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Research Article

Variational Iteration Method for Solving the Generalized Degasperis-Procesi Equation

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We introduce the variational iteration method for solving the generalized Degasperis-Procesi equation. Firstly, according to the variational iteration, the Lagrange multiplier is found after making the correction functional. Furthermore, several approximations of $u_{n+1}(x,t)$ which is converged to $u(x,t)$ are obtained, and the exact solutions of Degasperis-Procesi equation will be obtained by using the traditional variational iteration method with a suitable initial approximation $u_0(x,t)$. Finally, after giving the perturbation item, the approximate solution for original equation will be expressed specifically.

1. Introduction

The theory of soliton has extensive applications in physics, mechanics, and combustion science. In recent years, many researchers studied the soliton theory in the fields of shock wave [1, 2], light scattering, quantum mechanics, atmospheric physics, neural networks, explosion, and combustion [3]. There are many new methods for searching the soliton solution of nonlinear evolution equations such as hyperbolic tangent function method [4], the homogeneous balance methods [5], Jacobi elliptic function expansion method [3], and pseudo-spectral method [6].

The variational iteration method (VIM) was developed, in 1999, by He [7–13]. The VIM gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise, a few approximations can be used for numerical purposes. The Adomian decomposition method suffers from the complicated computational work needed for the derivation of Adomian polynomials for nonlinear terms. The VIM has no specific requirements, such as linearization, small parameters for nonlinear operators. Therefore, the VIM can overcome the foregoing restrictions and limitations of perturbation techniques, so that it provides us with a possibility to analyze strongly nonlinear problems. On the other hand, the VIM is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution [14]. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications. The VIM was successfully applied to study a variety of differential equations. It is based on Lagrange multiplier, and it has the merits of simplicity and easy execution. As a result, it has been proved by many authors to be a powerful mathematical tool for addressing various kinds of linear and nonlinear problem. For example, this method was used for solving nonlinear wave equations and the Laplace equation by Wazwaz [14]. The VIM for solving linear systems of ODEs with constant coefficients was studied by Khojasteh Salkuyeh [15]. Helmholtz equation was researched by Momani and Abuasad [16]. Geng [17] introduced the piecewise VIM for solving Riccati differential equation. Fractional vibration equation was researched by Das [18]. Furthermore, higher order boundary value problems were researched by Xu [19],Noor, and Mohyud-Din [20]. Noor et al. [21] applied a modified He's variational iteration method for solving singular fourth-order parabolic partial differential equations. The proposed modification is made by introducing He's polynomials in the correction functional. Ghorbani and Saberi-Nadjafi [22] modified the VIM by constructing an initial trial function without unknown parameters. Sevimlican [23] constructed approximate Green's function for a vector equation for the electric field by using VIM.

In this paper, we are concerned with the variational iterations method for solving the generalized Degasperis-Procesi equation. As a review, we will recall the VIM briefly in Section 2.

2. Variational Iteration Method

In this section, the basic concepts of variational iteration method (VIM) are introduced. Here, a description of method [7–15] is given to handle the general nonlinear problem. Consider the differential equation of the form

$$Lu(x,t) + Nu(x,t) = f(x,t), \quad (1)$$

where L is a linear operator, N is a nonlinear operator, and $f(x,t)$ is the inhomogeneous term. According to He's variational iteration method, we can construct a correction functional for (1) as follows:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) \\ &\quad + \int_0^t \lambda(\tau)\left(Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - f(x,\tau)\right) d\tau, \end{aligned} \quad (2)$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory [12, 24]. Here $\tilde{u}_n$ is considered as a restricted variation [14, 25] which means $\delta\tilde{u}_n = 0$; the subscript n denotes the nth approximations. The successive approximations $u_{n+1}(x,t)$, of the solution $u(x,t)$, can be obtained after using the obtained Lagrange multiplier and the zeroth approximation $u_0(x,t)$, which are selected from any function that satisfies the initial conditions. With λ determined, several approximations $u_{n+1}(x,t)$, $n \geqslant 0$ follow. Consequently, the exaction solution may be obtained as

$$u(x,t) = \lim_{n\to+\infty} u_{n+1}(x,t). \quad (3)$$

In fact, the VIM depends on the suitable selection of the initial approximation $u_0(x,t)$. Moreover, we use a well-known, powerful tool to prove the convergence of the sequence obtained via the VIM and its rate. It is the Banach's fixed point theorem that follows.

Theorem 1 (Banach's fixed point theorem). *Assume that X is a Banach space and*

$$A : X \longrightarrow X \quad (4)$$

is a nonlinear mapping, and suppose that

$$\|A[u] - A[v]\| \leqslant \alpha \|u - v\|, \quad u, v \in X \quad (5)$$

for some constant $\alpha < 1$. Then A has a unique fixed point. Furthermore, the sequence

$$u_{n+1} = A[u_n], \quad (6)$$

with an arbitrary choice of $u_0 \in X$ converges to the fixed point of A.

According to Theorem 1, for the nonlinear mapping

$$\begin{aligned} A[u_n(x,t)] &= u_n(x,t) \\ &\quad + \int_0^t \lambda(\tau)\{Lu_n(x,\tau) + N(x,\tau) - f(x,\tau)\} d\tau \end{aligned} \quad (7)$$

a sufficient condition for the convergence of the variational iteration method is strict contraction of A. Furthermore, the sequence (2) converges to the fixed point of A which is also the solution of problem (1). Some modifications to prove the convergence speed and to lengthen the interval of convergence for VIM series solution are suggested in [17, 26–30].

3. The Variational Iteration of Generalized Degasperis-Procesi Equation

Degasperis and Procesi consider the following family of third-order dispersive conservation laws [31],

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \quad (8)$$

where α, γ, c_0, c_1, c_2, and c_3 are real constants. In this family, only three equations satisfy asymptotic integrability conditions [31]. That is, if $c_0 = 1$, $c_1 = -1/2$, $c_2 = 0$, $c_3 = 0$, $\alpha^2 = 0$, and $\gamma = 1$, (8) is the KdV equation

$$u_t + u_x + uu_x + u_{xxx} = 0. \quad (9)$$

If $c_0 = 0$, $c_1 = -3/2$, $c_2 = 1/2$, $c_3 = 1$, $\alpha^2 = 1$, and $\gamma = 0$, (8) is the Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \quad (10)$$

If $c_0 = 0$, $c_1 = -2$, $c_2 = 1$, $c_3 = 1$, $\alpha^2 = 1$, and $\gamma = 0$, (8) is the Degasperis-Procesi equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \quad (11)$$

It should be mentioned that both C-H and D-P equations are derived as members of a one-parameter family of asymptotic shallow water approximations to the Euler equations. It shows that the two equations are physically relevant; otherwise, the D-P equation would be of purely theoretical interest.

Variational iteration method for KdV-Burgers and Lax's seventh-order KdV equations has been studied by Soliman [32]. In this paper, we consider the generalized Degasperis-Procesi equation

$$\begin{aligned} u_t - u_{xxt} + 4uu_x - 3uu_{xx} - uu_{xxx} \\ = f(u, u_x, u_t, u_{xx}, u_{xxx}, u_{xxt}) \end{aligned} \quad (12)$$

that was proposed in [33]. f is the generalized perturbation item. We suppose f is a sufficiently smooth function of the variable.

Step 1. Make the independent variable transformation:

$$\xi = k(x - \omega t) + \xi_0. \tag{13}$$

Here, $\xi_0 \in C$ is an arbitrary complex number. k is wave number; ω is wave velocity. Substituting (13) into (12), we have

$$-\omega u' + k^2 \omega u''' + 4uu' - 3kuu'' - k^2 uu''' = f_1. \tag{14}$$

Here, u' is the derivative of u with respect to ξ; that is, $u' = du/d\xi$. $f_1 = f_1(u, u', u'', u''')$.

Step 2. From [34], we find the special solution, when f is identical to 0:

$$u_0(\xi) = -\frac{1}{2} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(x - \omega t) + \xi_0\right]. \tag{15}$$

Remark 2. Notice that $\xi_0 \in C$, $i\tanh(i\xi) = -\tan\xi$, $\tanh(\xi + (\pi i/2)) = \coth\xi$, $i\coth(i\xi) = \cot\xi$, and $\tanh[(1/2)(\xi + (i/2)\pi)] = \tanh\xi + i\,\text{sech}\xi$, where $i = \sqrt{-1}$. These solutions contain the other four types of forms named $\coth\xi$, $\tan\xi$, $\cot\xi$, and $\tanh\xi + i\,\text{sech}\xi$.

Step 3. Make the correction functional

$$\begin{aligned} u_{n+1}(\xi) &= u_n(\xi) - \int_0^\xi \lambda(s)\left[-\omega\tilde{u}_n' + k^2\omega u_n''' + 4\tilde{u}_n\tilde{u}_n' \right. \\ &\quad \left. - 3k\tilde{u}_n\tilde{u}_n'' - k^2\tilde{u}_n\tilde{u}_n''' - \tilde{f}_1\right] ds. \end{aligned} \tag{16}$$

Here, $\tilde{u}_n$, $\tilde{u}_n'$, $\tilde{u}_n''$, and $\tilde{u}_n'''$ are considered as a restricted variation [35]. That is,

$$\delta\tilde{u}_n = \delta\tilde{u}_n' = \delta\tilde{u}_n'' = \delta\tilde{u}_n''' = 0. \tag{17}$$

Step 4. Under the above condition, make the correct functional stationary with respect to u_n; noticing that $\delta u_n(0) = 0$, we have

$$\begin{aligned} \delta u_{n+1}(\xi) &= \delta u_n(\xi) - \delta\int_0^\xi \lambda(s)\left[-\omega\tilde{u}_n' + k^2\omega u_n''' + 4\tilde{u}_n\tilde{u}_n' \right. \\ &\quad \left. - 3k^2\tilde{u}_n\tilde{u}_n'' - k^2\tilde{u}_n\tilde{u}_n''' - \tilde{f}\right] ds \\ &= \delta u_n(\xi) - \left[\lambda\delta u''(s)\big|_{s=\xi} - \lambda'(s)\,\delta u'(s)\big|_{s=\xi} \right. \\ &\quad \left. + \lambda^2(s)\,\delta u(s)\big|_{s=\xi} - \int_0^\xi \lambda'''(s)\,\delta u\, ds\right] = 0. \end{aligned} \tag{18}$$

For arbitrary δu_{n+1}, from the above relation, we obtain the Euler-Language equation:

$$\begin{aligned} 1 - k^2\omega\lambda''(s)\big|_{s=\xi} &= 0, \\ k^2\omega\lambda'''(s)\big|_{s=\xi} &= 0, \\ k^2\omega\lambda'(s)\big|_{s=\xi} &= 0, \\ k^2\omega\lambda(s)\big|_{s=\xi} &= 0. \end{aligned} \tag{19}$$

Solve (19), we derive

$$\lambda(s) = -\frac{1}{2}\cdot\frac{1}{k^2\omega}(s - \xi)^2. \tag{20}$$

Substituted (20) into (16), we have the integration form:

$$\begin{aligned} u_{n+1} &= u_n \\ &\quad + \int_0^\xi \frac{1}{2k^2\omega}(s - \xi)^2 \\ &\qquad \times \left[k^2\omega u_n''' - \omega u_n' + 4u_n u_n' \right. \\ &\qquad \left. - 3ku_n u_n'' - k^2 u_n u_n''' - f_1\right] ds, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{21}$$

From the above solution procedure, we can see that the approximate solutions converge to its exact solution. That is, $u(x,t) = \lim_{n\to\infty} u_n(x,t)$, $u_n(x,t)$ is the approximate solution with arbitrary degree of accurate solitary wave of Degasperis-Procesi equation.

Step 5. Calculation of the approximate solution.

According to the integration form (21),we can calculate the approximate solution. Firstly, let (15) be the zero-order approximate solution:

$$u_0(\xi) = -\frac{1}{2} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(x - \omega t) + \xi_0\right]. \tag{22}$$

Substitute (15) into (21). We obtain the one-order approximate solution $u_1(\xi)$

$$\begin{aligned} u_1(\xi) &= u_0(\xi) + \int_0^\xi \frac{1}{2k^2\omega}(s - \xi)^2(-f_1(u_0))\, ds \\ &= -\frac{1}{2} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(x - \omega t) + \xi_0\right] \\ &\quad - \int_0^\xi \frac{1}{2k^2\omega}(s - \xi)^2 f_1\left(u_0(s), u_0'(s), u_0''(s), u_0'''(s)\right) ds \\ &= -\frac{1}{2} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(x - \omega t) + \xi_0\right] + v_0(\xi), \end{aligned} \tag{23}$$

TABLE 1: Numerical example for the solution of DP equation, $\xi_0 = 0$, $\omega = 1$, $t = 0$, and $\varepsilon = 0.1$.

x	Initial solution u_0	1th approximate solution u_1	Absolute error
0.1	−0.498752	−0.498762	$-4.16770 * 10^{-6}$
0.11	−0.498491	−0.498497	$5.54752 * 10^{-6}$
0.12	−0.498204	−0.498211	$7.2026 * 10^{-6}$
0.13	−0.497893	−0.497902	$9.15804 * 10^{-6}$
0.14	−0.497558	−0.497694	$1.53894 * 10^{-5}$
0.15	−0.497198	−0.497212	$1.407042 * 10^{-5}$
0.16	−0.496814	−0.496831	$1.706776 * 10^{-5}$
0.17	−0.496405	−0.496426	$2.04856 * 10^{-5}$
0.18	−0.495972	−0.495996	$2.43198 * 10^{-5}$
0.19	−0.495515	−0.495543	$2.86050 * 10^{-5}$
0.2	−0.495033	−0.495066	$3.33668 * 10^{-5}$

TABLE 2: Numerical example for the solution of DP equation, $\xi_0 = 0$, $\omega = 1$, $t = 0$, and $\varepsilon = 0.05$.

x	Initial solution u_0	1th approximate solution u_1	Absolute error
0.1	−0.498752	−0.4987524	$2.08385 * 10^{-6}$
0.11	−0.498491	−0.498493	$2.77376 * 10^{-6}$
0.12	−0.498204	−0.4982076	$3.6013 * 10^{-6}$
0.13	−0.497893	−0.4978976	$4.57902 * 10^{-6}$
0.14	−0.497558	−0.4975637	$5.71902 * 10^{-6}$
0.15	−0.497198	−0.497205	$7.03521 * 10^{-6}$
0.16	−0.496814	−0.4968225	$8.5388 * 10^{-6}$
0.17	−0.496405	−0.496415	$1.02428 * 10^{-5}$
0.18	−0.495972	−0.4959842	$1.21599 * 10^{-5}$
0.19	−0.495515	−0.495529	$1.43025 * 10^{-5}$
0.2	−0.495033	−0.495049	$1.66834 * 10^{-5}$

in which $v_0(\xi) = -\int_0^{\xi}(1/2k^2\omega)(s-\xi)^2 f_1(u_0(s), u_0'(s), u_0''(s), u_0'''(s))ds$.

Then, substitute (23) into (21). We can obtain the second-order approximate solution $u_2(\xi)$:

$$\begin{aligned}
u_2(\xi) = {} & u_1(\xi) + \int_0^{\xi} \frac{1}{2k^2\omega}(s-\xi)^2 \\
& \times \left[k^2\omega u_1'''(s) - \omega u_1'(s) + 4u_1(s)u_1'(s)\right. \\
& - 3k^2 u_1(s)u_1''(s) - k^2 u_1(s) u_1'''(s) \\
& \left. - f_1\left(u_1(s), u_1'(s), u_1''(s), u_1'''(s)\right)\right] ds \\
= {} & -\frac{1}{2} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(x-\omega t) + \xi_0\right] + v_0(\xi) \\
& + \int_0^{\xi} \frac{1}{2k^2\omega}(s-\xi)^2 \\
& \times \left[k^2\omega(u_0(s) + v_0(s))'''\right. \\
& - \omega(u_0(s) + v_0(s))' + 4(u_0(s) + v_0(s)) \\
& \times (u_0(s) + v_0(s))' \\
& - 3k^2(u_0(s) + v_0(s))(u_0(s) + v_0(s))'' \\
& - k^2(u_0(s) + v_0(s))(u_0(s) + v_0(s))''' \\
& - f_1\left(u_0(s) + v_0(s), u_0'(s) + v_0'(s), u_0''(s)\right. \\
& \left.\left. + v_0''(s), u_0'''(s) + v_0'''(s)\right)\right] ds.
\end{aligned} \tag{24}$$

Using the same method, we can get the higher order approximate solution.

4. The Optical Soliton Perturbation Solution and the Numerical Example

Specially, we set

$$f = \varepsilon g(u) = \varepsilon u, \quad 0 < \varepsilon \ll 1. \tag{25}$$

Then (12) change to

$$u_t - u_{xxt} + 4uu_x - 3uu_{xx} - uu_{xxx} = \varepsilon g(u) = \varepsilon u. \tag{26}$$

We obtain the zero-order and the one-order approximate solution of (20)

$$\begin{aligned} u_0(\xi) &= -\frac{1}{2} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(x-\omega t)+\xi_0\right] \\ &= -\frac{1}{2} + \frac{1}{2}\tanh^2(\xi), \quad k = \frac{1}{2}, \\ u_1(\xi) &= u_0(\xi) + \int_0^{\xi} \frac{1}{2k^2\omega}(s-\xi)^2(-\varepsilon u(s))\,ds \\ &= -\frac{1}{2} + \frac{1}{2}\tanh^2\xi \\ &\quad - \frac{\varepsilon}{2k^2\omega}\int_0^{\xi} s^2\left(-\frac{1}{2}+\frac{1}{2}\tanh^2 s\right)ds \\ &\quad + \frac{\varepsilon\xi}{2k^2\omega}\int_0^{\xi} s\left(-\frac{1}{2}+\frac{1}{2}\tanh^2 s\right)ds \\ &\quad - \frac{\varepsilon\xi^2}{2k^2\omega}\int_0^{\xi}\left(-\frac{1}{2}+\frac{1}{2}\tanh^2 s\right)ds \\ &= -\frac{1}{2} + \frac{1}{2}\tanh^2\xi + \frac{\varepsilon}{2k^2\omega}\int_0^{\xi}\ln\cosh s\,ds \\ &= -\frac{1}{2} + \frac{1}{2}\tanh^2\left[\frac{1}{2}(x-\omega t)+\xi_0\right] \\ &\quad + \frac{2\varepsilon}{\omega}\int_0^{\xi}\ln\cosh s\,ds, \quad k = \frac{1}{2}. \end{aligned} \tag{27}$$

We set

$$\begin{aligned} \xi_0 &= 0, \quad \omega = 1, \quad t = 0, \quad \varepsilon = 0.1, \\ \xi_0 &= 0, \quad \omega = 1, \quad t = 0, \quad \varepsilon = 0.05; \end{aligned} \tag{28}$$

we will obtain the following numerical example. See Tables 1 and 2.

5. Conclusion

By the analysis of structure on the left side of (8) and the properties of f about the variable and the analytical variational iteration formula, we can prove that the sequence of functions of $\{u_n(\xi)\}$ decided by (21) is uniform convergence. So the limit function of $\{u_n(\xi)\}$ is the solution of the equation. Moreover, the zero-order approximate solution $u_0(\xi)$ is the soliton of (8) in which $f = 0$; it should be specially pointed out that the more accurate the identification of the multiplier is, the faster the approximations converge to their exact solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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