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Research Article

A Note on Gronwall's Inequality on Time Scales

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This paper gives a new version of Gronwall's inequality on time scales. The method used in the proof is much different from that in the literature. Finally, an application is presented to show the feasibility of the obtained Gronwall's inequality.

1. Introduction and Motivation

Recently, an interesting field of research is to study the dynamic equations on time scales, which have been extensively studied. For example, one can see [1–17] and references cited therein. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The *forward* and *backward jump operators* are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s > t\}$. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be *left dense* if $\rho(t) = t$ and *right dense* if $t < \inf \mathbb{T}$ and $\sigma(t) = t$. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ defined by $\mu(t) = \sigma(t) - t$ is called graininess. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* provided g is continuous at right-dense points. The set of all such rd-continuous functions is denoted by $\mathscr{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. A function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1+\mu(t)p(t) \neq 0$ for $t \in \mathbb{T}$. Denote $\mathscr{R}^+(\mathbb{T}, \mathbb{R}) := \{p \in \mathscr{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}) : 1+\mu(t)p(t) > 0\}$.

One of important topics is the differential inequalities on time scales. A nonlinear version of Gronwall's inequality is presented in [2, Theorem 6.4, pp 256]. This version is stated as follows.

Theorem A. *Let $y, f \in \mathscr{C}_{rd}\mathscr{R}(\mathbb{T}, \mathbb{R})$, $p(t) \in \mathscr{R}^+(\mathbb{T}, \mathbb{R})$, and $p(t) \geq 0$. Then*

$$y(t) \leq f(t) + \int_{t_0}^{t} y(s)\, p(s)\, \Delta s, \quad \forall t \in \mathbb{T}, \tag{1}$$

implies

$$y(t) \leq f(t) + \int_{t_0}^{t} e_p(t, \sigma(s))\, f(s)\, p(s)\, \Delta s, \quad \forall t \in \mathbb{T}. \tag{2}$$

Taking $f(t) \equiv \alpha$, a classical version of Gronwall's inequality follows (see [2, Corollary 6.7, pp 257]).

Theorem B. *Let $p \in \mathscr{R}^+(\mathbb{T}, \mathbb{R})$, $p(t) \geq 0$, $y \in \mathscr{C}_{rd}\mathscr{R}(\mathbb{T}, \mathbb{R})$, and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(s)\, p(s)\, \Delta s, \quad \forall t \in \mathbb{T}, \tag{3}$$

implies

$$y(t) \leq \alpha e_p(t, t_0), \quad \forall t \in \mathbb{T}. \tag{4}$$

This paper presents a new version of Gronwall's inequality as follows.

Theorem 1. *Let $-p \in \mathscr{R}^+(\mathbb{T}, \mathbb{R})$ and $y \in \mathscr{C}_{rd}\mathscr{R}(\mathbb{T}, \mathbb{R})$. Suppose that $p(t) \geq 0$, $y(t) \geq 0$, and $\alpha > 0$. Then*

$$y(t) \leq \alpha + \left| \int_{t_0}^{t} y(s)\, p(s)\, \Delta s \right|, \quad \forall t \in \mathbb{T}, \tag{5}$$

implies

$$y(t) \leq \begin{cases} \alpha e_p(t, t_0), & \text{for } t \in [t_0, +\infty)_{\mathbb{T}}, \\ \alpha e_{-p}(t, t_0), & \text{for } t \in (-\infty, t_0]_{\mathbb{T}}. \end{cases} \tag{6}$$

Remark 2. Note that, for $t \in (-\infty, t_0]_{\mathbb{T}}$, inequality (5) reduces to

$$y(t) \leq \alpha - \int_{t_0}^{t} y(s)\, p(s)\, \Delta s, \tag{7}$$

which is different from inequality (3) in Theorem B. Since Theorem B requires $p(t) \geq 0$, we see that Theorem B cannot be applied to (7). Moreover, the method used to prove Theorem A cannot be used to prove Theorem 1. To explain this, recall the proof of Theorem A in [2]. Let $z(t) = \int_{t_0}^{t} y(s)p(s)\Delta s$. Then $z(t_0) = 0$ and

$$\begin{aligned} z^{\Delta} &= y(t)\, p(t) \\ &\leq [f(t) + z(t)]\, p(t) = p(t)\, z(t) + p(t)\, f(t). \end{aligned} \tag{8}$$

By comparing theorem and variation of constants formula,

$$z(t) \leq \int_{t_0}^{t} e_p(t, \sigma(s))\, f(s)\, p(s)\, \Delta s, \tag{9}$$

and hence Theorem A follows in view of $y(t) \leq f(t) + z(t)$.

Now we try to adopt the same idea used in [2] to estimate inequality (7). Let $z(t) = \int_{t_0}^{t} y(s)p(s)\Delta s$. Then $z(t_0) = 0$ and

$$\begin{aligned} z^{\Delta} &= y(t)\, p(t) \leq [f(t) - z(t)]\, p(t) \\ &= -p(t)\, z(t) + p(t)\, f(t) \\ &= -p(t)\, z^{\sigma} + (1 + \mu(t)\, p(t))\, p(t)\, f(t). \end{aligned} \tag{10}$$

By comparing theorem and variation of constants formula, we have

$$z(t) \leq \int_{t_0}^{t} e_{\ominus p}(t, \sigma(s))\, f(s)\, p(s)\, \Delta s, \tag{11}$$

which implies

$$-z(t) \geq -\int_{t_0}^{t} e_{\ominus p}(t, \sigma(s))\, f(s)\, p(s)\, \Delta s. \tag{12}$$

If we were to use the same idea as in [2], we should combine (12) with

$$y(t) \leq f(t) - z(t). \tag{13}$$

However, on one side, $y(t) \leq \cdots$; on the other side, $f(t) - z(t) \geq \cdots$. These two inequalities cannot lead us anywhere.

Therefore, some novel proof is employed to prove Theorem 1. One can see the detailed proof in the next section.

2. Proof of Main Result

Before our proof of Theorem 1, we need some lemmas.

Lemma 3 (chain rule [2]). *Assume $g : \mathbb{T} \to \mathbf{X}$ is delta differentiable on $\mathbb{T}$. Assume further that $f : \mathbf{X} \to \mathbf{X}$ is continuously differentiable. Then $f \circ g : \mathbb{T} \to \mathbf{X}$ is delta differentiable and satisfies*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)\, g^{\Delta}(t)\right) dh \right\} g^{\Delta}(t). \tag{14}$$

Lemma 4. *Suppose that $g : \mathbb{T} \to \mathbb{R}^+$ is positive delta differentiable on $\mathbb{T}$ and $g^{\Delta}(t)/g(t)$ is regressive. Then $\xi_{\mu(t)}(g^{\Delta}(t)/g(t))$ is a preantiderivative of function $\mathrm{Log}[g(t)]$, where $\xi_h(z) = (1/h)\mathrm{Log}(1 + zh)$ and Log is the principal logarithm function.*

Proof. Let $f(x) = \mathrm{Log}\, x$. Obviously, $f : \mathbb{R}^+ \to \mathbb{R}$ is continuous on $\mathbb{R}^+$. To prove Lemma 4, it suffices to show that $[\mathrm{Log}[g(t)]]^{\Delta} = \xi_{\mu(t)}(g^{\Delta}(t)/g(t))$. In fact, by using Lemma 3, we have

$$\begin{aligned} &[\mathrm{Log}[g(t)]]^{\Delta} \\ &= (f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)\, g^{\Delta}(t)\right) dh \right\} g^{\Delta}(t) \\ &= \left\{ \int_0^1 \frac{1}{g(t) + h\mu(t)\, g^{\Delta}(t)} dh \right\} g^{\Delta}(t) \\ &= \left\{ \frac{1}{\mu(t)\, g^{\Delta}(t)} \mathrm{Log}\left[g(t) + h\mu(t)\, g^{\Delta}(t)\right] \Big|_{h=0}^{h=1} \right\} g^{\Delta}(t) \\ &= \begin{cases} \dfrac{1}{\mu(t)} \left\{ \mathrm{Log}\left[g(t) + \mu(t)\, g^{\Delta}(t)\right] - \mathrm{Log}\left[g(t)\right] \right\} \\ \quad \text{if } \mu(t) \neq 0, \\ \dfrac{g^{\Delta}(t)}{g(t)} \\ \quad \text{if } \mu(t) = 0 \end{cases} \\ &= \begin{cases} \dfrac{1}{\mu(t)} \left\{ \mathrm{Log} \dfrac{g(t) + \mu(t)\, g^{\Delta}(t)}{g(t)} \right\} & \text{if } \mu(t) \neq 0, \\ \dfrac{g^{\Delta}(t)}{g(t)} & \text{if } \mu(t) = 0 \end{cases} \\ &= \frac{1}{\mu(t)} \mathrm{Log} \left\{ 1 + \mu(t) \frac{g^{\Delta}(t)}{g(t)} \right\} \\ &= \xi_{\mu(t)} \left(\frac{g^{\Delta}(t)}{g(t)} \right). \end{aligned} \tag{15}$$

□

Proof of Theorem 1. To prove Theorem 1, we divide it into two cases.

Case 1. For $t \in [t_0, +\infty)_{\mathbb{T}}$, in this case, we have

$$y(t) \leq \alpha + \left| \int_{t_0}^{t} y(s)\, p(s)\, \Delta s \right| = \alpha + \int_{t_0}^{t} y(s)\, p(s)\, \Delta s, \quad \text{for } t \in [t_0, +\infty)_{\mathbb{T}}. \tag{16}$$

Hence, it is easy to conclude that $y(t) \leq \alpha e_p(t, t_0)$ for $t \in [t_0, +\infty)_{\mathbb{T}}$.

Case 2. For $t \in (-\infty, t_0]_{\mathbb{T}}$, let $z(t) = \int_{t_0}^{t} y(s)p(s)\Delta s$. For any $s \in [t, t_0]_{\mathbb{T}}$, we have

$$\begin{aligned} y(s) &\leq \alpha + \left| \int_{t_0}^{s} y(\tau)\, p(\tau)\, \Delta\tau \right| \\ &= \alpha - \int_{t_0}^{s} y(\tau)\, p(\tau)\, \Delta\tau = \alpha - z(s). \end{aligned} \tag{17}$$

Noting that $y \geq 0$, $p \geq 0$, $\alpha > 0$, we have $\alpha - z(s) > 0$. Thus, we have

$$\frac{y(s)}{\alpha - z(s)} \leq 1. \tag{18}$$

Multiplied by $-p(s)$ on both sides of the above inequality, it follows that

$$\frac{-y(s)\, p(s)}{\alpha - z(s)} \geq -p(s), \tag{19}$$

or

$$\frac{[\alpha - z(s)]^{\Delta}}{\alpha - z(s)} \geq -p(s). \tag{20}$$

Since $-p \in \mathcal{R}^+$, $-p \leq [\alpha - z(s)]^{\Delta}/(\alpha - z(s)) \in \mathcal{R}^+$. Using the fact that $\xi_{\mu(t)}(z)$ is nondecreasing with respect to z for $z \in \mathcal{R}^+$, we have

$$\xi_{\mu(s)}\left[\frac{(\alpha - z(s))^{\Delta}}{\alpha - z(s)}\right] \geq \xi_{\mu(s)}[-p(s)]. \tag{21}$$

An integration of the above inequality over $[t, t_0]_{\mathbb{T}}$ leads to

$$\int_{t}^{t_0} \xi_{\mu(s)}\left[\frac{(\alpha - z(s))^{\Delta}}{\alpha - z(s)}\right] \Delta s \geq \int_{t}^{t_0} \xi_{\mu(s)}[-p(s)]\, \Delta s. \tag{22}$$

It follows from Lemma 4 that

$$\mathrm{Log}[\alpha - z(s)]\Big|_{t}^{t_0} \geq \int_{t}^{t_0} \xi_{\mu(s)}[-p(s)]\, \Delta s, \tag{23}$$

or

$$\mathrm{Log}\, \alpha - \mathrm{Log}[\alpha - z(t)] \geq \int_{t}^{t_0} \xi_{\mu(t)}[-p(s)]\, \Delta s, \tag{24}$$

which leads to

$$\begin{aligned} \alpha - z(t) &\leq \alpha \exp\left(-\int_{t}^{t_0} \xi_{\mu(s)}[-p(s)]\, \Delta s\right) \\ &= \alpha \exp\left(\int_{t_0}^{t} \xi_{\mu(s)}[-p(s)]\, \Delta s\right) = \alpha e_{-p}(t, t_0). \end{aligned} \tag{25}$$

Therefore, $y(t) \leq \alpha - z(t) \leq e_{-p}(t, t_0)$ for $t \in (-\infty, t_0]_{\mathbb{T}}$. This completes the proof of Theorem 1. □

3. An Application

Inequality (5) has many potential applications. For instance, it can be used to study the property of the solutions to the dynamic systems. Consider the following linear system:

$$x^{\Delta} = A(t)\, x. \tag{26}$$

Let $X(t, t_0, x_0)$ and $X(t, t_0, \tilde{x}_0)$ be two solutions of (26) satisfying the initial conditions $X(t_0) = x_0$ and $X(t_0) = \tilde{x}_0$, respectively.

Theorem 5. *Suppose that $A(t)$ is bounded on $\mathbb{T}$. Then one has*

$$\begin{aligned} &\|X(t, t_0, x_0) - X(t, t_0, \tilde{x}_0)\| \\ &\quad \leq \begin{cases} \|x_0 - \tilde{x}_0\|\, e_{p_1}(t, t_0), & \textit{for } t \in [t_0, +\infty)_{\mathbb{T}}, \\ \|x_0 - \tilde{x}_0\|\, e_{-p_1}(t, t_0), & \textit{for } t \in (-\infty, t_0]_{\mathbb{T}}, \end{cases} \end{aligned} \tag{27}$$

where $p_1(t) \equiv M$.

Proof. Integrating (7) over $[t_0, t]$, we have

$$\begin{aligned} &X(t, t_0, x_0) \\ &\quad = x_0 + \int_{t_0}^{t} [A(s)\, X(s, t_0, x_0) + f(s, X(s, t_0, x_0))]\, \Delta s. \end{aligned} \tag{28}$$

Denoting $M = \sup_{t \in \mathbb{T}} \|A(t)\|$, simple computation leads us to

$$\begin{aligned} &\|X(t, t_0, x_0) - X(t, t_0, \tilde{x}_0)\| \\ &\quad \leq \|x_0 - \tilde{x}_0\| + M \left| \int_{t_0}^{t} \|X(s, t_0, x_0) - X(s, t_0, \tilde{x}_0)\|\, \Delta s \right|. \end{aligned} \tag{29}$$

By Theorem 1, it follows from (29) that

$$\begin{aligned} &\|X(t, t_0, x_0) - X(t, t_0, \tilde{x}_0)\| \\ &\quad \leq \begin{cases} \|x_0 - \tilde{x}_0\|\, e_{p_1}(t, t_0), & \text{for } t \in [t_0, +\infty)_{\mathbb{T}}, \\ \|x_0 - \tilde{x}_0\|\, e_{-p_1}(t, t_0), & \text{for } t \in (-\infty, t_0]_{\mathbb{T}}. \end{cases} \end{aligned} \tag{30}$$

□

Remark 6. One can see that, for the case $t \in (-\infty, t_0]_{\mathbb{T}}$, (29) reduces to

$$\begin{aligned} &\|X(t, t_0, x_0) - X(t, t_0, \tilde{x}_0)\| \\ &\leq \|x_0 - \tilde{x}_0\| - M \int_{t_0}^{t} [\|X(s, t_0, x_0) - X(s, t_0, \tilde{x}_0)\|]\, \Delta s. \end{aligned} \tag{31}$$

As you see, Theorem B cannot be used to (31) because the essential condition in Theorem B is $p(t) \geq 0$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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