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Research Article

On Harmonious Labeling of Corona Graphs

Martin Bača[1] and Maged Z. Youssef[2,3]

[1] *Department of Applied Mathematics and Informatics, Technical University, 042 00 Košice, Slovakia*
[2] *Department of Mathematics and Statistics, College of Sciences, Al-Imam Mohammed Ibn Saud Islamic University, Riyadh 11623, Saudi Arabia*
[3] *Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo 11566, Egypt*

Correspondence should be addressed to Martin Bača; martin.baca@tuke.sk

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A graph G with q edges is said to be harmonious, if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. In this paper, we study the existence of harmonious labeling for the corona graphs of a cycle and a graph G and for the corona graph of K_2 and a tree.

1. Introduction

Harmonious graphs naturally arose in the study of modular version of error-correcting codes and channel assignment problems. Graham and Sloane [1] defined a (p, q)-graph G of order p and size q to be *harmonious*, if there is an injective function $f : V(G) \to \mathbb{Z}_q$, where $\mathbb{Z}_q$ is the group of integers modulo q, such that the induced function $f^* : E(G) \to \mathbb{Z}_q$, defined by $f^*(xy) = f(x) + f(y)$ for each edge $xy \in E(G)$, is a bijection.

The function f is called *harmonious labeling* and the image of f denoted by $\mathrm{Im}(f)$ is called the corresponding set of vertex labels.

When G is a tree or, in general for a graph G with $p = q+1$, exactly one label may be used on two vertices.

Graham and Sloane [1] proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by 2^k, then q is divisible by 2^{k+1}. This necessary condition is called the harmonious parity condition. They also proved that if f is harmonious labeling of a graph G of size q, then so is $af + b$ labeling, where a is an invertible element of $\mathbb{Z}_q$ and b is any element of $\mathbb{Z}_q$.

Chang et al. [2] define an injective labeling f of a graph G with q edges to be *strongly c-harmonious*, if the vertex labels are from the set $\{0, 1, \ldots, q-1\}$ and the edge labels are from the set $\{f^*(xy) = f(x) + f(y) : xy \in E(G)\} = \{c, c+1, \ldots, c+q-1\}$. Grace [3, 4] called such labeling *sequential*. In the case of a tree, Grace allows the vertex labels to range from 0 up to q. Strongly 1-harmonious graph is called strongly harmonious.

By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. More than 50 papers have been published on harmonious labeling and comprehensive information can be found in [5]. Similarly, labeling of special types of crown graphs is examined in [6].

In this paper, we study the existence of harmonious labeling for the graphs obtained by corona operation between a cycle and a graph G and also between K_2 and a tree or K_2 and a unicyclic graph.

2. Main Results

In this section, we present the results related to corona graphs. The corona operation between two graphs was introduced by Frucht and Harary [7]. Given two graphs G of order p and H, the *corona* of G with H, denoted by $G \odot H$, is the graph with $V(G \odot H) = V(G) \cup \bigcup_{i=1}^{p} V(H_i)$, and $E(G \odot H) = E(G) \cup \bigcup_{i=1}^{p}(E(H_i) \cup \{(v_i, u) : v_i \in V(G) \text{ and } u \in V(H_i)\})$. In other words, a corona graph is obtained from two graphs, G of order p and H, taking one copy of G and p copies of H and joining by an edge the ith vertex of G to every vertex in the ith copy of H.

Grace [4] showed that $C_{2n+1} \odot K_1$ is harmonious and conjectured that $C_{2n} \odot K_1$ is harmonious. This conjecture has been proved by Liu and Zhang [8] and Liu [9]. Singh in [10, 11] has proved that $C_n \odot K_2$ and $C_n \odot K_3$ are sequential for all odd $n > 1$. Santhosh [12] has shown that $C_n \odot P_4$ is sequential for all odd $n \geq 3$.

The *join* of two graphs G and H, denoted by $G + H$, is the graph where $V(G) \cap V(H) = \emptyset$ and each vertex of G is adjacent to all vertices of H. When $H = K_1$, this is the corona graph $K_1 \odot G$.

Graham and Sloane [1] showed harmonious labeling of the join of the path P_n and K_1, that is, the *fan* $F_n = P_n + K_1$, and harmonious labeling of the *double fan* $P_n + \overline{K_2}$. Later, Chang et al. [2] gave harmonious labeling of the join of the star S_n and K_1.

The next result shows that if join of a graph G and K_1 is strongly harmonious, then the corona of a cycle and the graph G admitted harmonious labeling.

Theorem 1. *Let G be a graph of order p and size q. If $G + K_1$ is strongly harmonious with the 0 label on the vertex of K_1, then $C_n \odot G$ is harmonious for all odd $n \geq 3$.*

Proof. Let G be a (p, q)-graph and $G + K_1$ strongly harmonious with the 0 label on the vertex $x \in K_1$. Then, there exists labeling $f : V(G+K_1) \to \{0, \ldots, p+q-1\}$ such that $f(x) = 0$ and the edge labels are from the set $\{f^*(uv) = f(u) + f(v) : uv \in E(G + K_1)\} = \{1, 2, \ldots, p + q\}$.

Now, for n odd, $n \geq 3$, we consider the corona graph $C_n \odot G$ with $n(p+1)$ vertices and $\Gamma = n(p+q+1)$ edges. Denote the vertices and edges of the cycle C_n such that $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(C_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. By the symbol y^i, we denote a vertex in the ith copy of G, denoted by G_i, corresponding to the vertex y in G; that is, $y \in V(G)$ and $y^i \in V(G_i)$.

We define the vertex labeling $g : V(C_n \odot G) \to \{0, 1, \ldots, \Gamma - 1\}$ in the following:

$$g(x_i) = (p+q+1)(i-1), \quad \text{for } 1 \leq i \leq n,$$
$$g(y^i) = f(y) + (p+q+1)(i-1), \quad \text{for } 1 \leq i \leq n. \tag{1}$$

If we denote the join graph $G + K_1$ as $G + \{x\}$, then the set of all edge labels of the ith copy of $G + \{x\}$ consists of the consecutive integers $g^*(E(G_i + \{x_i\})) = \{2(p+q+1)(i-1) + 1, 2(p+q+1)(i-1) + 2, \ldots, 2(p+q+1)(i-1) + p + q\}$, $1 \leq i \leq n$. For edge labels of the cycle C_n, we have $g^*(x_i x_{i+1}) = g(x_i) + g(x_{i+1}) = (p+q+1)(2i-1)$, for $1 \leq i \leq n-1$, and $g^*(x_n x_1) = g(x_n) + g(x_1) = (p+q+1)(n-1)$.

It is not difficult to see that, for $1 \leq i \leq (n-1)/2$, it is true that

(i) $1 + \max\{g^*(E(G_i + \{x_i\}))\} = (p+q+1)(2i-1) = g^*(x_i x_{i+1})$;

(ii) $1 + g^*(x_i x_{i+1}) = \min\{g^*(E(G_{((n+1)/2)+i} + \{x_{((n+1)/2)+i}\}))\} = (p+q+1)(2i-1) + 1 \pmod{\Gamma}$;

(iii) $1 + \max\{g^*(E(G_{((n+1)/2)+i} + \{x_{((n+1)/2)+i}\}))\} = 2(p+q+1)i \pmod{\Gamma}$ and it is equal to $g^*(x_{((n+1)/2)+i} x_{((n+1)/2)+i+1}) \pmod{\Gamma}$;

(iv) $1 + g^*(x_{((n+1)/2)+i} x_{((n+1)/2)+i+1}) \pmod{\Gamma}$ is equal to the $\min\{g^*(E(G_{i+1} + \{x_{i+1}\}))\} = 2(p+q+1)i + 1$.

Moreover, $1 + \max\{g^*(E(G_{(n+1)/2} + \{x_{(n+1)/2}\}))\} = 0 \pmod{\Gamma}$ and it is equal to $g^*(x_{(n+1)/2} x_{(n+3)/2}) = 0 \pmod{\Gamma}$.

Thus, under the induced mapping g^*, all the resulting edge labels are distinct and they get the consecutive integers from 0 up to $n(p+q+1) - 1 \pmod{\Gamma}$. This concludes the proof. □

Graham and Sloane [1] have proved that the fans $F_m = P_m + K_1$, $m \leq 7$, and the wheels $W_m = C_m + K_1$, $m \not\equiv 2 \pmod 3$, are strongly harmonious with the 0 label on the vertex of K_1. In light of these results and Theorem 1, we have the following corollaries.

Corollary 2. *Let $C_n \odot P_m$ be the corona graph of a cycle C_n and a path P_m. Then, $C_n \odot P_m$ is harmonious for all odd $n \geq 3$ and $1 \leq m \leq 7$.*

Corollary 3. *Let $C_n \odot C_m$ be the corona graph of two cycles. Then, $C_n \odot C_m$ is harmonious for all odd $n \geq 3$ and $m \not\equiv 2 \pmod 3$.*

Shee [13] has shown that the complete tripartite graph $K_{1,m,k} = K_{m,k} + K_1$, $m, k \geq 1$, is strongly harmonious, while Gnanajothi [14] proved that $K_{1,1,m,k} = K_{1,m,k} + K_1$, $m, k \geq 1$, is also strongly harmonious. In both cases, the vertex of K_1 is labeled by the 0 label. Thus, with respect to Theorem 1, we obtain the following.

Corollary 4. *For $m, k \geq 1$ and odd $n \geq 3$, the corona graph $C_n \odot K_{m,k}$ is harmonious.*

Corollary 5. *For $m, k \geq 1$ and odd $n \geq 3$, the corona graph $C_n \odot K_{1,m,k}$ is harmonious.*

Let one consider the graphs obtained by corona operation between the single edge K_2 and a tree.

Theorem 6. *If T is a strongly c-harmonious tree of odd size q and $c = (q+1)/2$, then the corona graph $K_2 \odot T$ is also strongly c-harmonious.*

Proof. Let T be a tree of size q with strongly c-harmonious labeling $f : V(T) \to \{0, 1, \ldots, q\}$, where the edge labels are from the set of consecutive integers $\{f^*(e) : e \in E(T)\} = \{c, c+1, \ldots, c+q-1\}$.

Consider the corona graph $K_2 \odot T$ with vertices $x_1, x_2 \in V(K_2)$ and vertices $y^i \in V(T_i)$, $i = 1, 2$, corresponding to the vertices $y \in T$, where the vertex x_i is incident to every vertex in T_i for $i = 1, 2$.

Define now new vertex labeling $g : V(K_2 \odot T) \to \{0, 1, \ldots, 4q+2\}$ such that

$$g(x_i) = \begin{cases} c+q, & \text{for } i = 1, \\ q+1, & \text{for } i = 2, \end{cases}$$

$$g(y^i) = \begin{cases} f(y), & \text{for } i = 1 \text{ and every } y \in T, \\ f(y) + c + q + 1, & \text{for } i = 2 \text{ and every } y \in T. \end{cases} \tag{2}$$

Thus, $\operatorname{Im}(g) = \{0, 1, 2, \ldots, q, q + 1\} \cup \{c + q, c + q + 1, c + q + 2, \ldots, c + 2q, c + 2q + 1\}$ and, for the edge labels, we have

$$\begin{aligned}
&\{g^*(e) : e \in E(T_1)\} = \{c, c+1, c+2, \ldots, c+q-1\},\\
&\{g^*(x_1y^1) = g(x_1) + g(y^1) : y^1 \in V(T_1)\}\\
&\quad = \{c+q, c+q+1, \ldots, c+2q\},\\
&g^*(x_1x_2) = g(x_1) + g(x_2) = c + 2q + 1,\\
&\{g^*(x_2y^2) = g(x_2) + g(y^2) : y^2 \in V(T_2)\}\\
&\quad = \{c+2q+2, c+2q+3, \ldots, c+3q+2\},\\
&\{g^*(e) : e \in E(T_2)\}\\
&\quad = \{3c+2q+2, 3c+2q+3, \ldots, 3c+3q+1\}.
\end{aligned} \tag{3}$$

We can see that edge labels form the set of consecutive integers from c up to $3c+3q+1$ if and only if $\max\{g^*(x_2y^2) = g(x_2) + g(y^2) : y^2 \in V(T_2)\} + 1 = \min\{g^*(e) : e \in E(T_2)\}$; that is, $c = (q + 1)/2$. □

We know that every caterpillar Cat_p admits strongly c-harmonious labeling. As an illustration, Figure 1 provides an example of the strongly 5-harmonious labeling of Cat_{10}.

As an immediate consequence of Theorem 6, we can state the following corollary.

Corollary 7. *Let* Cat_{q+1} *be a caterpillar of odd size q. If* Cat_{q+1} *admits strongly* $(q+1)/2$*-harmonious labeling, then the corona graph* $K_2 \odot \text{Cat}_{q+1}$ *also admits strongly* $(q+1)/2$*-harmonious labeling.*

Theorem 8. *Let G be a unicyclic graph of odd size q. If G is a strongly c-harmonious and* $c = (q-1)/2$*, then the corona graph* $K_2 \odot G$ *is also strongly c-harmonious.*

Proof. Let G be a connected (p, q)-graph containing exactly one cycle. Clearly, $p = q$. Let $f : V(G) \to \{0, 1, \ldots, q-1\}$ be strongly c-harmonious labeling with the edge labels from the set of consecutive integers $\{f^*(e) : e \in E(G)\} = \{c, c+1, \ldots, c+q-1\}$.

If x_1 and x_2 are the vertices of K_2 and if by the symbol y^i we mean a vertex in the ith copy of G corresponding to the vertex $y \in V(G)$, then sets of vertices and edges of the corona graph $K_2 \odot G$ are as follows: $V(K_2 \odot G) = V(K_2) \cup V(G_1) \cup V(G_2)$, $E(K_2 \odot G) = \{x_1x_2\} \cup E(G_1) \cup \{x_1y^1 : y^1 \in V(G_1)\} \cup E(G_2) \cup \{x_2y^2 : y^2 \in V(G_2)\}$.

Define new vertex labeling $g : V(K_2 \odot G) \to \{0, 1, \ldots, 4q\}$ in the following:

$$\begin{aligned}
g(x_i) &= \begin{cases} c+q, & \text{for } i = 1,\\ q, & \text{for } i = 2,\end{cases}\\
g(y^i) &= \begin{cases} f(y), & \text{for } i = 1 \text{ and every } y \in G,\\ f(y)+c+q+1, & \text{for } i = 2 \text{ and every } y \in G.\end{cases}
\end{aligned} \tag{4}$$

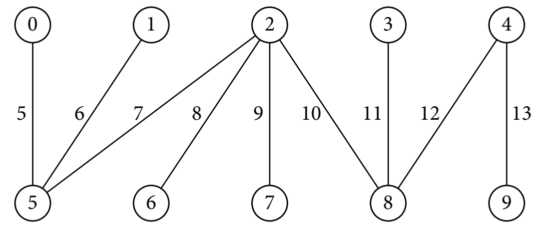

Figure 1: Strongly 5-harmonious labeling of the caterpillar Cat_{10}.

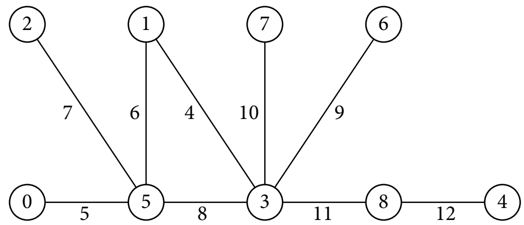

Figure 2: Strongly 4-harmonious labeling of a unicyclic graph.

The image of the vertex labeling g is a union of two sets of consecutive integers $\operatorname{Im}(g) = \{0, 1, 2, \ldots, q\} \cup \{c+q, c+q+1, c+q+2, \ldots, c+2q\}$. Observe that the edge labels are

$$\begin{aligned}
&\{g^*(e) : e \in E(G_1)\} = \{c, c+1, c+2, \ldots, c+q-1\},\\
&\{g^*(x_1y^1) = g(x_1) + g(y^1) : y^1 \in V(G_1)\}\\
&\quad = \{c+q, c+q+1, \ldots, c+2q-1\},\\
&g^*(x_1x_2) = g(x_1) + g(x_2) = c + 2q,\\
&\{g^*(x_2y^2) = g(x_2) + g(y^2) : y^2 \in V(T_2)\}\\
&\quad = \{c+2q+1, c+2q+2, \ldots, c+3q\},\\
&\{g^*(e) : e \in E(T_2)\}\\
&\quad = \{3c+2q+2, 3c+2q+3, \ldots, 3c+3q+1\}.
\end{aligned} \tag{5}$$

The edge labels form the set of consecutive integers from c up to $3c + 3q + 1$ if and only if $c + 3q + 1 = 3c + 2q + 2$. It is true if $c = (q-1)/2$. Thus, the labeling g is strongly $(q-1)/2$-harmonious labeling of the corona graph $K_2 \odot G$. □

An example of the strongly 4-harmonious unicyclic graph is presented in Figure 2.

We know that every odd cycle C_{2n+1} admits strongly n-harmonious labeling. As consequence of Theorem 8, we have the following.

Corollary 9. *The corona graph* $K_2 \odot C_{2n+1}$*,* $n \geq 1$*, is strongly n-harmonious.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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