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Research Article

Solving a Class of Singularly Perturbed Partial Differential Equation by Using the Perturbation Method and Reproducing Kernel Method

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We give the analytical solution and the series expansion solution of a class of singularly perturbed partial differential equation (SPPDE) by combining traditional perturbation method (PM) and reproducing kernel method (RKM). The numerical example is studied to demonstrate the accuracy of the present method. Results obtained by the method indicate the method is simple and effective.

1. Introduction

Singularly perturbed problems (SPPs) arise very frequently in many branches of mathematics such as fluid mechanics and chemical reactor theory. It is well known that the solutions of SPPs exhibit a multiscale character. So there are some major computation difficulties. In recent years, many special methods have been developed to deal with SPPs. Many papers [1–4] are devoted to SPPs of ordinary differential equation and the authors discussed the situation and width of boundary layer(s) and give some effective numerical algorithms. But few papers [5–7] deal with SPPDE.

The reproducing kernel Hilbert function space has been shown in [8–10] to solve a large class of linear and nonlinear problems effectively. However, in [8–10], it cannot be used directly to SPPs. The aim of this work is to fill this gap. In this paper, we solve a class of SPPs in reproducing kernel space. By using a traditional perturbation method and RKM, the series expansion solution of a class of SPPDE is given. The main contribution of this paper is to use RKM in SPPDE. The reason why we use this method is that we aim to solve some problems in many areas of science and improve high precision.

Let us consider the following SPPDE:

$$\begin{gathered}\varepsilon \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = F(x,t), \quad t \in [0,1],\ x \in [0,1], \\ U(0,t) = 0, \quad U(x,0) = \nu(x), \quad x \in [0,1],\end{gathered} \tag{1}$$

where $\varepsilon \ll 1$ is a positive number, functions $f(x,t)$ and $\nu(x)$ are sufficiently smooth, and $\nu(0) = 0$. Under suitable continuity and compatibility conditions, the problem (1) has a unique solution $U(x,t)$. In [5–7], we notice that a small variation in the parameter ε produces a large variation in the solution. It is quite well known that solution of such problems involves boundary layers.

2. Perturbation Method

Let $u(x,t) = U(x,t) - \nu(x)$; (1) can be equivalently turned into

$$\begin{gathered}\varepsilon \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u\frac{d\nu}{dx} + \nu\frac{\partial u}{\partial x} = f(x,t), \quad t \in [0,1],\ x \in [0,1], \\ u(0,t) = u(x,0) = 0, \quad t \in [0,1],\ x \in [0,1],\end{gathered} \tag{2}$$

where

$$f(x,t) = F(x,t) - v\frac{dv}{dx}. \tag{3}$$

In view of the traditional perturbation method [11], we use the parameter ε to expand the solution

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots. \tag{4}$$

Substituting (4) into (2), we get

$$\begin{aligned}
&\varepsilon\left(\frac{\partial u_0}{\partial t} + \varepsilon\frac{\partial u_1}{\partial t} + \varepsilon^2\frac{\partial u_2}{\partial t} + \cdots\right) + \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\right)\\
&\quad\times\left(\frac{\partial u_0}{\partial x} + \varepsilon\frac{\partial u_1}{\partial x} + \varepsilon^2\frac{\partial u_2}{\partial x} + \cdots\right)\\
&\quad+\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\right)\frac{dv}{dx}\\
&\quad+ v\left(\frac{\partial u_0}{\partial x} + \varepsilon\frac{\partial u_1}{\partial x} + \varepsilon^2\frac{\partial u_2}{\partial x} + \cdots\right) = f(x,t)
\end{aligned} \tag{5}$$

and equating coefficients of the identical powers of ε yields the following equations:

$$\begin{gathered}
\varepsilon^0: u_0\frac{\partial u_0}{\partial x} + u_0\frac{dv}{dx} + v\frac{\partial u_0}{\partial x} = f(x,t)\Big|_{\varepsilon=0},\\
u_0(x,0) = u_0(0,t) = 0,
\end{gathered} \tag{6}$$

$$\begin{gathered}
\varepsilon^1: \frac{\partial u_0}{\partial t} + (u_0 + v)\frac{\partial u_1}{\partial x} + u_1\left(\frac{\partial u_0}{\partial x} + \frac{dv}{dx}\right) = \frac{\partial f(x,t)}{\partial \varepsilon}\Big|_{\varepsilon=0},\\
u_1(x,0) = u_1(0,t) = 0,
\end{gathered} \tag{7}$$

$$\begin{gathered}
\varepsilon^2: \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + (u_0 + v)\frac{\partial u_2}{\partial x} + u_2\left(\frac{\partial u_0}{\partial x} + \frac{dv}{dx}\right)\\
= \frac{\partial^2 f(x,t)}{\partial \varepsilon^2}\Big|_{\varepsilon=0}, \quad u_2(x,0) = u_2(0,t) = 0,
\end{gathered} \tag{8}$$

$$\begin{gathered}
\varepsilon^3: \frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + (u_0 + v)\frac{\partial u_3}{\partial x} + u_3\left(\frac{\partial u_0}{\partial x} + \frac{dv}{dx}\right)\\
= \frac{\partial^3 f(x,t)}{\partial \varepsilon^3}\Big|_{\varepsilon=0}, \quad u_3(x,0) = u_3(0,t) = 0,\\
\vdots
\end{gathered} \tag{9}$$

Next, we use the reproducing kernel method to solve each of the equations above, after obtaining all of $u_0, u_1, u_2, u_3, \ldots$, from (6), (7), (8),. . ., because of $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$; therefore, the analytical solution of (2) is obtained. Now, let us introduce how to use the reproducing kernel method to solve (6), (7), (8),. . ..

3. Reproducing Kernel Method

For getting $u_0, u_1, u_2, \ldots$ from (6), (7), (8),. . ., we let

$$\begin{gathered}
(L_0 u_0)(x,t) = u_0\frac{dv}{dx} + v\frac{\partial u_0}{\partial x},\\
f_0(x,t,u_0) = f(x,t)\big|_{\varepsilon=0} - u_0\frac{\partial u_0}{\partial x},\\
(L_j u_j)(x,t) = (u_0 + v)\frac{\partial u_j}{\partial x} + u_j\left(\frac{\partial u_0}{\partial x} + \frac{dv}{dx}\right),\\
j = 1, 2, \ldots,\\
f_1(x,t) = \frac{\partial f(x,t)}{\partial \varepsilon}\Big|_{\varepsilon=0} - \frac{\partial u_0}{\partial t},\\
f_j(x,t) = \frac{\partial^j f(x,t)}{j!\partial \varepsilon^j}\Big|_{\varepsilon=0} - \frac{\partial u_{j-1}}{\partial t} - \sum_{k=1}^{j-1} u_k\frac{\partial u_{j-k}}{\partial x},\\
j = 2, 3, \ldots.
\end{gathered} \tag{10}$$

Equation (6) can be converted into the following equivalent form:

$$(L_0 u_0)(t,x) = f_0(x,t,u_0). \tag{11}$$

Equations (7), (8),. . ., can be converted into the following equivalent form:

$$(L_j u_j)(t,x) = f_j(x,t), \quad j = 1, 2, \ldots. \tag{12}$$

Be aimed at with the purpose of solving (11) and (12), we need to introduce the reproducing kernel space, previously. Like in [12], we give the reproducing kernel spaces $W_2^2[0,1]$:

$$\begin{aligned}
W_2^2[0,1] = \{u \mid u, u' &\text{ is one-variable absolutely continuous}\\
&\text{function, } u'' \in L^2[0,1],\ u(0) = 0\}.
\end{aligned} \tag{13}$$

Then, we define the inner product of $W_2^2[0,1]$. Consider the following:

$$\begin{aligned}
\langle u(x), v(x)\rangle &= u(0)v(0) + u'(0)v'(0)\\
&\quad+ \int_0^b u''(x)v''(x)\,dy.
\end{aligned} \tag{14}$$

From [13], we can prove $W_2^2[0,1]$ is a reproducing kernel Hilbert space, and the reproducing kernel of it is

$$R_x^{\{2\}}(y) = \begin{cases} 1 - \dfrac{y^3}{6} + \dfrac{1}{2}xy(2+y), & x < y,\\[2mm] 1 - \dfrac{x^3}{6} + \dfrac{x^2 y}{2} + xy, & y < x. \end{cases} \tag{15}$$

After all of these, we introduce the reproducing kernel space $W_2(D)$ [14]

$$W_2(D) = W_2^2[0.1] \otimes W_2^2[0.1] = \Big\{ u(x,t) \mid \frac{\partial^{n+m}}{\partial x^n \partial t^m} u(x,t) \text{ are two-vatiable complete continuous functions, } n = 0,1, m = 0,1 \ \frac{\partial^{p+q}}{\partial x^p \partial t^q} u(x,t) \in L^2(D),\ p = 0,1,2, q = 0,1,2,\ u(x,0) = u(0,t) = 0 \Big\} \tag{16}$$

and the inner product of it; see [15], and the reproducing kernel of $W_2(D)$ is

$$K_{(\xi,\eta)}(t,x) = R_\xi^{\{2\}}(t) R_\eta^{\{2\}}(x). \tag{17}$$

Similar to the definition of $W_2(D)$, we can define $W_1(D)$ and it is the reproducing kernel $\overline{K}_{(\xi,\eta)}(t,x) = R_\xi^{\{1\}}(t) R_\eta^{\{1\}}(x)$, where $W_2^1[0,1]$ are also a reproducing kernel space with the reproducing kernel $R_x^{\{1\}}(y)$ (see [16–18]).

It is easy to prove L_j $(j = 0,1,2,\ldots)$ is a linear bounded operator, because the problem (1) has a unique solution $U(x,t)$; in other words, L_j is also a invertible operator, so [19] if $L_j u(x,t) = f_j(x,t,u)$ $(j = 0,1,2,\ldots)$, where $u(x,t) \in W_2(D)$ and $f_j(x,t,u) \in W_1(D)$, L_j^{-1} is existent and $\{x_i,t_i\}_{i=1}^\infty$ is countable dense points in D. Let $\overline{\psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x,t)$, where the β_{ik} are the coefficients resulting from Gram-Schmidt orthonormalization and $\psi_i(x,t) = (L_{j(y,s)} K_{(x,t)}(y,s))(x_i,t_i)$, $i = 1,2,\ldots$; then

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f_j(x_k, t_k, u(x_k,t_k)) \overline{\psi}_i(x,t) \tag{18}$$

is an analytical solution of equation $L_j u(x,t) = f_j(x,t,u)$.

(i) Linear Problem. Suppose equation $L_j u(x,t) = f_j(x,t,u)$ is a linear problem; that is, $f_j(x,t,u) = f_j(x,t)$; we define an approximate solution $u_n(x,t)$ by

$$u_n(x,t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f_j(x_k,t_k) \overline{\psi}_i(x,t). \tag{19}$$

Theorem 1 (see [20–22] convergence analysis). *Let $\varepsilon_n^2 = \|u(x,t) - u_n(x,t)\|^2$; then the sequence of real numbers ε_n is monotonously decreasing and $\varepsilon_n \to 0$ and the sequence $u_n(x,t)$ is convergent uniformly to $u(x,t)$.*

(ii) Nonlinear Problem (see [23]*).* Suppose equation $L_j u(x,t) = f_j(x,t,u)$ is a nonlinear problem; that is, $f_j(x,t,u) = N(u) + F_j(x,t)$, where $N : W_2(D) \to W_1(D)$ is a nonlinear operator; we give an iterative sequence $u_n(x,t)$:

$u_{0,*}(x,t)$ is the solution of the linear equation $L_j u = F_j(x,t)$;

$u_{n+1,*}(x,t)$ is the solution of the linear equation $L_j u = N(u_{n,*}) + F_j(x,t)$, $n = 0,1,2,\ldots$.

Lemma 2. *If $u_{n,*}(x \cdot t) \to u(x,t)$, then $u(x,t)$ is the solution of equation $L_j u(x,t) = f_j(x,t,u)$.*

Theorem 3. *Suppose the nonlinear operator $A \triangleq (L_j^{-1} N) : W_1(D) \to W_2(D)$ satisfies contractive mapping principle; that is,*

$$\|A(u) - A(v)\| \le \lambda \|u - v\|, \quad \lambda < 1; \tag{20}$$

and then $u_{n,*}(x \cdot t)$ is convergent.

Using reproducing kernel method, we can get

$$u_0(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f_0(x_k,t_k,u_0(x_k,t_k)) \overline{\psi_{i0}}(x,t), \quad u_j(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f_j(x_k,t_k) \overline{\psi_i}(x,t), \quad j = 1,2,\ldots. \tag{21}$$

Therefore, the analytical solution of (2) is obtained.

$$u(x,t) = \sum_{j=0}^{\infty} \varepsilon^j u_j(x,t). \tag{22}$$

In calculation, we use

$$u_{n,m,l}(x) = \sum_{i=1}^{l} \sum_{k=1}^{i} \beta_{ik} g(x_k,t_k,(u_0)_{n-1}(x_k,t_k)) \overline{\psi_{i0}}(x,t) + \sum_{j=1}^{m} \varepsilon^j \sum_{i=1}^{l} \sum_{k=1}^{i} \beta_{ik} f_j(x_k,t_k) \overline{\psi_i}(x,t) \tag{23}$$

as the approximation solution of (2).

4. Numerical Experiment

Example 4. Considering a nonlinear advection equation with perturbation term

$$\varepsilon \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x,t), \quad (x,t) \in D = [0,1] \times [0,1], \quad u(0,t) = \varepsilon t, \quad u(x,0) = x, \tag{24}$$

where $f(x,t) = \varepsilon^2 + \varepsilon t + x$, $u_T(x,t) = \varepsilon t + x$ is the true solution, and $u_{n,m}(x,t)$ is the approximate solution (Table 2). When we take $m = 3$, $n = 2$, and $l = 2$, the numerical results are given in Table 1.

5. Conclusions and Remarks

In this paper, the combination of traditional perturbation and reproducing kernel space methods was employed successfully for solving nonlinear advection equation with singular term. The numerical results show that the present method is an accurate and reliable. Moreover, the method is also effective solving other nonlinear singular perturbation problems.

TABLE 1: Comparison of the absolute error $\varepsilon = 1 \times 10^{-3}$.

(x, t)	$u_T(t, x)$	Approximate solution	Absolute error
(0.0001, 0.0001)	0.0001001	0.0001	1×10^{-7}
(0.0200, 0.0200)	0.0200200	0.0200	2×10^{-5}
(0.0050, 0.0050)	0.0050050	0.0050	5×10^{-6}
(0.8100, 0.8100)	0.8108100	0.8100	8×10^{-4}
(0.2000, 0.2000)	0.2002000	0.2000	2×10^{-4}
(0.5500, 0.5500)	0.5505500	0.5500	5×10^{-4}
(0.0330, 0.0330)	0.03303300	0.0330	3×10^{-5}
(1, 0)	1	1	0

TABLE 2: Comparison of the absolute error $\varepsilon = 1 \times 10^{-4}$.

(x, t)	$u_T(t, x)$	Approximate solution	Absolute error
(0, 0)	0	0	0
(0.01, 0.01)	0.010001	0.01	1×10^{-6}
(0.03, 0.03)	0.030003	0.03	3×10^{-6}
(0.05, 0.05)	0.050005	0.05	5×10^{-6}
(0.06, 0.06)	0.060006	0.06	6×10^{-6}
(0.08, 0.08)	0.080008	0.08	8×10^{-6}
(0.10, 0.10)	0.100010	0.10	1×10^{-5}

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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