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Research Article

Stability and Hopf-Bifurcating Periodic Solution for Delayed Hopfield Neural Networks with *n* Neuron

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We consider a system of delay differential equations which represents the general model of a Hopfield neural networks type. We construct some new sufficient conditions for local asymptotic stability about the trivial equilibrium based on the connection weights and delays of the neural system. We also investigate the occurrence of an Andronov-Hopf bifurcation about the trivial equilibrium. Finally, the simulating results demonstrate the validity and feasibility of our theoretical results.

1. Introduction

Analysis of neural networks from the viewpoint of nonlinear dynamics is helpful in solving problems of theoretical and practical importance. The Hopfield neural networks (HNNs) have diverse applications in many areas such as classification, associative memory, pattern recognition, parallel computations, and optimization [1–5]. These vast applications have been the focus of detailed studies by researchers. So, we believe that the study of these neural networks models is important. Actually, in some artificial neural network applications such as content-addressable memories, information is stored as stable equilibrium points of the system. Retrieval occurs when the system is initialized within the basin of attraction of one of the equilibria and the network is allowed to be stabilized in its steady state. Time delay has important influences on the dynamical behavior of neural networks. Marcus and Westervelt [6] first considered the effect of including discrete time delays in the connection terms to represent the time of propagation between neurons. They found out that the delay can destabilize the network as a whole and create oscillatory behavior.

The study of the local asymptotic stability and Andronov-Hopf bifurcations of neural network models with multiple time delays are complex. In order to reach a deep and clear understanding of the dynamics of such models, most researchers have limited their study to the models with a single delay [7, 8]. In some papers, multiple delays are considered but there are no self-connection terms and moreover the systems with two delays have been generally investigated [9–11]. For example, Liao et al. [10] investigated the stability of a two-neuron system with different time delay as follows:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -x_1(t) + a_1 \tanh\left(x_2(t) - b_2 x_2(t-\tau)\right), \\ \frac{dx_2(t)}{dt} &= -x_2(t) + a_2 \tanh\left(x_1(t) - b_1 x_1(t-\sigma)\right). \end{aligned} \quad (1)$$

They showed that $(0,0)$ is a unique fixed point of the mentioned system if $a_1 a_2 (1-b_1)(1-b_2) \leq 1$. They estimated the length of delays for which local asymptotical stability is preserved. So, they achieved a delay-dependent stability condition with delayed system. As another example, Olien and Bélair [8] investigated a system with two delays; that is,

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{2} a_{ij} f\left(u_j\left(t - \tau_j\right)\right), \quad i = 1, 2. \quad (2)$$

They discussed several cases, such as $\tau_1 = \tau_2$ and $a_{11} = a_{22} = 0$. They obtained some sufficient conditions for the stability of the stationary point (0, 0) of the latest system and showed that this system may undergo some bifurcations at certain values of the parameters.

Songa et al. investigated the stability and Hopf bifurcation in an unidirectional ring of n neurons [12] but the model considered here is more general than the one in Song's studies. In fact, we have considered a Hopfield neural network with arbitrary neurons in which each neuron is bidirectionally connected to all the others. The Lyapunov stability theorem is used to establish the sufficient condition for the asymptotic stability of the equilibrium point in recent studies but, here, we obtain sufficient conditions for local asymptotic stability based on analyzing the associated characteristic transcendental equation. In this paper, delay-independent and delay-dependent sufficient conditions for local asymptotic stability are obtained and the Andronov-Hopf bifurcation for delayed Hopfield neural networks with n neuron is studied.

2. Local Analysis of a Neural Network with Delays

Consider the following delayed neural network described as:

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{n} a_{ij} f\left(u_j\left(t-\tau_j\right)\right), \quad i = 1, 2, \ldots, n, \tag{3}$$

where $u_i(t)$ represents the activation state of ith neuron $(i = 1, 2, \ldots, n)$ at time t, a_{ij} is the weight of synaptic connections from ith neuron to jth neuron, and $\tau_j \geq 0$ is the time delay. In system (3), each neuron is connected not only to itself but also to the other neuron via a nonlinear sigmoidal function f which is a typical transmitting function among neurons. The initial value is assumed to be

$$u_i(\theta) = \varphi_i(\theta) \quad \text{for } \theta \in [-k, 0], \tag{4}$$

where $\varphi_i(\theta) \in C([-k, 0], \mathbb{R})$, $i = 1, 2, \ldots, n$, and $k = \max_{1\leq j\leq n} \tau_j$. The natural phase space for (3) is the the Banach space $C = C([-k, 0], \mathbb{R}^n)$ of continuous functions defined on $[-k, 0]$ equipped with the supremum norm $\|\varphi\| = \sup_{-k\leq s\leq 0}\|\varphi(s)\|$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous; then the solutions of (3) define the continuous semiflow

$$\begin{gathered} \Phi : \mathbb{R}^+ \times C \longmapsto C \\ (t, \varphi) \longmapsto x_t^{\varphi}. \end{gathered} \tag{5}$$

A function $\widehat{\xi} \in C$ is an equilibrium point (or stationary point) of Φ if $\widehat{\xi}(s) = (\xi_1, \xi_2, \ldots, \xi_n)$ for all $-k \leq s \leq 0$, satisfying $-\xi_i + \sum_{j=1}^{n} a_{ij} f(\xi_j) = 0$, $i = 1, 2, \ldots, n$.

Suppose that $f \in C^1(\mathbb{R})$, $f(0) = 0$, and $uf(u) > 0$ for $u \neq 0$. It is clear that the origin of the state space is a stationary point of system (3). For stability analysis, the system (3) has been linearized about the origin of state space and the following system of linearized equations is obtained:

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{n} \alpha_{ij} u_j\left(t-\tau_j\right), \quad i = 1, 2, \ldots, n, \tag{6}$$

where $\alpha_{ij} = a_{ij} f'(0)$, $i, j = 1, 2, \ldots, n$. The associated characteristic equation of system (6) is as follows:

$$\begin{vmatrix} \lambda + 1 - \alpha_{11} e^{-\lambda\tau_1} & -\alpha_{12} e^{-\lambda\tau_2} & \cdots & -\alpha_{1n} e^{-\lambda\tau_n} \\ -\alpha_{21} e^{-\lambda\tau_1} & \lambda + 1 - \alpha_{22} e^{-\lambda\tau_2} & \cdots & -\alpha_{2n} e^{-\lambda\tau_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1} e^{-\lambda\tau_1} & -\alpha_{n2} e^{-\lambda\tau_2} & \cdots & \lambda + 1 - \alpha_{nn} e^{-\lambda\tau_n} \end{vmatrix} = 0. \tag{7}$$

The zero solution of system (3) is stable if and only if all roots λ of characteristic equation (3) have negative real parts. In this paper, we will find some conditions which ensure that all roots of characteristic equation (3) have negative real parts. The characteristic equation of the linearized system (3) about the origin of state space is a transcendental equation involving exponential functions and it is difficult to find all values of parameter τ such that all the characteristic roots have negative real parts. If A, B, Λ are defined as

$$\begin{gathered} A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{33} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{33} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}, \\ B = \begin{pmatrix} e^{-\lambda\tau_1} & 0 & 0 & 0 \\ 0 & e^{-\lambda\tau_2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\lambda\tau_n} \end{pmatrix}, \\ \Lambda = \lambda + 1, \end{gathered} \tag{8}$$

then, the characteristic equation (7) can be written as follows:

$$\det(\Lambda I - AB) = 0. \tag{9}$$

Now, motivating Leverrier's method, we propose the following formula for characteristic equation (9):

$$\det(\Lambda I - AB) = \Lambda^n + h_1 \Lambda^{n-1} + h_2 \Lambda^{n-2} + \cdots + h_{n-1}\Lambda + h_n = 0, \tag{10}$$

where $h_k = -(1/k)(S_k + S_{k-1}h_1 + \cdots + S_1 h_{k-1})$, $S_k = \operatorname{tr}((AB)^k)$, for $k = 1, 2, \ldots, n$.

Suppose that $\tau_k = \tau$, $\overline{S}_k = \operatorname{tr}(A^k)$, and $\overline{h}_k = -(1/k)(\overline{S}_k + \overline{S}_{k-1}\overline{h}_1 + \cdots + \overline{S}_1\overline{h}_{k-1})$ hold for $k = 1, 2, \ldots, n$; then it is clear that matrix B will have the following form:

$$B = \begin{pmatrix} e^{-\lambda\tau} & 0 & 0 & 0 \\ 0 & e^{-\lambda\tau} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\lambda\tau} \end{pmatrix} = e^{-\lambda\tau} I_{n\times n},$$

$$S_k = \operatorname{tr}\left((AB)^k\right) = \operatorname{tr}\left(\left(e^{-\lambda\tau}A\right)^k\right) = e^{-k\lambda\tau}\overline{S}_k,$$
$$h_k = -\frac{1}{k}e^{-k\lambda\tau}\left(\overline{S}_k + \overline{S}_{k-1}\overline{h}_1 + \cdots + \overline{S}_1\overline{h}_{k-1}\right) = e^{-k\lambda\tau}\overline{h}_k, \quad k = 1, 2, \ldots, n. \tag{11}$$

Therefore, if $\tau_k = \tau$ for $k = 1, 2, \ldots, n$, it is easy to verify that (10) can be simplified to

$$\det(\Lambda I - AB) = \Lambda^n + \overline{h}_1 e^{-\lambda\tau}\Lambda^{n-1} + \overline{h}_2 e^{-2\lambda\tau}\Lambda^{n-2} + \cdots + \overline{h}_{n-1}e^{-(n-1)\lambda\tau}\Lambda + e^{-n\lambda\tau}\overline{h}_n = 0. \tag{12}$$

Suppose that $\beta_1, \beta_2, \ldots, \beta_n$ are the eigenvalues of the matrix A and $\mathscr{P}_j(\lambda,\tau) = \lambda + 1 - \beta_j e^{-\lambda\tau}$; then it is easy to see that formula (12) can be rewritten as follows:

$$\begin{aligned}\mathscr{P}(\lambda,\tau) &= \det(\Lambda I - AB) \\ &= \left(\Lambda - \beta_1 e^{-\lambda\tau}\right)\left(\Lambda - \beta_2 e^{-\lambda\tau}\right)\cdots\left(\Lambda - \beta_n e^{-\lambda\tau}\right) \\ &= \left(\lambda + 1 - \beta_1 e^{-\lambda\tau}\right)\left(\lambda + 1 - \beta_2 e^{-\lambda\tau}\right)\cdots\left(\lambda + 1 - \beta_n e^{-\lambda\tau}\right) \\ &= \mathscr{P}_1(\lambda,\tau)\mathscr{P}_2(\lambda,\tau)\cdots\mathscr{P}_n(\lambda,\tau) = 0.\end{aligned} \tag{13}$$

Having applied formula (13), the sufficient conditions for local stability of system (3) are obtained.

Theorem 1. *Suppose that the eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$ of the matrix A are real; that is,*

$$\beta_j = a_j \quad a_j \in \mathbb{R}, \quad j = 1, 2, \ldots, n. \tag{14}$$

(1) *If $\max_{j=1,2,\ldots,n}|a_j| < 1$, then, for any arbitrary amount of τ $(\tau \geq 0)$, all the roots of characteristic equation (13) have negative real part, and hence the zero solution of system (6) is locally asymptotically stable.*

(2) *If $0 \leq \tau \leq \min_{j=1,2,\ldots,n}\{\pi/2|a_j|\}$, then matrix A being a Hurwitz matrix implies that all the roots of characteristic equation (13) have negative real part and hence the zero solution of system (6) is locally asymptotically stable.*

Proof. We suppose that $\lambda = \mu + i\omega$ is a root of characteristic equation (13). λ is a root of (13) if and only if λ satisfies

$$\begin{aligned}\mathscr{P}(\lambda,\tau) &= \mathscr{P}(\mu + i\omega, \tau) \\ &= \mathscr{P}_1(\mu + i\omega,\tau)\mathscr{P}_2(\mu + i\omega,\tau)\cdots\mathscr{P}_n(\mu + i\omega,\tau) \\ &= 0.\end{aligned} \tag{15}$$

Therefore, there must exist some j $(1 \leq j \leq n)$, such that

$$\mathscr{P}_j(\mu + i\omega,\tau) = \mu + i\omega + 1 - a_j e^{-(\mu+i\omega)\tau} = 0. \tag{16}$$

Let $R_j(\mu,\tau)$ and $I_j(\mu,\tau)$ be the real and imaginary parts of (16), respectively; we have

$$R_j(\mu,\tau) = \mu + 1 - a_j e^{-\mu\tau}\cos(\omega\tau) = 0, \tag{17}$$

$$I_j(\mu,\tau) = \omega + a_j e^{-\mu\tau}\sin(\omega\tau) = 0. \tag{18}$$

For proving the first part of the theorem, suppose that $\mu \geq 0$; then, similar to the proof of the theorems of Gupta et al. [13], we can prove that $|R_j(\mu,\tau)| > 0$. Thus, we have demonstrated that if $|R_j(\mu,\tau)| = 0$, then $\mu < 0$. It completes the proof.

For proving the second part of the theorem, suppose that $0 \leq \tau \leq \min_{j=1,2,\ldots,n}\{\pi/2|a_j|\}$ and matrix A is a Hurwitz matrix (i.e., $a_j < 0$). We show that $\mu < 0$. For this purpose, suppose $\mu \geq 0$. From (18), we have

$$|\omega| \leq |a_j|. \tag{19}$$

When $0 \leq \tau \leq \min_{j=1,2,\ldots,n}\{\pi/2|a_j|\}$, by (19), we have $0 \leq |\omega|\tau \leq |a_j|\tau \leq \pi/2$. Hence, $0 \leq \cos(\omega\tau) \leq 1$. From $\mu \geq 0$, $a_j < 0$, and $0 \leq \cos(\omega\tau) \leq 1$, we have

$$\mu + 1 - a_j e^{-\mu\tau}\cos(\omega\tau) > 0. \tag{20}$$

So, $R_j(\mu,\tau) > 0$ and this result is in contradiction with (17). So, $\mu \ngeq 0$. It completes the proof. □

Theorem 2. *Let $A_{n\times n} = (\alpha_{ij})$ be a real matrix consisting of the coefficient of system (6) with p real eigenvalues $\beta_j = a_j$, $j = 1, \ldots, p$, and $2q$ complex eigenvalues $\beta_j^{\pm} = a_j \pm ib_j$, $j = p + 1, \ldots, n - q$.*

(1) *If $\max_{j=1,2,\ldots,p}|a_j| < 1$ and $\max_{j=p+1,\ldots,n-q}\{|a_j| + |b_j|\} < 1$, then, for any arbitrary amount of τ $(\tau \geq 0)$, all the roots of characteristic equation (13) have negative real part and hence the zero solution of system (6) is locally asymptotically stable.*

(2) *If $0 \leq \tau \leq \min\{\min_{j=1,2,\ldots,p}\{\pi/2|a_j|\}, \min_{j=p+1,\ldots,n-q}\{\pi/4(|a_j| + |b_j|)\}\}$, and $|a_j| \geq |b_j|$, $j = p+1, \ldots, n-q$, then matrix A being a Hurwitz matrix implies that all the roots of characteristic equation (13) have negative real part and hence the zero solution of system (6) is locally asymptotically stable.*

Proof. In this case, formula (13) will be transformed to the following form:

$$\mathcal{P}(\lambda,\tau) = \mathcal{P}_1(\lambda,\tau)\cdots\mathcal{P}_p(\lambda,\tau)\,\mathcal{P}^{\pm}_{p+1}(\lambda,\tau)\cdots\mathcal{P}^{\pm}_{n-q}(\lambda,\tau) = 0, \quad (21)$$

such that

$$\mathcal{P}_j(\lambda,\tau) = \lambda + 1 - a_j e^{-\lambda\tau} = 0, \quad j = 1,\ldots,p$$
$$\mathcal{P}^{\pm}_j(\lambda,\tau) = \lambda + 1 - (a \pm ib)\,e^{-\lambda\tau} = 0 \quad (22)$$
$$j = p+1,\ldots,n-q.$$

Let $\lambda = \mu + i\omega$ be a root of characteristic equation (21). λ is a root of (21) if and only if λ satisfies

$$\mathcal{P}(\lambda,\tau) = \mathcal{P}_1(\mu+i\omega,\tau)\cdots\mathcal{P}_p(\mu+i\omega,\tau) \times \mathcal{P}^{\pm}_{p+1}(\mu+i\omega,\tau)\cdots\mathcal{P}^{\pm}_{n-q}(\mu+i\omega,\tau) = 0. \quad (23)$$

Therefore, there must exist some $1 \le j \le p$ such that $\mathcal{P}_j(\mu + i\omega,\tau) = \mu + i\omega + 1 - \beta_j e^{-(\mu+i\omega)\tau} = 0$ or there must exist some $p+1 \le j \le n-q$ such that $\mathcal{P}^{\pm}_j(\lambda,\tau) = \mu + i\omega + 1 - (a_j \pm ib_j)e^{-(\mu+i\omega)\tau} = 0$.

For proving the first part of the theorem, suppose that $\mu \ge 0$; then, similar to the proof of the theorems of Gupta et al. [13], we can prove that $|R_j(\mu,\tau)| > 0$. Thus, we have demonstrated that if $|R_j(\mu,\tau)| = 0$, then $\mu < 0$. It completes the proof.

For proving the sccond part of the theorem, suppose that matrix A is a Hurwitz matrix (i.e., $a_j < 0$).

If $\mathcal{P}_j(\lambda,\tau) = 0$ $(j = 1,\ldots,p)$, similar to Theorem 1, assuming $0 \le \tau \le \min_{j=1,2,\ldots,n}\{\pi/2|a_j|\}$, we will conclude that $\mu < 0$.

If

$$\mathcal{P}^{\pm}_j(\lambda,\tau) = \mu + i\omega + 1 - (a \pm ib)\,e^{-(\mu+i\omega)\tau} = 0, \quad (24)$$

we show that $\mu < 0$. For this purpose, suppose that $\mu \ge 0$ and let $R_j(\mu,\tau)$ and $I_j(\mu,\tau)$ be the real and imaginary parts of (24), respectively; we have

$$R^{\pm}_j(\mu,\tau) = \mu + 1 - a_j e^{-\mu\tau}\cos(\omega\tau) \mp b_j e^{-\mu\tau}\sin(\omega\tau) = 0, \quad (25)$$

$$I^{\pm}_j(\mu,\tau) = \omega + a_j e^{-\mu\tau}\sin(\omega\tau) \mp b_j e^{-\mu\tau}\cos(\omega\tau) = 0. \quad (26)$$

From (26), we have

$$|\omega| \le |a_j| + |b_j|. \quad (27)$$

Since $0 \le \tau \le \min_{j=p+1,\ldots,n-q}\{\pi/4(|a_j|+|b_j|)\}$, by (27), we have $0 \le |\omega|\tau \le (|a_j| + |b_j|)\tau \le \pi/4$. Hence, $|\cos(\omega\tau)| \ge |\sin(\omega\tau)|$. From $\mu \ge 0$, $a_j < 0$, $|a_j| \ge |b_j|$, and $|\cos(\omega\tau)| \ge |\sin(\omega\tau)|$, we have

$$\mu + 1 - a_j e^{-\mu\tau}\cos(\omega\tau) \mp b_j e^{-\mu\tau}\sin(\omega\tau) > 0. \quad (28)$$

So, $R^{\pm}_j(\mu,\tau) > 0$ and this is in contradiction with (25). So, $\mu \ngeq 0$. It completes the proof. □

Theorem 3. *Let $A_{n\times n} = (\alpha_{ij})$ be a real matrix consisting of the coefficient of system (6). There exist $\tau^* > 0$ such that if $\tau < \tau^*$, then matrix A being a Hurwitz matrix implies that all the roots of characteristic equation (13) have negative real part and hence the zero solution of system (6) is locally asymptotically stable.*

Proof. Suppose that matrix A is a Hurwitz matrix. When all of the eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$ of the matrix A are real (i.e., $\beta_j = a_j$, $a_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$.), by Theorem 1, we know that $\tau^* = \min_{j=1,2,\ldots,n}\{\pi/2|a_j|\}$.

Let $A_{n\times n} = (\alpha_{ij})$ be a real matrix with p real eigenvalues $\beta_j = a_j$, $j = 1,\ldots,p$, and $2q$ complex eigenvalues $\beta^{\pm}_j = a_j \pm ib_j$, $j = p+1,\ldots,n-q$, and let matrix A be a Hurwitz matrix (i.e., $a_j < 0$). If $|a_j| \ge |b_j|$, $j = p+1,\ldots,n-q$, then, by Theorem 2, we know that $\tau^* = \min\{\min_{j=1,2,\ldots,p}\{\pi/2|a_j|\}, \min_{j=p+1,\ldots,n-q}\{\pi/4(|a_j|+|b_j|)\}\}$ and else if there exist k, $p+1 \le k \le n-q$, such that $|a_k| < |b_k|$ and

$$\mathcal{P}^{\pm}_k(\lambda,\tau) = \mu + i\omega + 1 - (a \pm ib)\,e^{-(\mu+i\omega)\tau} = 0, \quad (29)$$

we show that $\mu < 0$. For this purpose, suppose that $\mu \ge 0$ and let $R_k(\mu,\tau)$ and $I_k(\mu,\tau)$ be the real and imaginary parts of (29), respectively; we have

$$R^{\pm}_k(\mu,\tau) = \mu + 1 - a_k e^{-\mu\tau}\cos(\omega\tau) \mp b_k e^{-\mu\tau}\sin(\omega\tau) = 0, \quad (30)$$

$$I^{\pm}_k(\mu,\tau) = \omega + a_k e^{-\mu\tau}\sin(\omega\tau) \mp b_k e^{-\mu\tau}\cos(\omega\tau) = 0. \quad (31)$$

From (31), we have

$$|\omega| \le |a_k| + |b_k|. \quad (32)$$

Supposing that $0 \le \tau \le \min_{j=p+1,\ldots,n-q}\{\pi/4(|a_j| + |b_j|)\}$, by (32), we have $0 \le |\omega|\tau \le (|a_j| + |b_j|)\tau \le \pi/4$. Hence, $|\cos(\omega\tau)| \ge |\sin(\omega\tau)|$ and

$$\cos(\omega\tau) \ge \frac{1}{2}. \quad (33)$$

Again, suppose that $0 \le \tau \le \arcsin(-a_k/2|b_k|)/(|a_k| + |b_k|)$. From (32), we have

$$|\omega\tau| \le \arcsin\left(\frac{-a_k}{2|b_k|}\right) \quad (34)$$

and therefore $|\sin(\omega\tau)| \le (-a_k/2|b_k|)$. So,

$$|b_k|\,|\sin(\omega\tau)| \le \left(\frac{-a_k}{2}\right). \quad (35)$$

On the other hand, from (33), we have

$$-a_k\cos(\omega\tau) \ge \left(\frac{-a_k}{2}\right). \quad (36)$$

From (35) and (36), we have

$$|b_k|\,|\sin(\omega\tau)| \le -a_k\cos(\omega\tau). \quad (37)$$

Thus, from $\mu \geq 0$ and (37), we have

$$\mu + 1 - a_k e^{-\mu\tau} \cos(\omega\tau) \mp b_k e^{-\mu\tau} \sin(\omega\tau) > 0. \quad (38)$$

So, $R_k^{\pm}(\mu, \tau) > 0$ and this is in contradiction with (30). So, $\mu \ngeq 0$. Therefore, for this case, $\tau^* = \min\{\min_{j=1,2,\dots,p}\{\pi/2|a_j|\}, \min_{j=p+1,\dots,n-q}\{\pi/4(|a_j| + |b_j|)\}, \min_{j=p+1,\dots,n-q}\{\arcsin(-a_k/2|b_k|)/(|a_k| + |b_k|)\}\}$. It completes the proof. □

3. Boundedness

Proposition 4. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\sup_{y\in\mathbb{R}}|f(y)| \leq M$, $\tau = \max_{1\leq j\leq n}\tau_j$, and $u : [t_0 - \tau, \infty) \to \mathbb{R}^n$ is a solution of* (3) *with $u(t_0) = 0$, then*

$$\max_{t>t_0}\|u(t)\|_2 \leq M\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\right)^2}. \quad (39)$$

Proof. Let $x : \mathbb{R} \to \mathbb{R}^n$ be the solution of the initial value problem

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij}M, \quad x_i(0) = 0, \qquad i = 1, 2, \dots, n, \quad t \in \mathbb{R}. \quad (40)$$

Then, $x_i(t) = (\sum_{j=1}^{n} a_{ij}M)(1 - e^{-(t-t_0)})$, $i = 1, 2, \dots, n$, $t \in \mathbb{R}$. Clearly, if $u : [t_0 - \tau, \infty) \to \mathbb{R}^n$ is a solution of (3), then

$$\begin{aligned} \dot{u}_i(t) &= -u_i(t) + \sum_{j=1}^{n} a_{ij} f\left(u_j\left(t - \tau_j\right)\right) \\ &\leq -u_i(t) + \sum_{j=1}^{n} a_{ij}M, \quad i = 1, 2, \dots, n, \ t \in \mathbb{R}. \end{aligned} \quad (41)$$

In consequence, Corollary 6.2 of Chapter I in [14] implies that

$$u_i(t) \leq x_i(t) \leq \sum_{j=1}^{n} a_{ij}M, \quad i = 1, 2, \dots, n, \ t > t_0. \quad (42)$$

Therefore, $\max_{t>t_0}\|u(t)\|_2 \leq M\sqrt{\sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij})^2}$. □

The lower bound can be verified analogously.

Corollary 5. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded map with $\sup_{y\in\mathbb{R}}|f(y)| \leq M$ in addition and $p : \mathbb{R} \to \mathbb{R}$ is a periodic solution of* (3) *so that 0 is in the range of p, then*

$$\max_{t\in\mathbb{R}}\|P(t)\|_2 \leq M\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\right)^2}. \quad (43)$$

4. Bifurcation Analysis

Mathematical model is generally the first approximation of any considered real systems. More realistic models should include some of the earlier states of the system; that is, the model should include time delay. In this section, we will consider the effect of the time delay involved in the feedback control. The main attention here will be focused on Andronov-Hopf bifurcation. We know that the number of the eigenvalues of the characteristic equation (13) with negative real parts, counting multiplicities, can change only when the eigenvalues become purely imaginary pairs as the time delay τ and the components of A are varied. Note that these components are independent of the delay. As seen in (13), when $\beta_j \neq 1$, $j = 1, 2, \dots, n$, none of the roots of $\mathcal{P}(\lambda, \tau)$ is zero. Thus, the trivial equilibrium $u = \mathbf{0}$ becomes unstable only when (13) has at least a pair of purely imaginary roots $\pm i\omega$ (i is the imaginary unit) at which an Andronov-Hopf bifurcation occurs. We will determine if the solution curve of the characteristic equation (13) crosses the imaginary axis. We regard the time delay τ as the parameter for considering the Hopf-bifurcation aspect of the trivial equilibrium of the system (3).

Let $A_{n\times n} = (\alpha_{ij})$ be a real matrix consisting of the coefficient of system (6) with p distinct real eigenvalues $\beta_j = a_j \neq 1$, $j = 1, \dots, p$, and $2q$ distinct complex eigenvalues $\beta_j^{\pm} = a_j \pm ib_j$, $j = p + 1, \dots, n - q$. Moreover, suppose that $\tau_i = \tau$ holds for $i = 1, 2, \dots, n$ in system (6). Therefore, the characteristic equation (13) will be transformed to the form (21). Suppose that $\lambda = i\omega$ is a root of (21). This supposition is true if and only if ω satisfies the following equation for some τ:

$$\begin{aligned} \mathcal{P}(i\omega, \tau) = \mathcal{P}_1(i\omega, \tau) \cdots \mathcal{P}_p(i\omega, \tau) & \\ \times \mathcal{P}_{p+1}^{\pm}(i\omega, \tau) \cdots \mathcal{P}_{n-q}^{\pm}(i\omega, \tau) &= 0. \end{aligned} \quad (44)$$

Therefore, there must exist some $1 \leq r \leq p$, such that

$$\mathcal{P}_r(i\omega, \tau) = i\omega + 1 - a_r e^{-(i\omega)\tau} = 0, \quad (45)$$

or there must exist some $p + 1 \leq s \leq n - q$, such that $\mathcal{P}_s^{\pm}(i\omega, \tau) = 0$; that is,

$$\mathcal{P}_s^{\pm}(\lambda, \tau) = i\omega + 1 - \left(a_s \pm ib_s\right) e^{-(i\omega)\tau} = 0. \quad (46)$$

Theorem 6. *Let $A_{n\times n} = (\alpha_{ij})$ be a real matrix consisting of the coefficient of system* (6) *with p distinct real eigenvalues $\beta_j = a_j \neq 1$, $j = 1, \dots, p$, and $2q$ distinct complex eigenvalues $\beta_j^{\pm} = a_j \pm ib_j$, $j = p + 1, \dots, n - q$. If one of the following cases is satisfied*

(1) *for $k = r$ (r introduced above), $a_k < -1$, $\max_{j=p+1,\dots,n-q}\{a_j^2 + b_j^2\} < 1$, and $\max_{j=1,2,\dots,p, j\neq k}|a_j| < 1$, or*

(2) *for $k = s$ (s introduced above), $a_k^2 + b_k^2 > 1$, $\max_{j=1,2,\dots,p}|a_j| < 1$, and $\max_{j=p+1,\dots,n-q, j\neq k}\{a_j^2 + b_j^2\} < 1$,*

then there exist some $\tau^* > 0$ such that the origin of state space of system (3), *when $\tau_k = \tau$ for $k = 1, 2, \ldots, n$, is locally asymptotically stable when $\tau < \tau^*$. Moreover, an Andronov-Hopf bifurcation occurs at the origin of state space of system* (3) *when $\tau = \tau^*$.*

Proof. In the first case, setting the real and imaginary parts of (45) to zero, we have

$$\cos(\omega\tau) = \frac{1}{a_k}, \qquad \sin(\omega\tau) = -\frac{\omega}{a_k}. \tag{47}$$

Taking squares and adding the two above equations, we have $\omega^2 = a_k^2 - 1$. By $a_k < -1$, there is one positive root of $\omega^2 = a_k^2 - 1$. We denote the positive root by $\omega^* = \sqrt{a_k^2 - 1}$. The unique solution $\theta = \omega\tau \in [0, 2\pi]$ of (47) is $\theta = \omega\tau = \arccos(1/a_k)$ since $\sin(\omega\tau) = (-\omega/a_k) > 0$. Therefore, for the imaginary root $\lambda = i\omega$ of (45), we have a sequence $\{\tau^j\}_0^\infty$ as follows:

$$\tau^j = \frac{1}{\omega}\left[\arccos\left(\frac{1}{a_k}\right) + 2j\pi\right], \quad j = 0, 1, 2, 3, \ldots. \tag{48}$$

Assuming that τ^* is the minimum value associated with the imaginary solution $i\omega^*$ of the characteristic equation (45) found above, we determine $\text{sign}\{d\,\text{Re}(\lambda)/d\tau|_{\tau=\tau^*}\}$, where sign is the sign function and $\text{Re}(\lambda)$ is the real part of λ. We assume that $\lambda(\tau) = \mu(\tau) + \omega(\tau)$ is a solution of (21). Thus, $\mu(\tau^*) = 0$ and $\omega(\tau^*) = \omega^*$. Taking derivative of (21) with respect to τ, we have

$$\begin{aligned}
\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau^*}
&= -\left.\frac{(\partial/\tau)\mathscr{P}(\lambda,\tau)}{(\partial/\lambda)\mathscr{P}(\lambda,\tau)}\right|_{\tau=\tau^*} \\
&= -\left(\prod_{\substack{j=1\\ j\neq k}}^{p}\mathscr{P}_j(i\omega^*,\tau^*) \times \prod_{j=p+1}^{n-q}\mathscr{P}_j^{\pm}(i\omega^*,\tau^*)\frac{d}{d\tau}\mathscr{P}_k(i\omega^*,\tau^*)\right) \\
&\quad \times \left(\prod_{\substack{j=1\\ j\neq k}}^{p}\mathscr{P}_j(i\omega^*,\tau^*) \times \prod_{j=p+1}^{n-q}\mathscr{P}_j^{\pm}(i\omega^*,\tau^*)\frac{d}{d\lambda}\mathscr{P}_k(i\omega^*,\tau^*)\right)^{-1} \\
&= -\left.\frac{(\partial/\partial\tau)\mathscr{P}_k(\lambda,\tau)}{(\partial/\partial\lambda)\mathscr{P}_k(\lambda,\tau)}\right|_{\tau=\tau^*} \\
&= -\left.\frac{\lambda a_k e^{-\lambda\tau}}{1 + \tau a_k e^{-\lambda\tau}}\right|_{\tau=\tau^*}.
\end{aligned} \tag{49}$$

From (45) (with $\lambda = i\omega$), we have $e^{-\lambda\tau} = (1+\lambda)/a$. Thus, $(d\lambda/d\tau)^{-1} = -(1/\lambda a_k e^{-\lambda\tau}) - (\tau/\lambda)$. Evaluating $(d\lambda/d\tau)^{-1}$ at $\tau = \tau^*$ (i.e., $\lambda = i\omega^*$) and taking the real part, we have

$$\text{Re}\left[\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\tau=\tau^*}\right] = \frac{1}{1 + (\omega^*)^2} = \frac{1}{a_k^2} > 0. \tag{50}$$

So, $d\,\text{Re}(\lambda)/d\tau$ is positive at $\tau = \tau^*$. Thus, the solution curve of the characteristic equation (45) crosses the imaginary axis. This shows that an Andronov-Hopf bifurcation occurs at $\tau = \tau^* > 0$. When $\tau < \tau^*$, the origin of state space of system (3) is locally asymptotically stable by continuity. Note that if $a_k > 1$, then (45) has a real root $\lambda > 0$ and, in this case, the origin of the state space of system (3) is unstable. On the other hand, as a result of the abovementioned formulas, we can say that the characteristic equation (21) with condition (i) has a simple pair of purely imaginary roots $\pm i\omega^*$ at each τ^j, $j = 0, 1, 2, \ldots$, where τ^j was presented above.

In the second case, setting the real and imaginary parts of (46) to zero results in

$$\begin{aligned}
\cos(\omega\tau) &= \frac{a_k \mp b_k\omega}{a_k^2 + b_k^2}, \\
\sin(\omega\tau) &= \frac{-a_k\omega \mp b_k}{a_k^2 + b_k^2}.
\end{aligned} \tag{51}$$

Taking squares and adding the two equations above together, we have $\omega^2 = a_k^2 + b_k^2 - 1$. By $a_k^2 + b_k^2 > 1$, there is one positive root of $\omega^2 = a_k^2 + b_k^2 - 1$. By $\omega^* = \sqrt{a_k^2 + b_k^2 - 1}$, we denote the positive root. The unique solution $\theta = \omega\tau \in [0, 2\pi]$ of (51) is $\theta = \omega\tau = \arccos((a_k \mp b_k\omega)/(a_k^2 + b_k^2))$ if $\sin(\omega\tau) > 0$, that is, if $-a_k\omega \mp b_k > 0$ and $\theta = \omega\tau = 2\pi - \arccos((a_k - b_k\omega)/(a_k^2 + b_k^2))$, if $\sin(\omega\tau) < 0$, that is, if $-a_k\omega \mp b_k < 0$. Therefore, for the imaginary root $\lambda = i\omega$ of (46), we have two sequences $\{\tau^{1,j}\}_0^\infty$ and $\{\tau^{2,j}\}_0^\infty$ as follows:

$$\begin{aligned}
\tau^{1,j} &= \frac{1}{\omega}\left[\arccos\left(\frac{a_k \mp bk\omega}{a_k^2 + b_k^2}\right) + 2j\pi\right], \\
&\quad - a_k\omega \mp b_k > 0, \quad j = 0, 1, 2, 3, \ldots \\
\tau^{2,j} &= \frac{1}{\omega}\left[2\pi - \arccos\left(\frac{a_k \mp b_k\omega}{a_k^2 + b_k^2}\right) + 2j\pi\right], \\
&\quad - a_k\omega \mp b_k < 0, \quad j = 0, 1, 2, 3, \ldots.
\end{aligned} \tag{52}$$

Assuming that $\tau^* = \min_{j=0,1,2,\ldots}\{\tau^{1,j}, \tau^{2,j}\}$, that is, τ^* is the minimum value associated with the imaginary solution $i\omega^*$ of the characteristic equation (46) found earlier, we determine $\text{sign}\{(d\,\text{Re}(\lambda)/d\tau)|_{\tau=\tau^*}\}$, where sign is the sign function and $\text{Re}(\lambda)$ is the real part of λ. We assume that $\lambda(\tau) = \mu(\tau) + \omega(\tau)$ is a solution of (46). Thus, $\mu(\tau^*) = 0$ and $\omega(\tau^*) = \omega^*$. Similar to the previous case, taking derivative of (46) with respect to τ, we have

$$\frac{d\lambda}{d\tau} = -\frac{\lambda(a_k \pm ib_k)e^{-\lambda\tau}}{1 + \tau(a_k \pm ib_k)e^{-\lambda\tau}}. \tag{53}$$

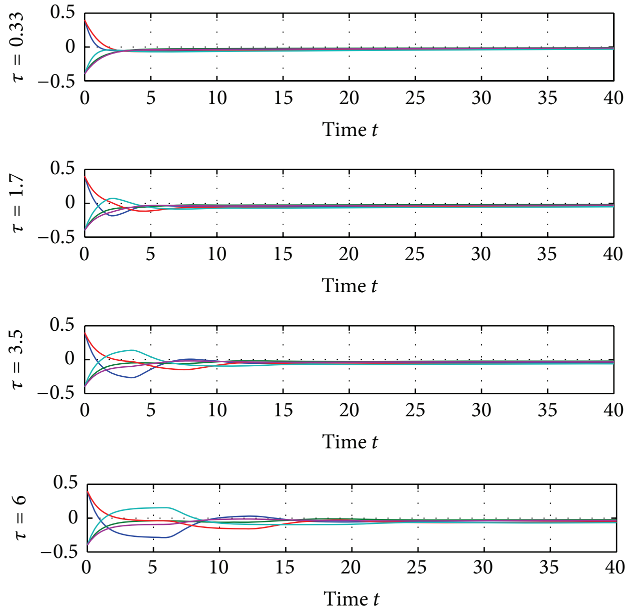

FIGURE 1: Time histories of system (7) when $n = 5$ (Example 1).

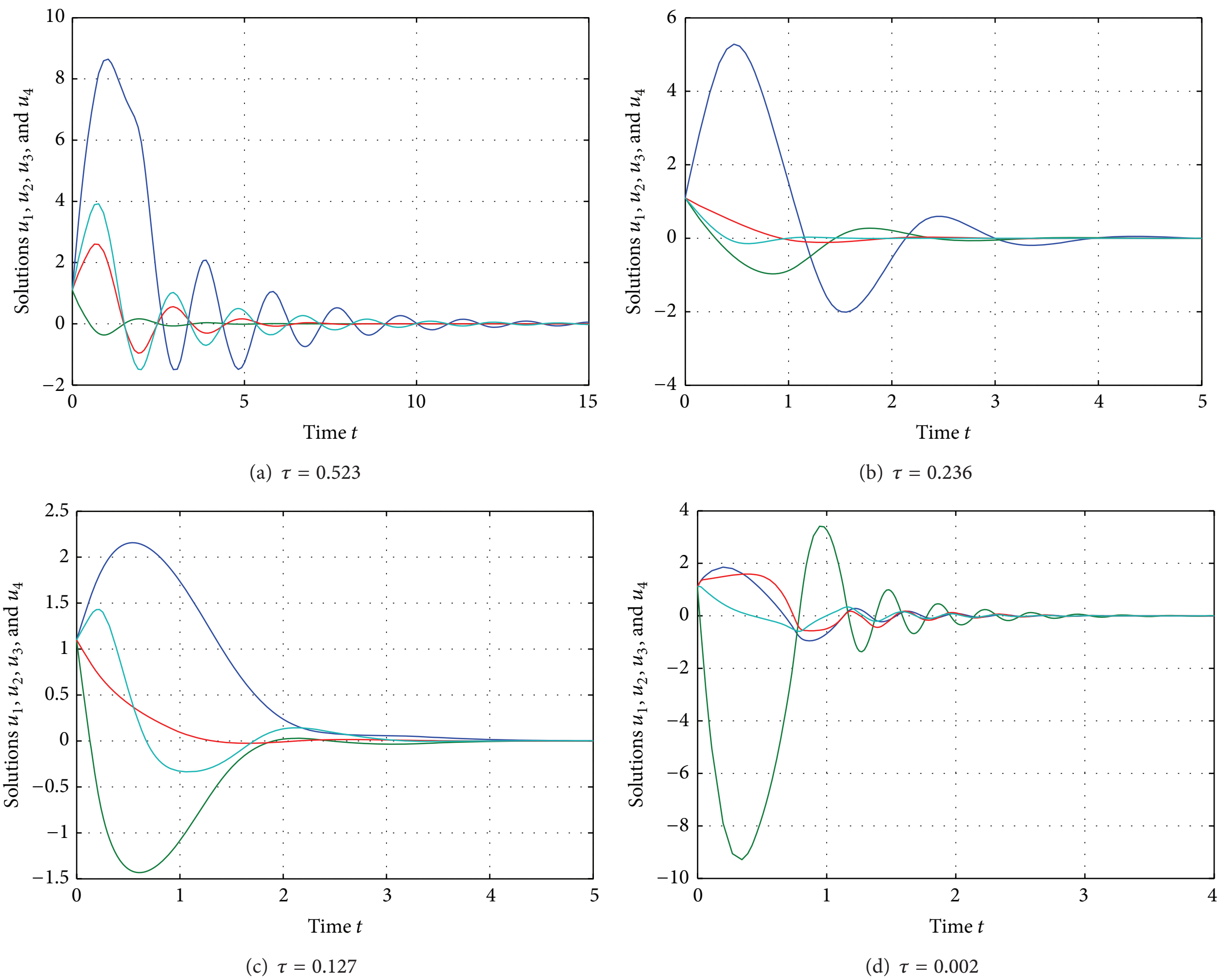

FIGURE 2: Time histories of system (3) when $n = 4$ (Example 2).

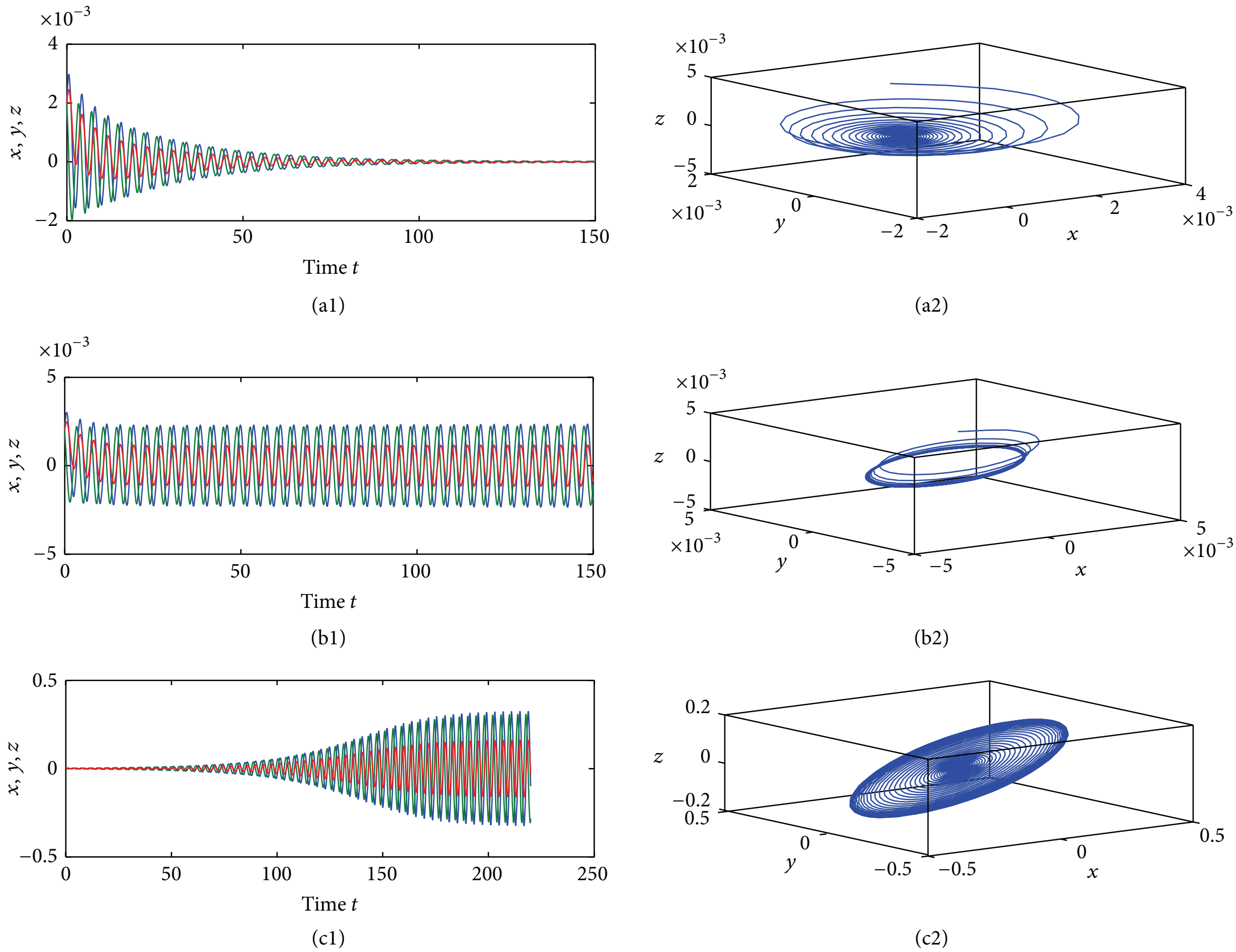

FIGURE 3: Time histories (a1, b1, and c1) and phase trajectories (a2, b2, and c2) of system (3) before, during, and after Andronov-Hopf bifurcation. In (a1) and (a2), $\tau = 0.193814883934925 < 0.213814883934925 = \tau^*$, and then the origin is locally asymptotically stable. In (b1) and (b2), $\tau = 0.213814883934925 = \tau^*$, and then there exists a periodic solution near the origin. In (c1) and (c2), $\tau = 0.233814883934925 > 0.213814883934925 = \tau^*$, and then there also exists a periodic solution near the origin, which implies that the origin is unstable (Example 3).

From (46), we have $e^{-\lambda\tau} = (1+\lambda)/(a_k \pm ib_k)$. Thus, $(d\lambda/d\tau)^{-1} = -(1/\lambda(a_k \pm ib_k)e^{-\lambda\tau}) - (\tau/\lambda)$ is derived. Evaluating $(d\lambda/d\tau)^{-1}$ at $\tau = \tau^*$ (i.e., $\lambda = i\omega^*$) and taking the real part, we have

$$\operatorname{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau^*}\right] = \frac{1}{1+(\omega^*)^2} = \frac{1}{{a_k}^2 + {b_k}^2} > 0. \quad (54)$$

So, $d\operatorname{Re}(\lambda)/d\tau$ is positive at $\tau = \tau^*$. Thus, the solution curve of the characteristic equation (46) crosses the imaginary axis. This shows that an Andronov-Hopf bifurcation occurs at $\tau = \tau^* > 0$. When $\tau < \tau^*$, the origin of state space of system (3) is locally asymptotically stable by continuity. As a result of the abovementioned formulas, we can say that the characteristic equation (21) with condition (ii) has a simple pair of purely imaginary roots $\pm i\omega^*$ at each $\tau^{1,j}$ and $\tau^{2,j}$, $j = 0, 1, 2, \ldots$, where $\tau^{1,j}$ and $\tau^{2,j}$ are given in (52). □

5. Numerical Simulation

For the numerical simulation, a program has been developed in Matlab.

Example 1. We consider system (3) with $n = 5$, $f(x) = \tanh(x)$, and $\tau = 0.3$. Note that $\alpha_{ij} = a_{ij} f'(0) = a_{ij}$ and the parameters were chosen as follows:

$$A = \begin{bmatrix} 0.24 & 0.3 & 0.02 & 0.4 & 0.1 \\ 0.2 & 0.02 & 0.02 & 0.1 & 0.2 \\ 0.5 & 0.01 & 0.1 & 0.2 & 0.5 \\ 0.31 & 0.0025 & 0.52 & 0.21 & 0.21 \\ 0.17 & 0.1 & 0.04 & 0.3 & 0.053 \end{bmatrix}. \quad (55)$$

This gives $\beta_1 = 0.97$, $\beta_{2,3} = -0.23 \mp 0.28i$, $\beta_4 = -0.11$, and $\beta_5 = 0.002$ (approximately). Therefore, the conditions of Theorem 2 part 1 are satisfied. Solutions $u_1, u_2, \ldots, u_5$ in Figure 1 are plotted with respect to t.

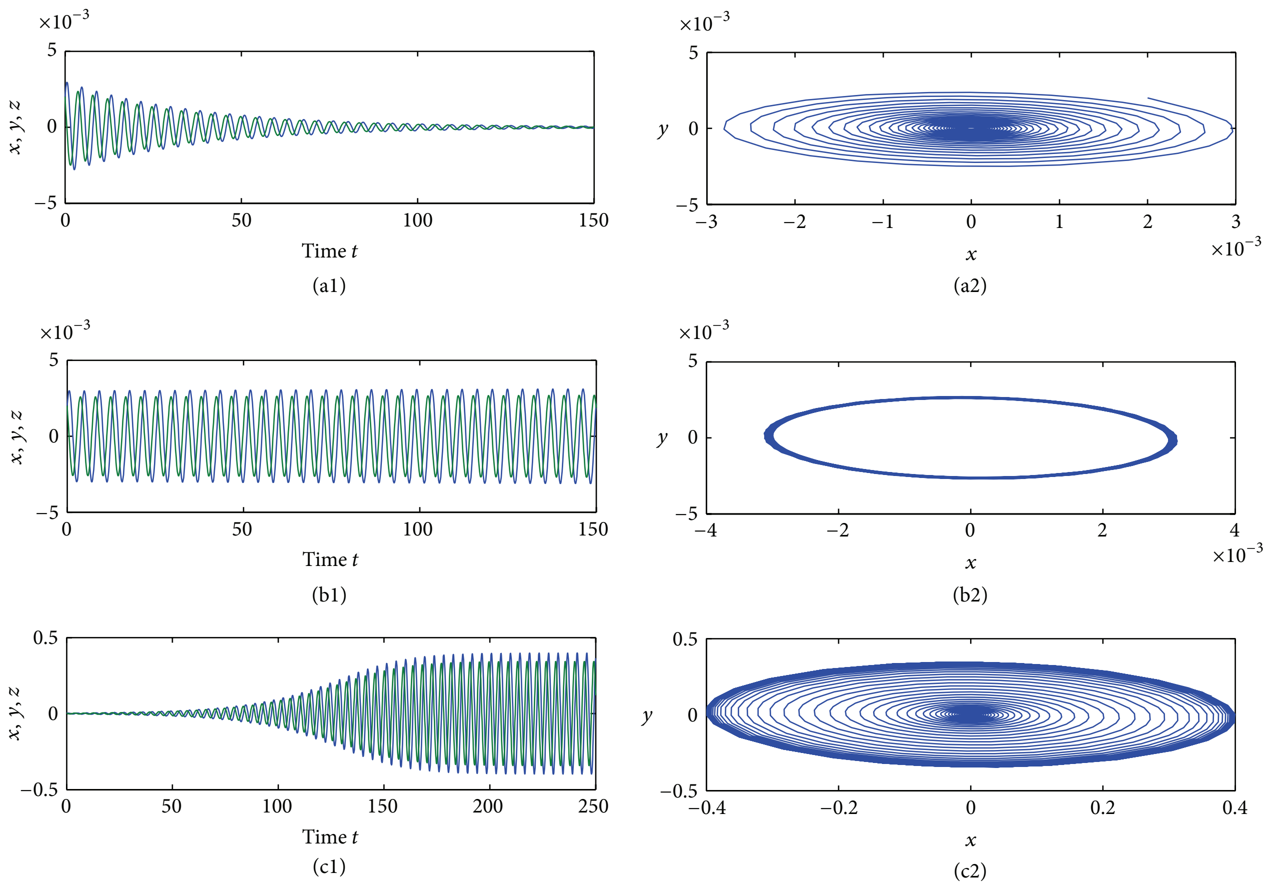

Figure 4: Time histories (a1, b1, and c1) and phase trajectories (a2, b2, and c2) of system (3) before, during, and after Andronov-Hopf bifurcation. In (a1) and (a2), $\tau = 0.233164320601144 < 0.253164320601144 = \tau^*$, and then the origin is locally asymptotically stable. In (b1) and (b2), $\tau = 0.253164320601144 = \tau^*$, and then there exists a periodic solution near the origin. In (c1) and (c2), $\tau = 0.283164320601144 > 0.253164320601144 = \tau^*$, and then there also exists a periodic solution near the origin, which implies that the origin is unstable (Example 4).

Example 2. We consider system (3) with $n = 5$ and $f(x) = \tanh(x)$. Note that $\alpha_{ij} = a_{ij} f'(0) = a_{ij}$ and the parameters were chosen as

$$A = \begin{bmatrix} -1.7 & 10 & 7 & 3 \\ 0 & -2.1 & a_{23} & a_{24} \\ 0 & a_{32} & -1.5 & a_{34} \\ 0 & 11 & 1 & -3 \end{bmatrix}. \quad (56)$$

If $a_{23} = a_{24} = a_{34} = 0$ and $a_{32} = 7$, then τ^* in Theorem 3 is 0.5236. Solutions u_1, u_2, u_3, u_4 in Figure 2(a) are plotted for $\tau = 5230$ with respect to t. If $a_{23} = -6$, $a_{24} = 4$, $a_{32} = 0.4$, and $a_{34} = 0.4$, then τ^* in Theorem 3 is 0.2366. Solutions u_1, u_2, u_3, u_4 in Figure 2(b) are plotted for $\tau = 2360$ with respect to t.

Let

$$A = \begin{bmatrix} -1 & a_{12} & a_{13} & a_{14} \\ a_{21} & -2 & -8 & a_{24} \\ a_{31} & a_{32} & -1.5 & -0.4 \\ 2 & 4 & 1 & -3 \end{bmatrix}. \quad (57)$$

If $a_{12} = 0$, $a_{13} = 7$, $a_{14} = 0$, $a_{21} = 0$, $a_{24} = 0$, $a_{31} = 0$, and $a_{32} = 0.4$, then τ^* in Theorem 3 is 0.1274. Solutions u_1, u_2, u_3, u_4 in Figure 2(c) are plotted for $\tau = 1270$ with respect to t. If $a_{12} = 3$, $a_{13} = 1$, $a_{14} = 12$, $a_{21} = 9$, $a_{24} = -80$, $a_{31} = 11$, and $a_{32} = 7$, then τ^* in Theorem 3 is 0.0021. Solutions u_1, u_2, u_3, u_4 in Figure 2(d) are plotted for $\tau = 0020$ with respect to t.

Example 3. We consider system (3) with $n = 3$ and $f(x) = \tanh(x)$. Note that $\alpha_{ij} = a_{ij} f'(0) = a_{ij}$ and the parameters were chosen as $\alpha_{11} = 0.5$, $\alpha_{12} = 2$, $\alpha_{13} = \alpha_{31} = 0$, $\alpha_{21} = -2.5$, $\alpha_{22} = 0.3$, $\alpha_{23} = 1.4$, $\alpha_{32} = 1$, and $\alpha_{33} = 0.7$. This gives $\beta_1 = 0.775053638587553$ and $\beta_{2,3} = 0.362473180706224 \mp 1.901773087956392i$ (approximately). Therefore, the conditions of Theorem 6 are satisfied and there exists some $\tau^* > 0$ such that the origin of state space of system (3) is locally asymptotically stable when $\tau < \tau^*$. Moreover, an Andronov-Hopf bifurcation occurs at the origin of state space of system (3) when $\tau = \tau^*$. In this example, the value obtained for τ^* is 0.213814883934925. With the abovementioned values of the parameters, Figure 3 is obtained.

Example 4. We consider system (3) with $n = 2$ and $f(x) = \tanh(x)$. Note that $\alpha_{ij} = a_{ij} f'(0) = a_{ij}$ and the parameters were chosen as $\alpha_{11} = 0.5$, $\alpha_{12} = 2$, $\alpha_{21} = -1.5$, $\alpha_{22} = 0.3$. This gives $\beta_{2,3} = 0.400000000000000 \mp 1.729161646579058i$ (approximately). Therefore, the conditions of Theorem 6 are satisfied and there exists some $\tau^* > 0$ such that the origin of state space of system (3) is locally asymptotically stable when $\tau < \tau^*$. Moreover, an Andronov-Hopf bifurcation occurs at the origin of state space of system (3) when $\tau = \tau^*$. In this example, the value obtained for τ^* is 0.213814883934925. With the abovementioned values of the parameters, Figure 4 is obtained.

6. Conclusions

In this paper, we have studied the stability and numerical solutions of a Hopfield delayed neural network system which is more general than the models applied by earlier researchers. In fact, our focus here is on a Hopfield neural network with arbitrary neurons in which each neuron is bidirectionally connected to all the others. By analyzing the associated characteristic transcendental equation, some delay-dependent and delay-independent conditions, which can easily be examined, were established to guarantee the origins of the state space of the model to have local asymptotical stability. Since the characteristic equation of the linearized system at the zero solution involves exponential functions, it is too difficult to investigate the conditions under which the entire characteristic roots have negative real parts. Here, we have reached conditions under which the stability of a matrix consisting of the coefficient of system guarantees the asymptotic stability of the origin of the state space of the network. Also, by considering the feasibility and ease of analyzing the stability of the aforementioned matrix, our approach can be considered as highly practical. Furthermore, we have investigated the occurrence of the Andronov-Hopf bifurcation in the above stated system. Simulation examples have been employed to illustrate the theories. Motivated by the novel method proposed by Leonov and Kuznetsov [15] about finding hidden attractors exploited for nonlinear dynamic systems, we plan to employ their ideas and findings in our upcoming surveys on the Hopfield neural network. To this end, Lyapunov values will undoubtedly play a very practical and significant role.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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