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Research Article

Riemann-Liouville and Higher Dimensional Hardy Operators for NonNegative Decreasing Function in $L^{p(\cdot)}$ Spaces

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One-weight inequalities with general weights for Riemann-Liouville transform and n-dimensional fractional integral operator in variable exponent Lebesgue spaces defined on $\mathbb{R}^n$ are investigated. In particular, we derive necessary and sufficient conditions governing one-weight inequalities for these operators on the cone of nonnegative decreasing functions in $L^{p(x)}$ spaces.

1. Introduction

We derive necessary and sufficient conditions governing the one-weight inequality for the Riemann-Liouville operator

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad 0 < \alpha < 1 \tag{1}$$

and n-dimensional fractional integral operator

$$I_\alpha g(x) = \frac{1}{|x|^\alpha} \int_{|y|<|x|} \frac{g(t)}{|x-t|^{n-\alpha}} dt \quad 0 < \alpha < n, \tag{2}$$

on the cone of nonnegative decreasing function in $L^{p(x)}$ spaces.

In the last two decades a considerable interest of researchers was attracted to the investigation of the mapping properties of integral operators in so-called Nakano spaces $L^{p(\cdot)}$ (see, e.g., the monographs [1, 2] and references therein). Mathematical problems related to these spaces arise in applications to mechanics of the continuum medium. For example, Ružicka [3] studied the problems in the so-called rheological and electrorheological fluids, which lead to spaces with variable exponent.

Weighted estimates for the Hardy transform

$$(H_1 f)(x) = \int_0^x f(t)\, dt, \quad x > 0, \tag{3}$$

in $L^{p(\cdot)}$ spaces were derived in the papers [4] for power-type weights and in [5–9] for general weights. The Hardy inequality for nonnegative decreasing functions was studied in [10, 11]. Furthermore Hardy type inequality was studied in [12, 13] by Rafeiro and Samko in Lebesgue spaces with variable exponent.

Weighted problems for the Riemann-Liouville transform in $L^{p(x)}$ spaces were explored in the papers [5, 14–16] (see also the monograph [17]).

Historically, one and two weight Hardy inequalities on the cone of nonnegative decreasing functions defined on $\mathbb{R}_+$ in the classical Lebesgue spaces were characterized by Arino and Muckenhoupt [18] and Sawyer [19], respectively.

It should be emphasized that the operator $I_\alpha f(x)$ is the weighted truncated potential. The trace inequity for this operator in the classical Lebesgue spaces was established by Sawyer [20] (see also the monograph [21], Ch.6 for related topics).

In general, the modular inequality

$$\int_0^1 \left| \int_0^x f(t)\, dt \right|^{q(x)} v(x)\, dx \le c \int_0^1 |f(t)|^{p(t)} w(t)\, dt \tag{$*$}$$

for the Hardy operator is not valid (see [22], Corollary 2.3, for details). Namely, the following fact holds: if there exists a positive constant c such that inequality $(*)$ is true for all

$f \geq 0$, where q; p; w; and v are nonnegative measurable functions, then there exists $b \in [0,1]$ such that $w(t) > 0$ for almost every $t < b$; $v(x) = 0$ for almost every $x > b$, and $p(t)$ and $q(x)$ take the same constant values a.e. for $t \in (0;b)$ and $x \in (0;b) \cap \{v \neq 0\}$.

To get the main result we use the following pointwise inequalities:

$$c_1 (Tf)(x) \leq (R_\alpha f)(x) \leq c_2 (Tf)(x), \quad c_3 (Hg)(x) \leq (I_\alpha g)(x) \leq c_4 (Hg)(x), \tag{4}$$

for nonnegative decreasing functions, where c_1, c_2, c_3, and c_4 are constants and are independent of f, g, and x, and

$$Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad Hg(x) = \frac{1}{|x|^n}\int_{|y|<|x|} g(y)\,dy. \tag{5}$$

In the sequel by the symbol $Tf \approx Tg$ we mean that there are positive constants c_1 and c_2 such that $c_1 Tf(x) \leq Tg(x) \leq c_2 Tf(x)$. Constants in inequalities will be mainly denoted by c or C; the symbol $\mathbb{R}_+$ means the interval $(0,+\infty)$.

2. Preliminaries

We say that a radial function $f : \mathbb{R}^n \to \mathbb{R}_+$ is decreasing if there is a decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(|x|) = f(x)$, $x \in \mathbb{R}^n$. We will denote g again by f. Let $p : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function, satisfying the conditions $p^- = \operatorname{essinf}_{x\in\mathbb{R}^n} p(x) > 0$, $p^+ = \operatorname{esssup}_{x\in\mathbb{R}^n} p(x) < \infty$.

Given $p : \mathbb{R}^n \to \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$ and a nonnegative measurable function (weight) u in $\mathbb{R}^n$, let us define the following local oscillation of p:

$$\varphi_{p(\cdot),u}(\delta) = \operatorname*{esssup}_{x\in B(0,\delta)\cap \operatorname{supp} u} p(x) - \operatorname*{essinf}_{x\in B(0,\delta)\cap \operatorname{supp} u} p(x), \tag{6}$$

where $B(0,\delta)$ is the ball with center 0 and radius δ.

We observe that $\varphi_{p(\cdot),u}(\delta)$ is nondecreasing and positive function such that

$$\lim_{\delta\to\infty} \varphi_{p(\cdot),u}(\delta) = p_u^+ - p_u^-, \tag{7}$$

where p_u^+ and p_u^- denote the essential infimum and supremum of p on the support of u, respectively.

By the similar manner (see [10]) the function $\psi_{p(\cdot),u}(\eta)$ is defined for an exponent $p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and weight v on $\mathbb{R}_+$:

$$\psi_{p(\cdot),v}(\eta) = \operatorname*{esssup}_{x\in(0,\eta)\cap \operatorname{supp} v} p(x) - \operatorname*{essinf}_{x\in(0,\eta)\cap \operatorname{supp} v} p(x). \tag{8}$$

Let $D(\mathbb{R}_+)$ be the class of nonnegative decreasing functions on $\mathbb{R}_+$ and let $DR(\mathbb{R}^n)$ be the class of all nonnegative radially decreasing functions on $\mathbb{R}^n$. Suppose that u is measurable a.e. positive function (weight) on $\mathbb{R}^n$. We denote by $L^{p(x)}(u,\mathbb{R}^n)$ the class of all nonnegative functions on $\mathbb{R}^n$ for which

$$S_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} u(x)\,d\mu(x) < \infty. \tag{9}$$

For essential properties of $L^{p(x)}$ spaces we refer to the papers [23, 24] and the monographs [1, 2].

Under the symbol $L^{p(x)}_{\text{dec}}(u,\mathbb{R}_+)$ we mean the class of nonnegative decreasing functions on $\mathbb{R}_+$ from $L^{p(x)}(u,\mathbb{R}^n) \cap DR(\mathbb{R}^n)$.

Now we list the well-known results regarding one-weight inequality for the operator T. For the following statement we refer to [18].

Theorem A. *Let r be constant such that $0 < r < \infty$. Then the inequity*

$$\int_0^\infty v(x)(Tf(x))^r dx \leq c\int_0^\infty v(x)(f(x))^r dx, \quad f \in L^r(v,\mathbb{R}_+),\ f\downarrow \tag{10}$$

for a weight v holds, if and only if there exists a positive constant C such that for all $s > 0$

$$\int_s^\infty \left(\frac{s}{x}\right)^r v(x)\,dx \leq C\int_0^s v(x)\,dx. \tag{11}$$

Condition (11) is called B_r condition and was introduced in [18].

Theorem B (see [10]). *Let v be a weight on $(0,\infty)$ and $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\psi_{p(\cdot),v(0^+)} = 0$. The following facts are equivalent:*

(a) *there exists a positive constant c such that, for any $f \in D(\mathbb{R}_+)$,*

$$\int_0^\infty (Tf(x))^{p(x)} v(x)\,dx \leq C\int_0^\infty (f(x))^{p(x)} v(x)\,dx; \tag{12}$$

(b) *for any $r, s > 0$,*

$$\int_r^\infty \left(\frac{r}{sx}\right)^{p(x)} v(x)\,dx \leq C\int_0^r \frac{v(x)}{s^{p(x)}}dx; \tag{13}$$

(c) *$p_{|_{\operatorname{supp} v}} \equiv p_0$ a.e. and $v \in B_{p_0}$.*

Proposition 1. *For the operators T, H, R_α, and I_α, the following relations hold:*

(a)

$$R_\alpha f \approx Tf, \quad 0 < \alpha < 1,\ f \in D(\mathbb{R}_+); \tag{14}$$

(b)

$$I_\alpha g \approx Hg, \quad 0 < \alpha < n,\ g \in DR(\mathbb{R}^n). \tag{15}$$

Proof. (a) Upper estimate: represent $R_\alpha f$ as follows:

$$R_\alpha f(x) = \frac{1}{x^\alpha}\int_0^{x/2} \frac{f(t)}{(x-t)^{1-\alpha}}dt + \frac{1}{x^\alpha}\int_{x/2}^x \frac{f(t)}{(x-t)^{1-\alpha}}dt = S_1(x) + S_2(x). \tag{16}$$

Observe that if $t < x/2$, then $x/2 < x - t$. Hence

$$S_1(x) \le c\frac{1}{x}\int_0^{x/2} f(t)\,dt \le cTf(x), \tag{17}$$

where the positive constant c does not depend on f and x. Using the fact that f is decreasing we find that

$$S_2(x) \le cf\left(\frac{x}{2}\right) \le cTf(x). \tag{18}$$

Lower estimate follows immediately by using the fact that f is nonnegative and the obvious estimate $x - t \le x$ and $0 < t < x$.

(b) Upper estimate: let us represent the operator I_α as follows:

$$\begin{aligned} I_\alpha g(x) &= \frac{1}{|x|^\alpha}\int_{|y|<|x|/2}\frac{g(y)}{|x-y|^{n-\alpha}}dy \\ &\quad + \frac{1}{|x|^\alpha}\int_{|x|/2<|y|<|x|}\frac{g(y)}{|x-y|^{n-\alpha}}dy \\ &=: S_1'(x) + S_2'(x). \end{aligned} \tag{19}$$

Since $|x|/2 \le |x-y|$ for $|y| < |x|/2$ we have that

$$S_1'(x) \le \frac{c}{|x|^n}\int_{|y|<|x|/2} g(y)\,dy \le cHg(x). \tag{20}$$

Taking into account the fact that f is radially decreasing on $\mathbb{R}^n$ we find that there is a decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$S_2'(x) \le f\left(\frac{|x|}{2}\right)\cdot\frac{1}{|x|^\alpha}\int_{|x|/2<|y|<|x|}|x-y|^{\alpha-n}dy. \tag{21}$$

Let $F_x = \{y : |x|/2 < |y| < |x|\}$. Then we have

$$\begin{aligned} &\int_{F_x}|x-y|^{\alpha-n}dy \\ &= \int_0^\infty \left|\left\{y \in F_x : |x-y|^{\alpha-n} > t\right\}\right| dt \\ &\le \int_0^{|x|^{\alpha-n}} \left|\left\{y \in F_x : |x-y|^{\alpha-n} > t\right\}\right| dt \\ &\quad + \int_{|x|^{\alpha-n}}^\infty \left|\left\{y \in F_x : |x-y|^{\alpha-n} > t\right\}\right| dt \\ &=: I_1 + I_2. \end{aligned} \tag{22}$$

It is easy to see that

$$I_1 \le \int_0^{|x|^{\alpha-n}} |B(0,|x|)|\,dt = c|x|^\alpha; \tag{23}$$

while using the fact that $n/(n-\alpha) > 1$ we find that

$$\begin{aligned} I_2 &\le \int_{|x|^{\alpha-n}}^\infty \left|\left\{y \in F_x : |x-y| \le t^{1/(\alpha-n)}\right\}\right| dt \\ &\le c\int_{|x|^{\alpha-n}}^\infty t^{n/(\alpha-n)}dt = c_{\alpha,n}|x|^\alpha. \end{aligned} \tag{24}$$

Finally we conclude that

$$S_2'(x) \le cf\left(\frac{|x|}{2}\right) \le cHf(x). \tag{25}$$

Lower estimate follows immediately by using the fact that f is nonnegative and the obvious estimate $|x-y| \le |x|$, where $0 < |y| < |x|$. □

We will also need the following statement.

Lemma 2. *Let r be a constant such that $0 < r < \infty$. Then the inequality*

$$\int_{\mathbb{R}^n}(Hf(x))^r u(x)\,dx \le C\int_{\mathbb{R}^n}(f(x))^r u(x)\,dx, \qquad f \in L^r_{dec}(u,\mathbb{R}^n), \tag{26}$$

holds, if and only if there exists a positive constant C such that, for all $s > 0$,

$$\begin{aligned} &\int_{|x|>s}\left(\frac{s}{|x|}\right)^r |x|^{r(1-n)}u(x)\,dx \\ &\le C\int_{|x|<s}|x|^{r(1-n)}\,u(x)\,dx. \end{aligned} \tag{27}$$

Proof. We will see that inequality (26) is equivalent to the inequality

$$\int_0^\infty \tilde{u}(t)\left(T\overline{f}(t)\right)^r dt \le C\int_0^\infty \tilde{u}(t)\left(\overline{f}(t)\right)^r dt, \tag{28}$$

where $\tilde{u}(t) = t^{(n-1)(1-r)}\overline{u}(t)$, $\overline{f}(t) = t^{n-1}f(t)$, and $\overline{u}(t) = \int_{S^{n-1}}u(t\overline{x})d\sigma(\overline{x})$.

Indeed, using polar coordinates in $\mathbb{R}^n$ we have

$$\begin{aligned} &\int_{\mathbb{R}^n}(Hf(x))^r u(x)\,dx \\ &= \int_{\mathbb{R}^n}u(x)\left(\frac{1}{t^n}\int_{|y|<t}f(y)\,dy\right)^r dx \\ &= \int_0^\infty t^{n-1}\left(\frac{1}{t^n}\int_{|y|<t}f(y)\,dy\right)^r\left(\int_{S^{n-1}}u(t\overline{x})\,d\sigma\overline{x}\right)dt \\ &= C\int_0^\infty t^{n-1}t^{-nr}t^r\left(\frac{1}{t}\int_0^t \tau^{n-1}f(\tau)\,d\tau\right)^r\overline{u}(t)\,dt \\ &= C\int_0^\infty t^{n-1}t^{r(1-n)}\overline{u}(t)\left(\frac{1}{t}\int_0^t\overline{f}(\tau)\,d\tau\right)^r dt \\ &\le C\int_0^\infty \tilde{u}(t)\left(\overline{f}(t)\right)^r dt \\ &= C\int_0^\infty t^{(n-1)(1-r)}t^{(n-1)r}\left(f(t)\right)^r dt \\ &= C\int_{\mathbb{R}^n}(f(x))^r u(x)\,dx. \end{aligned} \tag{29}$$

Conversely taking the test function $f_r(x) = \chi_{B(0,r)}(x)|x|^{1-n}$, $r > 0$, in modular inequality (26), one can easily obtain inequality (27). □

3. The Main Results

To formulate the main results we need to prove the following proposition.

Proposition 3. *Let u be a weight on $\mathbb{R}^n$ and $p:\mathbb{R}^n \to \mathbb{R}_+$ such that $0 < p^- \le p^+ < \infty$, and assume that $\varphi_{p(\cdot),u(0+)} = 0$. The following statements are equivalent:*

(a) *there exists a positive constant C such that, for any $f \in DR(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (Hf(x))^{p(x)} u(x)\,dx \le C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x)\,dx; \quad (30)$$

(b) *for any $r, s > 0$,*

$$\int_{|x|>r} \left(\frac{r}{s|x|^n}\right)^{p(x)} u(x)\,dx \le C \int_{B(0,r)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}}\,dx; \quad (31)$$

(c) $p_{|_{\mathrm{supp}\,u}} \equiv p_0$ *a.e. and* $u \in B_{p_0}$.

Proof. We use the arguments of [10]. To show that (a) implies (b) it is enough to test the modular inequality (30) for the function $f_{r,s}(x) = (1/s)\chi_{B(0,r)}(x)|x|^{1-n}$, $s, r > 0$. Indeed, it can be checked that

$$Hf_{r,s}(x) = \begin{cases} \dfrac{1}{|x|^n s} \displaystyle\int_{|y|\le|x|} |y|^{1-n}\,dy, & \text{if } |x| \le r; \\ \dfrac{1}{|x|^n s} \displaystyle\int_{|y|\le r} |y|^{1-n}\,dy, & \text{if } |x| > r. \end{cases} \quad (32)$$

Further, we find that

$$\begin{aligned} &\int_{|x|>r} u(x)\,(Hf_{r,s})^{p(x)}\,dx \\ &\le \int_{\mathbb{R}^n} u(x)\,(Hf_{r,s})^{p(x)}\,dx \\ &\le C \int_{\mathbb{R}^n} u(x) \left(\frac{1}{s}\chi_{B(0,r)}(x)\,|x|^{1-n}\right)^{p(x)} dx. \end{aligned} \quad (33)$$

Therefore

$$\int_{|x|>r} u(x) \left(\frac{r}{s|x|^n}\right)^{p(x)} dx \le C \int_{B(0,r)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}}\,dx. \quad (34)$$

To obtain (c) from (b) we are going to prove that condition (b) implies that $\varphi_{p(\cdot),u(\delta)}$ is a constant function; namely, $\varphi_{p(\cdot),u(\delta)} = p_u^+ - p_u^-$ for all $\delta > 0$. This fact and the hypothesis on $\varphi_{p(\cdot),u(\delta)}$ imply that $\varphi_{p(\cdot),u(\delta)} \equiv 0$, and hence, due to (7),

$$p_{|_{\mathrm{supp}\,u}} \equiv p_u^+ - p_u^- \equiv p_0 \quad \text{a.e.} \quad (35)$$

Finally (31) means that $u \in B_{p_0}$. Let us suppose that $\varphi_{p(\cdot),u}$ is not constant. Then one of the following conditions holds:

(i) there exists $\delta > 0$ such that

$$\alpha = \operatorname*{esssup}_{x\in B(0,\delta)\cap \mathrm{supp}\,u} p(x) < p_u^+ < \infty, \quad (36)$$

and, hence, there exists $\epsilon > 0$ such that

$$\left|\{|x| > \delta : p(x) \ge \alpha + \epsilon\} \cap \mathrm{supp}\,u\right| > 0, \quad (37)$$

or

(ii) there exists $\delta > 0$ such that

$$\beta = \operatorname*{essinf}_{x\in B(0,\delta)\cap \mathrm{supp}\,u} p(x) > p_u^- > 0, \quad (38)$$

and then, for some $\epsilon > 0$,

$$\left|\{|x| > \delta : p(x) \le \beta - \epsilon\} \cap \mathrm{supp}\,u\right| > 0. \quad (39)$$

In case (i) we observe that condition (b), for $r = \delta$, implies that

$$\int_{|x|>\delta} \left(\frac{\delta}{s}\right)^{p(x)} \frac{u(x)}{|x|^{np(x)}}\,dx \le C \int_{B(0,\delta)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}}\,dx. \quad (40)$$

Then using (36) we obtain, for $s < \min(1, \delta)$,

$$\begin{aligned} &\left(\frac{\delta}{s}\right)^{\alpha+\epsilon} \int_{\{|x|\ge\delta : p(x)\ge\alpha+\epsilon\}} \frac{u(x)}{|x|^{np(x)}}\,dx \\ &\le \frac{C}{s^{\alpha}} \int_{B(0,\delta)} u(x)\,|x|^{(1-n)p(x)}\,dx, \end{aligned} \quad (41)$$

which is clearly a contradiction if we let $s \downarrow 0$. Similarly in case (ii) let us consider the same condition (b), for $r = \delta$, and fix now $s > 1$. Taking into account (38) we find that

$$\begin{aligned} &\frac{1}{s^{\beta-\epsilon}} \int_{\{|x|\ge\delta : p(x)\le\beta-\epsilon\}} \left(\frac{\delta}{|x|^n}\right)^{p(x)} u(x)\,dx \\ &\le \frac{C}{s^{\beta}} \int_{B(0,\delta)} |x|^{(1-n)p(x)} u(x)\,dx, \end{aligned} \quad (42)$$

which is a contradiction if we let $s \uparrow \infty$.

Finally, the fact that condition (c) implies (a) follows from [18, Theorem 1.7]. □

Theorem 4. *Let u be a weight on $(0,\infty)$ and $p:\mathbb{R}_+ \to \mathbb{R}_+$ such that $0 < p^- \le p^+ < \infty$. Assume that $\psi_{p(\cdot),v(0^+)} = 0$. The following facts are equivalent:*

(i) *there exists a positive constant C such that, for any $f \in D(\mathbb{R}_+)$,*

$$\begin{aligned} &\int_{\mathbb{R}_+} (R_\alpha f(x))^{p(x)} u(x)\,dx \\ &\le C \int_{\mathbb{R}_+} (f(x))^{p(x)} u(x)\,dx; \end{aligned} \quad (43)$$

(ii) *condition* (13) *holds;*

(iii) *condition* (*c*) *of Theorem B is satisfied.*

Proof. Proof follows by using Theorem B and Proposition 1(a). □

Theorem 5. *Let u be a weight on $\mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}_+$ such that $0 < p^- \le p^+ < \infty$, and assume that $\varphi_{p(\cdot),u(0^+)} = 0$. The following facts are equivalent:*

(i) *there exists a positive constant C such that, for any $f \in DR(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (I_\alpha f(x))^{p(x)} u(x)\,dx \le C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x)\,dx; \quad (44)$$

(ii) *condition* (31) *holds;*

(iii) *condition (c) of Proposition 3 holds.*

Proof. Proof follows by using Propositions 3 and 1(b). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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