on Circuit Theory and Design (ECCTD '01)*, pp. 205–208, 2001.

[6] W. Cai and J. Wang, "Adaptive multiresolution collocation methods for initial boundary value problems of nonlinear PDEs," *SIAM Journal on Numerical Analysis*, vol. 33, no. 3, pp. 937–970, 1996.

[7] J. Wang, "Cubic spline wavelet bases of sobolev spaces and multilevel interpolation," *Applied and Computational Harmonic Analysis*, vol. 3, no. 2, pp. 154–163, 1996.

[8] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*, Van Nostrand Reinhold, New York, NY, USA, 1983.

[9] I. Daubechies, *Ten Lectures on Wavelet*, SIAM, Philadelphia, Pa, USA, 1992.

[10] G. Beylkin, "On the representation of operators in bases of compactly supported wavelets," *SIAM Journal on Numerical Analysis*, vol. 29, no. 6, pp. 1716–1740, 1992.

[11] D. Zhou and W. Cai, "A fast wavelet collocation method for high-speed circuit simulation," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 46, no. 8, pp. 920–930, 1999.

[12] N. Guan, K. Yashiro, and S. Ohkawa, "On a choice of wavelet bases in the wavelet transform approach," *IEEE Transactions on Antennas and Propagation*, vol. 48, no. 8, pp. 1186–1191, 2000.