Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 614019, 8 pages
http://dx.doi.org/10.1155/2014/614019

Research Article

n-Tupled Coincidence Point Theorems in Partially Ordered Metric Spaces for Compatible Mappings

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Received 24 November 2013; Accepted 27 January 2014; Published 12 March 2014

Academic Editor: Jen-Chih Yao

The intent of this paper is to introduce the notion of compatible mappings for n-tupled coincidence points due to (Imdad et al. (2013)). Related examples are also given to support our main results. Our results are the generalizations of the results of (Gnana Bhaskar and Lakshmikantham (2006), Lakshmikantham and Ćirić (2009), Choudhury and Kundu (2010), and Choudhary et al. (2013)).

1. Introduction

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. There exists vast literature on the topic and it is a very active field of research at present. A self-map T on a metric space X is said to have a fixed point $x \in X$ if $Tx = x$. Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tool for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities).

Existence of a fixed point for contraction type mappings in partially ordered metric spaces and applications has been considered by many authors; for detail, see [1–11]. In particular, Gnana Bhaskar and Lakshmikantham [12], Nieto and Rodriguez-Lopez [8], Ran and Recuring [13], and Agarwal et al. [9] presented some new results for contractions in partially ordered metric spaces.

Coupled fixed point problems belong to a category of problems in fixed point theory in which much interest has been generated recently after the publication of a coupled contraction theorem by Gnana Bhaskar and Lakshmikantham [12]. One of the reasons for this interest is the application of these results for proving the existence and uniqueness of the solution of differential equations, integral equations, the Volterra integral and Fredholm integral equations, and boundary value problems. For comprehensive description of such work, we refer to [1, 3–5, 7, 10–12, 14–18].

Common fixed point results for commuting maps in metric spaces were first deduced by Jungck [19]. The concept of commuting has been weakened in various directions and in several ways over the years. One such notion which is weaker than commuting is the concept of compatibility introduced by Jungck [20]. In common fixed point problems, this concept and its generalizations have been used extensively; for instance, see [3, 8, 9, 13–17, 20].

Most recently, Imdad et al. [21] introduced the notion of n-tupled coincidence point and proved n-tupled coincidence point theorems for commuting mappings in metric spaces. Motivated by this fact, we introduce the notion of compatibility for n-tupled coincidence points and prove n-tupled fixed point for compatible mappings satisfying contractive conditions in partially ordered metric spaces.

2. Preliminaries

Definition 1 (see [10]). Let $(X, \leq)$ be a partially ordered set equipped with a metric d such that (X, d) is a metric space.

Further, equip the product space $X \times X$ with the following partial ordering:

$$\text{for } (x, y), (u, v) \in X \times X, \\ \text{define } (u, v) \leq (x, y) \Longleftrightarrow x \geq u, \quad y \leq v. \tag{1}$$

Definition 2 (see [10]). Let $(X, \leq)$ be a partially ordered set and $F : X \to X$; then F enjoys the mixed monotone property if $F(x, y)$ is monotonically nondecreasing in x and monotonically nonincreasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y) \leq F(x_2, y), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1) \geq F(x, y_2). \tag{2}$$

Definition 3 (see [10]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$; then $(x, y) \in X \times X$ is called a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 4 (see [10]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$; then F enjoys the mixed g-monotone property if $F(x, y)$ is monotonically g-nondecreasing in x and monotonically g-nonincreasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \Longrightarrow F(x_1, y) \leq F(x_2, y), \\ \text{for any } y \in X, \\ y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \Longrightarrow F(x, y_1) \geq F(x, y_2), \\ \text{for any } x \in X. \tag{3}$$

Definition 5 (see [10]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$; then $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings F and g if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 6 (see [10]). Let $(X, \leq)$ be a partially ordered set; then $(x, y) \in X \times X$ is called a coupled fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y) = x$ and $gy = F(y, x) = y$.

Throughout the paper, r stands for a general even natural number.

Definition 7 (see [21]). Let $(X, \leq)$ be a partially ordered set and $F : \prod_{i=1}^{r} X^i \to X$; then F is said to have the mixed monotone property if F is nondecreasing in its odd position arguments and nonincreasing in its even positions arguments; that is, if,

(i) for all $x_1^1, x_2^1 \in X$, $x_1^1 \leq x_2^1 \Rightarrow F(x_1^1, x^2, x^3, \ldots, x^r) \leq F(x_2^1, x^2, x^3, \ldots, x^r)$,

(ii) for all $x_1^2, x_2^2 \in X$, $x_1^2 \leq x_2^2 \Rightarrow F(x^1, x_1^2, x^3, \ldots x^r) \geq F(x^1, x_2^2, x^3, \ldots, x^r)$,

(iii) for all $x_1^3, x_2^3 \in X$, $x_1^3 \leq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, x^4, \ldots, x^r) \leq F(x^1, x^2, x_2^3, x^4, \ldots, x^r)$,

for all $x_1^r, x_2^r \in X$, $x_1^r \leq x_2^r \Rightarrow F(x^1, x^2, x^3, \ldots, x_1^r) \geq F(x^1, x^2, x^3, \ldots, x_2^r)$.

Definition 8 (see [21]). Let $(X, \leq)$ be a partially ordered set and let $F : \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ be two mappings. Then the mapping F is said to have the mixed g-monotone property if F is g-nondecreasing in its odd position arguments and g-nonincreasing in its even positions arguments; that is, if,

(i) for all $x_1^1, x_2^1 \in X$, $gx_1^1 \leq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, \ldots, x^r) \leq F(x_2^1, x^2, x^3, \ldots, x^r)$,

(ii) for all $x_1^2, x_2^2 \in X$, $gx_1^2 \leq gx_2^2 \Rightarrow F(x^1, x_1^2, x^3, \ldots, x^r) \geq F(x^1, x_2^2, x^3, \ldots, x^r)$,

(iii) for all $x_1^3, x_2^3 \in X$, $gx_1^3 \leq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, \ldots, x^r) \leq F(x^1, x^2, x_2^3, \ldots, x^r)$,

for all $x_1^r, x_2^r \in X$, $gx_1^r \leq gx_2^r \Rightarrow F(x^1, x^2, x^3, \ldots, x_1^r) \geq F(x^1, x^2, x^3, \ldots, x_2^r)$.

Definition 9 (see [21]). Let X be a nonempty set. An element $(x^1, x^2, x^3, \ldots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the mapping $F : \prod_{i=1}^{r} X^i \to X$ if

$$\begin{aligned} x^1 &= F(x^1, x^2, x^3, \ldots, x^r), \\ x^2 &= F(x^2, x^3, \ldots, x^r, x^1), \\ x^3 &= F(x^3, \ldots, x^r, x^1, x^2), \\ &\vdots \\ x^r &= F(x^r, x^1, x^2, \ldots, x^{r-1}). \end{aligned} \tag{4}$$

Example 10. Let (R, d) be a partial ordered metric space under natural setting and let $F : \prod_{i=1}^{r} X^i \to X$ be mapping defined by $F(x^1, x^2, x^3, \ldots, x^r) = \sin(x^1 \cdot x^2 \cdot x^3 \cdot \cdots \cdot x^r)$, for any $x^1, x^2, x^3, \ldots, x^r \in X$; then $(0, 0, 0, \ldots, 0)$ is an r-tupled fixed point of F.

Definition 11 (see [21]). Let X be a nonempty set. An element $(x^1, x^2, x^3, \ldots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled coincidence point of the mappings $F : \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ if

$$\begin{aligned} gx^1 &= F(x^1, x^2, x^3, \ldots, x^r), \\ gx^2 &= F(x^2, x^3, \ldots, x^r, x^1), \\ gx^3 &= F(x^3, \ldots, x^r, x^1, x^2), \\ &\vdots \\ gx^r &= F(x^r, x^1, x^2, \ldots, x^{r-1}). \end{aligned} \tag{5}$$

Example 12. Let (R, d) be a partial ordered metric space under natural setting and let $F : \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ be mappings defined by

$$F\left(x^1, x^2, x^3, \ldots, x^r\right) = \sin x^1 \cdot \cos x^2 \cdot \sin x^3 \cdot \cos x^4 \cdot \cdots \cdot \sin x^{r-1} \cdot \cos x^r,$$
$$g(x) = \sin x, \tag{6}$$

for any $x^1, x^2, x^3, \ldots, x^r \in X$; then $\{(x^1, x^2, x^3, \ldots, x^r), x^i = m\pi, m \in N, 1 \le i \le r\}$ is an r-tupled coincidence point of F and g.

Definition 13 (see [21]). Let X be a nonempty set. An element $(x^1, x^2, x^3, \ldots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the mappings $F : \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ if

$$\begin{gathered} x^1 = gx^1 = F\left(x^1, x^2, x^3, \ldots, x^r\right), \\ x^2 = gx^2 = F\left(x^2, x^3, \ldots, x^r, x^1\right), \\ x^3 = gx^3 = F\left(x^3, \ldots, x^r, x^1, x^2\right), \\ \vdots \\ x^r = gx^r = F\left(x^r, x^1, x^2, \ldots, x^{r-1}\right). \end{gathered} \tag{7}$$

Now, we define the concept of compatible mappings for r-tupled mappings.

Definition 14. Let $(X, \le)$ be a partially ordered set; then the mappings $F : \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ are called compatible if

$$\begin{gathered} \lim_{n\to\infty} g\left(F\left(x_n^1, x_n^2, \ldots, x_n^r\right), F\left(gx_n^1, gx_n^2, \ldots, gx_n^r\right)\right) = 0, \\ \lim_{n\to\infty} g\left(F\left(x_n^2, x_n^3, \ldots, x_n^r, x_n^1\right), F\left(gx_n^2, gx_n^3, \ldots, gx_n^r, gx_n^1\right)\right) = 0, \\ \lim_{n\to\infty} g\left(F\left(x_n^3, \ldots, x_n^r, x_n^1, x_n^2\right), F\left(gx_n^3, \ldots, gx_n^r, gx_n^1, gx_n^2\right)\right) = 0, \\ \vdots \\ \lim_{n\to\infty} g\left(F\left(x_n^r, x_n^1, x_n^2 \ldots, x_n^{r-1}\right), F\left(gx_n^r, gx_n^1, gx_n^2, \ldots, gx_n^{r-1}\right)\right) = 0, \end{gathered} \tag{8}$$

whenever $\{x_n^1\}, \{x_n^2\}, \{x_n^3\}, \ldots, \{x_n^r\}$ are sequences in X such that

$$\begin{gathered} \lim_{n\to\infty} F\left(x_n^1, x_n^2, x_n^3, \ldots, x_n^r\right) = \lim_{n\to\infty} g\left(x_n^1\right) = x^1, \\ \lim_{n\to\infty} F\left(x_n^2, x_n^3, \ldots, x_n^r, x_n^1\right) = \lim_{n\to\infty} g\left(x_n^2\right) = x^2, \\ \lim_{n\to\infty} F\left(x_n^3, x_n^4, \ldots, x_n^1, x_n^2\right) = \lim_{n\to\infty} g\left(x_n^3\right) = x^3, \\ \vdots \\ \lim_{n\to\infty} F\left(x_n^r, x_n^1, x_n^2, \ldots, x_n^{r-1}\right) = \lim_{n\to\infty} g\left(x_n^r\right) = x^r, \end{gathered} \tag{9}$$

for some $x^1, x^2, x^3, \ldots, x^r \in X$.

3. Main Results

Recently, Imdad et al. [21] proved the following theorem.

Theorem 15. *Let $(X, \le)$ be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r\to t^+} \varphi(t) < t$ for each $t > 0$. Further let $F : \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property satisfying the following conditions:*

(i) $F(\prod_{i=1}^{r} X^i) \subseteq g(X)$,

(ii) *g is continuous and monotonically increasing,*

(iii) *(g, F) is a commuting pair,*

(iv) $d(F(x^1, x^2, x^3, \ldots, x^r), F(y^1, y^2, y^3, \ldots, y^r)) \le \varphi((1/r) \sum_{n=1}^{r} d(g(x^n), g(y^n)))$,

for all $x^1, x^2, x^3, \ldots, x^r$, $y^1, y^2, y^3, \ldots, y^r \in X$, with $gx^1 \le gy^1$, $gx^2 \ge gy^2$, $gx^3 \le gy^3, \ldots, gx^r \ge gy^r$. Also, suppose that either

(a) *F is continuous or*

(b) *X has the following properties:*

  (i) *If a nondecreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all $n \ge 0$.*

  (ii) *If a nonincreasing sequence $\{y_n\} \to y$, then $y \le y_n$ for all $n \ge 0$.*

If there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^r \in X$ such that

$$\begin{gathered} gx_0^1 \le F\left(x_0^1, x_0^2, x_0^3, \ldots, x_0^r\right), \\ gx_0^2 \ge F\left(x_0^2, x_0^3, \ldots, x_0^r, x_0^1\right), \\ gx_0^3 \le F\left(x_0^3, \ldots, x_0^r, x_0^1, x_0^2\right), \\ \vdots \\ gx_0^r \ge F\left(x_0^r, x_0^1, x_0^2, x_0^3, \ldots, x_0^{r-1}\right), \end{gathered} \tag{10}$$

then F and g have an r-tupled coincidence point; that is, there exist $x^1, x^2, x^3, \ldots, x^r \in X$ such that

$$\begin{aligned}
gx^1 &= F\left(x^1, x^2, x^3, \ldots, x^r\right), \\
gx^2 &= F\left(x^2, x^3, \ldots, x^r, x^1\right), \\
gx^3 &= F\left(x^3, \ldots, x^r, x^1, x^2\right), \\
&\vdots \\
gx^r &= F\left(x^r, x^1, x^2, x^3, \ldots, x^{r-1}\right).
\end{aligned} \tag{11}$$

Now, we prove our main results.

Theorem 16. *Let $(X, \leq)$ be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(t) < t$ for each $t > 0$. Further let $F : \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property satisfying the following conditions:*

(1) $F(\prod_{i=1}^{r} X^i) \subseteq g(X)$,

(2) *g is continuous and monotonically increasing,*

(3) *the pair (g, F) is compatible,*

(4) $d(F(x^1, x^2, x^3, \ldots, x^r), F(y^1, y^2, y^3, \ldots, y^r)) \leq \varphi((1/r) \sum_{n=1}^{r} d(g(x^n), g(y^n)))$,

for all $x^1, x^2, x^3, \ldots, x^r$, $y^1, y^2, y^3, \ldots, y^r \in X$, with $gx^1 \leq gy^1$, $gx^2 \geq gy^2$, $gx^3 \leq gy^3, \ldots, gx^r \geq gy^r$. Also, suppose that either

(a) *F is continuous or*

(b) *X has the following properties:*

  (i) *If a nondecreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \geq 0$.*

  (ii) *If a nonincreasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \geq 0$.*

If there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^r \in X$ such that

$$\begin{aligned}
gx_0^1 &\leq F\left(x_0^1, x_0^2, x_0^3, \ldots, x_0^r\right), \\
gx_0^2 &\geq F\left(x_0^2, x_0^3, \ldots, x_0^r, x_0^1\right), \\
gx_0^3 &\leq F\left(x_0^3, \ldots, x_0^r, x_0^1, x_0^2\right), \\
&\vdots \\
gx_0^r &\geq F\left(x_0^r, x_0^1, x_0^2, x_0^3, \ldots, x_0^{r-1}\right),
\end{aligned} \tag{12}$$

then F and g have an r-tupled coincidence point; that is, there exist $x^1, x^2, x^3, \ldots, x^r \in X$ such that

$$\begin{aligned}
gx^1 &= F\left(x^1, x^2, x^3, \ldots, x^r\right), \\
gx^2 &= F\left(x^2, x^3, \ldots, x^r, x^1\right), \\
gx^3 &= F\left(x^3, \ldots, x^r, x^1, x^2\right), \\
&\vdots \\
gx^r &= F\left(x^r, x^1, x^2, x^3, \ldots, x^{r-1}\right).
\end{aligned} \tag{13}$$

Proof. Starting with $x_0^1, x_0^2, x_0^3, \ldots, x_0^r \in X$, we define the sequences $\{x_n^1\}, \{x_n^2\}, \{x_n^3\}, \ldots, \{x_n^r\}$ in X as follows:

$$\begin{aligned}
gx_{n+1}^1 &= F\left(x_n^1, x_n^2, x_n^1, \ldots, x_n^r\right), \\
gx_{n+1}^2 &= F\left(x_n^2, x_n^3, \ldots, x_n^r, x_n^1\right), \\
gx_{n+1}^3 &= F\left(x_n^3, \ldots, x_n^r, x_n^1, x_n^2\right), \\
&\vdots \\
gx_{n+1}^r &= F\left(x_n^r, x_n^1, x_n^2, x_n^3, \ldots, x_n^{r-1}\right).
\end{aligned} \tag{14}$$

Now, we prove that, for all $n \geq 0$,

$$\begin{aligned}
&gx_n^1 \leq gx_{n+1}^1, \\
&gx_n^2 \geq gx_{n+1}^2, \\
&gx_n^3 \leq gx_{n+1}^3, \ldots, gx_n^r \geq gx_{n+1}^r,
\end{aligned} \tag{15}$$

$$\begin{aligned}
gx_0^1 &\leq F\left(x_0^1, x_0^2, x_0^3, \ldots, x_0^r\right) = gx_1^1, \\
gx_0^2 &\geq F\left(x_0^2, x_0^3, \ldots, x_0^r, x_0^1\right) = gx_1^2, \\
gx_0^3 &\leq F\left(x_0^3, \ldots, x_0^r, x_0^1, x_0^2\right) = gx_1^3, \\
&\vdots \\
gx_0^r &\geq F\left(x_0^r, x_0^1, x_0^2, x_0^3, \ldots, x_0^{r-1}\right) = gx_1^r.
\end{aligned} \tag{16}$$

So (15) holds for $n = 0$. Suppose (15) holds for some $n > 0$. Consider

$$\begin{aligned}
gx_{n+1}^1 &= F\left(x_n^1, x_n^2, x_n^3, \ldots, x_n^r\right) \\
&\leq F\left(x_{n+1}^1, x_n^2, x_n^3, \ldots, x_n^r\right) \\
&\leq F\left(x_{n+1}^1, x_{n+1}^2, x_n^3, \ldots, x_n^r\right) \\
&\leq F\left(x_{n+1}^1, x_{n+1}^2, x_{n+1}^3, \ldots, x_n^r\right) \\
&\leq F\left(x_{n+1}^1, x_{n+1}^2, \ldots, x_{n+1}^r\right) = gx_{n+2}^1,
\end{aligned}$$

$$\begin{aligned}
gx_{n+1}^2 &= F\left(x_n^2, x_n^3, \dots, x_n^r, x_n^1\right)\\
&\geq F\left(x_{n+1}^2, x_n^3, \dots, x_n^r, x_n^1\right)\\
&\geq F\left(x_{n+1}^2, x_{n+1}^3, \dots, x_n^r, x_n^1\right)\\
&\geq F\left(x_{n+1}^2, x_{n+1}^3, \dots, x_{n+1}^r, x_n^1\right)\\
&\geq F\left(x_{n+1}^1, x_{n+1}^2, \dots, x_{n+1}^r, x_{n+1}^1\right) = gx_{n+2}^2,\\
gx_{n+1}^3 &= F\left(x_n^3, \dots, x_n^r, x_n^1, x_n^2\right)\\
&\leq F\left(x_{n+1}^3, \dots, x_n^r, x_n^1, x_n^2\right)\\
&\leq F\left(x_{n+1}^3, x_{n+1}^4, \dots, x_n^r, x_n^1, x_n^2\right)\\
&\leq F\left(x_{n+1}^3, x_{n+1}^4, \dots, x_{n+1}^r, x_n^1, x_n^2\right)\\
&\leq F\left(x_{n+1}^3, x_{n+1}^4, x_n^5, \dots, x_{n+1}^r, x_{n+1}^1, x_n^2\right)\\
&\leq F\left(x_{n+1}^3, x_{n+1}^4, \dots, x_{n+1}^r, x_{n+1}^1, x_{n+1}^2\right) = gx_{n+2}^3,\\
&\vdots\\
gx_{n+1}^r &= F\left(x_n^r, x_n^1, x_n^2, \dots, x_n^{r-1}\right)\\
&\geq F\left(x_{n+1}^r, x_n^1, x_n^2, \dots, x_n^{r-1}\right)\\
&\geq F\left(x_{n+1}^r, x_{n+1}^1, x_n^2, x_n^3 \dots, x_n^{r-1}\right)\\
&\geq F\left(x_{n+1}^r, x_{n+1}^1, x_{n+1}^2 \dots, x_n^{r-1}\right)\\
&\geq F\left(x_{n+1}^r, x_{n+1}^1, x_n^2, \dots, x_{n+1}^{r-1}\right) = gx_{n+2}^r.
\end{aligned} \tag{17}$$

Thus by induction (15) holds for all $n \geq 0$. Using (14) and (15)

$$\begin{aligned}
&d\left(g\left(x_m^1\right), g\left(x_{m+1}^1\right)\right)\\
&\quad = d\left(F\left(x_{m-1}^1, x_{m-1}^2, \dots, x_{m-1}^r\right), F\left(x_m^1, x_m^2, \dots, x_m^r\right)\right)\\
&\quad \leq \varphi\left(\frac{1}{r}\sum_{n=1}^{r} d\left(g\left(x_{m-1}^n\right), g\left(x_m^n\right)\right)\right).
\end{aligned} \tag{18}$$

Similarly, we can inductively write

$$\begin{aligned}
d\left(g\left(x_m^2\right), g\left(x_{m+1}^2\right)\right) &\leq \varphi\left(\frac{1}{r}\sum_{n=1}^{r} d\left(g\left(x_{m-1}^n\right), g\left(x_m^n\right)\right)\right),\\
&\vdots\\
d\left(g\left(x_m^r\right), g\left(x_{m+1}^r\right)\right) &\leq \varphi\left(\frac{1}{r}\sum_{n=1}^{r} d\left(g\left(x_{m-1}^n\right), g\left(x_m^n\right)\right)\right).
\end{aligned} \tag{19}$$

Therefore, by putting

$$\begin{aligned}
\gamma_m = {} & d\left(g\left(x_m^1\right), g\left(x_{m+1}^1\right)\right) + d\left(g\left(x_m^2\right), g\left(x_{m+1}^2\right)\right)\\
& + \dots + d\left(g\left(x_m^r\right), g\left(x_{m+1}^r\right)\right),
\end{aligned} \tag{20}$$

we have

$$\begin{aligned}
\gamma_m &= d\left(g\left(x_m^1\right), g\left(x_{m+1}^1\right)\right) + d\left(g\left(x_m^2\right), g\left(x_{m+1}^2\right)\right)\\
&\quad + \dots + d\left(g\left(x_m^r\right), g\left(x_{m+1}^r\right)\right)\\
&\leq r\varphi\left(\frac{1}{r}\sum_{n=1}^{r} d\left(g\left(x_{m-1}^n\right), g\left(x_m^n\right)\right)\right)\\
&= r\varphi\left(\frac{1}{r}\gamma_{m-1}\right).
\end{aligned} \tag{21}$$

Since $\varphi(t) < t$ for all $t > 0$, $\gamma_m \leq \gamma_{m-1}$ for all m so that $\{\gamma_m\}$ is a nonincreasing sequence. Since it is bounded below, there are some $\gamma \geq 0$ such that

$$\lim_{n\to\infty} \gamma_m = +\gamma. \tag{22}$$

We will show that $\gamma = 0$. Suppose, if possible, $\gamma > 0$. Taking limit as $m \to \infty$ of both sides of (21) and keeping in mind our supposition that $\lim_{r\to t^+} \varphi(r)$ for all $t > 0$, we have

$$\gamma = \lim_{n\to\infty} \gamma_m \leq r\varphi\left(\frac{1}{r}\gamma_{m-1}\right) = r\varphi\left(\frac{1}{r}\gamma\right) < r\frac{\gamma}{r} = \gamma, \tag{23}$$

and this contradiction gives $\gamma = 0$ and hence

$$\begin{aligned}
\lim_{n\to\infty} \Big[& d\left(g\left(x_m^1\right), g\left(x_{m+1}^1\right)\right) + d\left(g\left(x_m^2\right), g\left(x_{m+1}^2\right)\right)\\
& + \dots + d\left(g\left(x_m^r\right), g\left(x_{m+1}^r\right)\right)\Big] = 0.
\end{aligned} \tag{24}$$

Next we show that all the sequences $\{g(x_m^1)\}, \{g(x_m^2)\}, \{g(x_m^3)\}, \dots$, and $\{g(x_m^r)\}$ are Cauchy sequences. If possible, suppose that at least one of $\{g(x_m^1)\}, \{g(x_m^2)\}, \dots$, and $\{g(x_m^r)\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and sequences of positive integers $\{l(k)\}$ and $\{m(k)\}$ such that, for all positive integers k, $m(k) > l(k) > k$,

$$\begin{aligned}
& d\left(gx_{l(k)}^1, gx_{m(k)}^1\right) + d\left(gx_{l(k)}^2, gx_{m(k)}^2\right)\\
&\quad + \dots + d\left(gx_{l(k)}^r, gx_{m(k)}^r\right) \geq \varepsilon,\\
& d\left(gx_{l(k)}^1, gx_{m(k)-1}^1\right) + d\left(gx_{l(k)}^2, gx_{m(k)-1}^2\right)\\
&\quad + \dots + d\left(gx_{l(k)}^r, gx_{m(k)-1}^r\right) < \varepsilon.
\end{aligned} \tag{25}$$

Now,

$$\begin{aligned}\varepsilon &\le d\left(gx^1_{l(k)}, gx^1_{m(k)}\right) + d\left(gx^2_{l(k)}, gx^2_{m(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)}, gx^r_{m(k)}\right) \\ &\le d\left(gx^1_{l(k)}, gx^1_{m(k)-1}\right) + d\left(gx^2_{l(k)}, gx^2_{m(k)-1}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)}, gx^r_{m(k)-1}\right) \\ &\quad + d\left(gx^1_{m(k)-1}, gx^1_{m(k)}\right) + d\left(gx^2_{m(k)-1}, gx^2_{m(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{m(k)-1}, gx^r_{m(k)}\right).\end{aligned} \tag{26}$$

That is,

$$\begin{aligned}\varepsilon &\le d\left(gx^1_{l(k)}, gx^1_{m(k)}\right) + d\left(gx^2_{l(k)}, gx^2_{m(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)}, gx^r_{m(k)}\right) \\ &\le \varepsilon + d\left(gx^1_{m(k)-1}, gx^1_{m(k)}\right) + d\left(gx^2_{m(k)-1}, gx^2_{m(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{m(k)-1}, gx^r_{m(k)}\right).\end{aligned} \tag{27}$$

Taking $k \to \infty$ in the above inequality and using (24), we have

$$\begin{aligned}\lim_{k\to\infty} &\left[d\left(gx^1_{l(k)}, gx^1_{m(k)}\right) + d\left(gx^2_{l(k)}, gx^2_{m(k)}\right) \right.\\ &\quad \left. + \cdots + d\left(gx^r_{l(k)}, gx^r_{m(k)}\right)\right] = \varepsilon.\end{aligned} \tag{28}$$

Again,

$$\begin{aligned}d&\left(gx^1_{l(k)+1}, gx^1_{m(k)+1}\right) + d\left(gx^2_{l(k)+1}, gx^2_{m(k)+1}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)+1}, gx^r_{m(k)+1}\right) \\ &\le d\left(gx^1_{l(k)+1}, gx^1_{l(k)}\right) + d\left(gx^2_{l(k)+1}, gx^2_{l(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)+1}, gx^r_{l(k)}\right) \\ &\quad + d\left(gx^1_{l(k)}, gx^1_{m(k)}\right) + d\left(gx^2_{l(k)}, gx^2_{m(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)}, gx^r_{m(k)}\right) \\ &\quad + d\left(gx^1_{m(k)}, gx^1_{m(k)+1}\right) + d\left(gx^2_{m(k)}, gx^2_{m(k)+1}\right) \\ &\quad + \cdots + d\left(gx^r_{m(k)}, gx^r_{m(k)+1}\right), \\ d&\left(gx^1_{l(k)}, gx^1_{m(k)}\right) + d\left(gx^2_{l(k)}, gx^2_{m(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)}, gx^r_{m(k)}\right) \\ &\le d\left(gx^1_{l(k)+1}, gx^1_{l(k)}\right) + d\left(gx^2_{l(k)+1}, gx^2_{l(k)}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)+1}, gx^r_{l(k)}\right) \\ &\quad + d\left(gx^1_{l(k)+1}, gx^1_{m(k)+1}\right) + d\left(gx^2_{l(k)+1}, gx^2_{m(k)+1}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)+1}, gx^r_{m(k)+1}\right) \\ &\quad + d\left(gx^1_{m(k)}, gx^1_{m(k)+1}\right) + d\left(gx^2_{m(k)}, gx^2_{m(k)+1}\right) \\ &\quad + \cdots + d\left(gx^r_{m(k)}, gx^r_{m(k)+1}\right).\end{aligned} \tag{29}$$

Taking limit as $k \to \infty$ in the above inequality and using (24) and (28), we have

$$\begin{aligned}\lim_{k\to\infty} &\left\{d\left(gx^1_{l(k)+1}, gx^1_{m(k)+1}\right) + d\left(gx^2_{l(k)+1}, gx^2_{m(k)+1}\right) \right.\\ &\quad \left. + \cdots + d\left(gx^r_{l(k)+1}, gx^r_{m(k)+1}\right)\right\} = \varepsilon.\end{aligned} \tag{30}$$

Now,

$$\begin{aligned}d&\left(gx^1_{l(k)+1}, gx^1_{m(k)+1}\right) + d\left(gx^2_{l(k)+1}, gx^2_{m(k)+1}\right) \\ &\quad + \cdots + d\left(gx^r_{l(k)+1}, gx^r_{m(k)+1}\right) \\ &= d\left(F\left(x^1_{l(k)}, x^2_{l(k)} \ldots, x^r_{l(k)}\right), F\left(x^1_{m(k)}, x^2_{m(k)} \ldots, x^r_{m(k)}\right)\right) \\ &\quad + d\left(F\left(x^2_{l(k)} \ldots, x^r_{l(k)}, x^1_{l(k)}\right), F\left(x^2_{m(k)} \ldots, x^r_{m(k)}, x^1_{m(k)}\right)\right) \\ &\quad + \cdots + d\left(F\left(x^r_{l(k)}, x^1_{l(k)}, \ldots, x^{r-1}_{l(k)}\right),\right. \\ &\qquad\qquad \left. F\left(x^2_{m(k)}, x^1_{m(k)}, \ldots, x^{r-1}_{m(k)}\right)\right) \\ &\le r\varphi\left(\frac{1}{r}\sum_{n=1}^{r} d\left(g\left(x^n_{l(k)}\right), g\left(x^n_{m(k)}\right)\right)\right).\end{aligned} \tag{31}$$

Letting $k \to \infty$ in the above inequality and using (28), (30), and the property of φ, we get

$$\varepsilon \le r\varphi\left(\frac{\varepsilon}{r}\right) < r\frac{\varepsilon}{r} = \varepsilon, \tag{32}$$

which is a contradiction. Therefore, $\{g(x^1_m)\}, \{g(x^2_m)\}, \{g(x^3_m)\}, \ldots, \{g(x^r_m)\}$ are Cauchy sequences. Since the metric space (X, d) is complete, there exist $x^1, x^2, \ldots, x^r \in X$ such that

$$\begin{aligned}\lim_{m\to\infty} g\left(x^1_m\right) &= x^1, \\ \lim_{m\to\infty} g\left(x^2_m\right) &= x^2, \\ &\vdots \\ \lim_{m\to\infty} g\left(x^r_m\right) &= x^r.\end{aligned} \tag{33}$$

As g is continuous, from (33), we have

$$\lim_{m\to\infty} g\left(g\left(x_m^1\right)\right) = g\left(x^1\right),$$
$$\lim_{m\to\infty} g\left(g\left(x_m^2\right)\right) = g\left(x^2\right),$$
$$\vdots \tag{34}$$
$$\lim_{m\to\infty} g\left(g\left(x_m^r\right)\right) = g\left(x^r\right).$$

By the compatibility of g and F, we have

$$\begin{aligned}
&\lim_{n\to\infty} d\left(g\left(F\left(x_m^1, x_m^2, \ldots, x_m^r\right)\right),\right.\\
&\qquad \left.F\left(g\left(x_m^1\right), g\left(x_m^2\right), \ldots, g\left(x_m^r\right)\right)\right) = 0,\\
&\lim_{n\to\infty} d\left(g\left(F\left(x_m^2, \ldots, x_m^r, x_m^1\right)\right),\right.\\
&\qquad \left.F\left(g\left(x_m^2\right), \ldots, g\left(x_m^r\right), g\left(x_m^1\right)\right)\right) = 0,\\
&\qquad\vdots\\
&\lim_{n\to\infty} d\left(g\left(\left(F\left(x_m^r, x_m^1, \ldots, x_m^{r-1}\right)\right)\right),\right.\\
&\qquad \left.F\left(g\left(x_m^r\right), g\left(x_m^1\right), \ldots, g\left(x_m^{r-1}\right)\right)\right) = 0.
\end{aligned} \tag{35}$$

Now, we show that F and g have an r-tupled coincidence point. To accomplish this, suppose (a) holds. That is, F is continuous. Then using (35) and (15), we see that

$$\begin{aligned}
&d\left(g\left(x^1\right), F\left(x^1, x^2, \ldots, x^r\right)\right)\\
&\quad = \lim_{n\to\infty} d\left(g\left(g\left(x_{m+1}^1\right)\right),\right.\\
&\qquad\qquad \left.F\left(g\left(x_m^1\right), g\left(x_m^2\right), \ldots, g\left(x_m^r\right)\right)\right)\\
&\quad = \lim_{n\to\infty} d\left(g\left(F\left(x_m^1, x_m^2, \ldots, x_m^r\right)\right),\right.\\
&\qquad\qquad \left.F\left(g\left(x_m^1\right), g\left(x_m^2\right), \ldots, g\left(x_m^r\right)\right)\right)\\
&\quad = 0,
\end{aligned} \tag{36}$$

which implies $g(x^1) = F(x^1, x^2, \ldots, x^r)$. Similarly, we can easily prove that $g(x^2) = F(x^2, \ldots, x^r, x^1), \ldots, g(x^r) = F(x^r, x^1, \ldots, x^{r-1})$. Hence $(x^1, x^2, \ldots, x^r) \in \prod_{i=1}^r X^i$ is an r-tupled coincidence point of the mappings F and g.

If (b) holds, since $g(x_m^i)$ is nondecreasing or nonincreasing as i is odd or even and $g(x_m^i) \to x^i$ as $m \to \infty$, we have $g(x_m^i) \le x^i$, when i is odd, while $g(x_m^i) \ge x^i$, when i is even. Since g is monotonically increasing,

$$\begin{aligned}
&g\left(g\left(x_m^i\right)\right) \le g\left(x^i\right), \quad \text{when } i \text{ is odd},\\
&g\left(g\left(x_m^i\right)\right) \ge g\left(x^i\right), \quad \text{when } i \text{ is even}.
\end{aligned} \tag{37}$$

Now, using triangle inequality together with (15), we get

$$\begin{aligned}
&d\left(g\left(x^1\right), F\left(x^1, x^2, \ldots, x^r\right)\right)\\
&\quad \le d\left(g\left(x^1\right), g\left(x_{m+1}^1\right)\right)\\
&\qquad + d\left(g\left(x_{m+1}^1\right), F\left(x^1, x^2, \ldots, x^r\right)\right)\\
&\quad \le d\left(g\left(x^1\right), g\left(x_{m+1}^1\right)\right)\\
&\qquad + d\left(g\left(F\left(x_m^1, x_m^2, \ldots, x_m^r\right)\right),\right.\\
&\qquad\qquad \left.F\left(g\left(x_m^1\right), g\left(x_m^2\right), \ldots, g\left(x_m^r\right)\right)\right)\\
&\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{38}$$

Therefore, $g(x^1) = F(x^1, x^2, \ldots, x^r)$. Similarly, we can prove $g(x^2) = F(x^2, \ldots, x^r, x^1), \ldots, g(x^r) = F(x^r, x^1, \ldots, x^{r-1})$. Thus the theorem follows. □

Now, we furnish an illustrative example to support our theorem.

Example 17. Let $X = R$ be complete metric space under usual metric and natural ordering $\le$ of real numbers. Define the mappings $F : \prod_{i=1}^r X^i \to X$ and $g : X \to X$ as follows:

$$\begin{aligned}
g(x) &= rx,\\
F\left(x^1, x^2, x^3, \ldots, x^r\right) &= \frac{x^1 - x^2 + x^3 - \cdots + x^{r-1} - x^r}{r+1}.
\end{aligned} \tag{39}$$

Set $\varphi(t) = t/(r+1)$; then we see that

$$\begin{aligned}
&d\left(F\left(x^1, x^2, \ldots, x^{r-1}, x^r\right), F\left(y^1, y^2, \ldots, y^{r-1}, y^r\right)\right)\\
&\quad \times d\left(\frac{x^1 - x^2 + x^3 - \cdots + x^{r-1} - x^r}{r+1},\right.\\
&\qquad\qquad \left.\frac{y^1 - y^2 + y^3 - \cdots + y^{r-1} - y^r}{r+1}\right)\\
&\quad \times \frac{1}{r+1}\left\lfloor\left(x^1 - x^2 + x^3 - \cdots + x^{r-1} - x^r\right)\right.\\
&\qquad\qquad \left.-\left(y^1 - y^2 + y^3 - \cdots + y^{r-1} - y^r\right)\right\rfloor\\
&\quad \le \frac{1}{r+1}\frac{1}{r}\left\{r\left(\lfloor x - y\rfloor + \left\lfloor x^2 - y^2\right\rfloor + \cdots + \left\lfloor x^r - y^r\right\rfloor\right)\right\}\\
&\quad \times \frac{1}{r+1}\left(\frac{1}{r}\sum_{n=1}^r d\left(gx^n, gy^n\right)\right)\\
&\quad = \varphi\left(\frac{1}{r}\sum_{n=1}^r d\left(gx^n, gy^n\right)\right).
\end{aligned} \tag{40}$$

Also, the pair (g, F) is compatible. Thus all the conditions of our Theorem 16 are satisfied (without order) and $(0, 0, \ldots, 0)$ is an r-tuple coincidence point of F and g.

Corollary 18. *Let $(X, \leq)$ be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Further let $F : \prod_{i=1}^{r} X^i \rightarrow X$ and $g : X \rightarrow X$ be two mappings satisfying all the conditions of Theorem 15 with a suitable replacement of condition (4) of Theorem 16 by*

$$d\left(F\left(x^1, x^2, \ldots, x^r\right), F\left(y^1, y^2, \ldots, y^r\right)\right) \leq \frac{k}{r}\sum_{n=1}^{r} d\left(g\left(x^n\right), g\left(y^n\right)\right), \quad k \in [0, 1). \tag{41}$$

Then F and g have an r-tupled coincidence point.

Proof. If we put $\varphi(t) = kt$ where $k \in [0, 1)$ in Theorem 15, then the result follows immediately. □

Conflict of Interests

The authors declare that they have no conflict of interests.

Acknowledgments

The authors would like to express their sincere thanks to the guest editor Professor Jen-Chih Yao and the anonymous reviewers for their valuable suggestions.

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