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Research Article

Growth Theorems for a Subclass of Strongly Spirallike Functions

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In this paper we consider a subclass of strongly spirallike functions on the unit disk D in the complex plane $\mathbb{C}$, namely, strongly almost spirallike functions of type β and order α. We obtain the growth results for strongly almost spirallike functions of type β and order α on the unit disk D in $\mathbb{C}$ by using subordination principles and the geometric properties of analytic mappings. Furthermore we get the growth theorems for strongly almost starlike functions of order α and strongly starlike functions on the unit disk D of $\mathbb{C}$. These growth results follow the deviation results of these functions.

1. Introduction

Growth theorems for univalent analytic functions are important parts in geometric function theories of one complex variable. In 1983, Duren [1] obtained the following well-known growth and deviation theorem.

Theorem 1 (see [1]). *If $f(z)$ is a normalized biholomorphic function on the unit disk D, then*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2},$$
$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}. \quad (1)$$

Many scholars tried to extend the beautiful results to the cases in several complex variables. However, Cartan [2] pointed out that the corresponding growth theorem does not hold in several complex variables. He suggested that we may consider the biholomorphic mappings with special geometrical characteristic, such as convex mappings and starlike mappings.

In 1991, Barnard et al. [3] obtained the growth theorems for starlike mappings on the unit ball B^n in $\mathbb{C}^n$ firstly. After that, there are a lot of followup studies. Gong et al. [4] extended the results to the cases on B^n and obtained the growth theorems for starlike mappings on the bounded convex Reinhardt domains B_p. Graham and Varolin [5] obtained the growth and covering theorems for normalized biholomorphic convex functions on the unit disk and also obtained the growth and covering theorems for normalized biholomorphic starlike functions on the unit disk by Alexander's theorem. Liu and Ren [6] obtained the growth theorems for starlike mappings on the general bounded starlike and circular domains in $\mathbb{C}^n$. Liu and Lu [7] obtained the growth theorems for starlike mappings of order α on the bounded starlike and circular domains. Feng and Lu [8] obtained the growth theorems for almost starlike mappings of order α on the bounded starlike and circular domains. Honda [9] obtained the growth theorems for normalized biholomorphic k-symmetric convex mappings on the unit ball in complex Banach spaces. In recent years, there are a lot of new results about the growth and covering theorems for the subclasses of biholomorphic mappings in several complex variables [10–12].

It can be seen that we can make a great breakthrough in the growth and covering theorems for the subclasses of biholomorphic mappings in several complex variables if we restrict the biholomorphic mappings with the geometrical characteristic. The mappings discussed focus on starlike mappings, convex mappings, and their subclasses.

In 1974, Suffridge extended starlike mappings and convex mappings and gave the definition of spirallike mappings. Gurganus [13] gave the definition of spirallike mappings of type β in several complex variables. Hamada and Kohr [14] obtained the growth theorems for spirallike mappings on some domains. Later Feng [15] gave the definition of almost spirallike mappings of type β and order α on the unit ball B^n in $\mathbb{C}^n$. Feng et al. [16] obtained the growth theorems for almost spirallike mappings of type β and order α on the unit ball in complex Banach spaces.

However, when we introduce the definition of the new subclasses of starlike mappings, convex mappings, and spirallike mappings, we always discuss them in $\mathbb{C}$ firstly.

In [17], Cai and Liu gave the definition of strongly almost spirallike functions of type β and order α on the unit disk. They also discussed their coefficient estimates.

In this paper, we mainly discuss the growth theorems for strongly almost spirallike functions of type β and order α on D, where D is the unit disk. Moreover we get the growth theorems for strongly almost starlike functions of order α and strongly starlike functions on D. At last, we obtain the deviation results of these functions.

Definition 2 (see [17]). Suppose that $f(z)$ is an analytic function on D, $\alpha \in [0,1)$, $\beta \in (-\pi/2, \pi/2)$, $c \in (0,1)$, and

$$\left|\frac{-\alpha + i\tan\beta}{1-\alpha} + \frac{1 - i\tan\beta}{1-\alpha} \cdot \frac{f(z)}{zf'(z)} - \frac{1+c^2}{1-c^2}\right| < \frac{2c}{1-c^2}, \quad z \in D \setminus \{0\}. \tag{2}$$

Then $f(z)$ is called a strongly almost spirallike function of type β and order α on D.

We can get the definition of strongly spirallike functions of type β [18], strongly almost starlike functions of order α [19], and strongly starlike functions on D [19] by setting $\alpha = 0$, $\beta = 0$, and $\alpha = \beta = 0$, respectively, in Definition 2.

In order to give the main results, we need the following lemmas.

Lemma 3 (see [1]). *Let $g(z)$ be an univalent analytic function on D. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$, $f(D) \subset g(D)$.*

Lemma 4 (see [20]). $|(z - z_1)/(z - z_2)| = k$ $(0 < k \neq 1, z_1 \neq z_2)$ *represents a circle whose center is z_0 and whose radius is ρ in $\mathbb{C}$, where*

$$z_0 = \frac{z_1 - k^2 z_2}{1-k^2}, \qquad \rho = \frac{k\,|z_1 - z_2|}{1-k^2}. \tag{3}$$

Lemma 5 (see [20]). *Let $f(z) : D \to D$ be an analytic function on D and $f(0) = 0$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for $\forall z \in D$.*

2. Main Results

Theorem 6. *Let $f(z)$ be a strongly almost spirallike function of type β and order α on D and $\alpha \in [1/2, 1)$, $\beta \in (-\pi/2, \pi/2)$, $c \in (0,1)$. Then*

$$\frac{1 - c^2|z|^2}{1 + m_1|z|^2 + n|z|} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1 + m_1|z|^2 + n|z|}{1 - m_2|z|^2}, \tag{4}$$

where

$$m_1 = c^2\left[2(1-\alpha)\cos\beta\left(|\sin\beta| + \cos\beta\right) - 1\right],$$
$$n = 2c(1-\alpha)\cos\beta, \qquad m_2 = c^2\left[1 - 4\alpha(1-\alpha)\cos^2\beta\right]. \tag{5}$$

Proof. Since $f(z)$ is a strongly almost spirallike function of type β and order α on D, we get

$$\left|\frac{-\alpha + i\tan\beta}{1-\alpha} + \frac{1 - i\tan\beta}{1-\alpha} \cdot \frac{f(z)}{zf'(z)} - \frac{1+c^2}{1-c^2}\right| < \frac{2c}{1-c^2}. \tag{6}$$

Let

$$p(z) = \frac{-\alpha + i\tan\beta}{1-\alpha} + \frac{1 - i\tan\beta}{1-\alpha} \cdot \frac{f(z)}{zf'(z)}. \tag{7}$$

Then

$$p(0) = 1, \qquad \left|p(z) - \frac{1+c^2}{1-c^2}\right| < \frac{2c}{1-c^2}, \tag{8}$$

so we have $p(z) \prec (1+cz)/(1-cz)$. Therefore we get that there exists an analytic function $w(z)$ on D which satisfies $p(z) = (1 + cw(z))/(1 - cw(z))$, where $w(0) = 0$, $|w(z)| < 1$. Then

$$\frac{-\alpha + i\tan\beta}{1-\alpha} + \frac{1 - i\tan\beta}{1-\alpha} \cdot \frac{f(z)}{zf'(z)} = \frac{1 + cw(z)}{1 - cw(z)}. \tag{9}$$

Immediately, we have

$$cw(z) = \frac{\left(f(z)/zf'(z)\right) - 1}{\left(f(z)/zf'(z)\right) + \left(\left(1 - 2\alpha + i\tan\beta\right)/\left(1 - i\tan\beta\right)\right)}. \tag{10}$$

It follows that

$$\left|\left[\frac{f(z)}{zf'(z)} - 1\right]\left[\frac{f(z)}{zf'(z)} - \frac{2\alpha - 1 - i\tan\beta}{1 - i\tan\beta}\right]^{-1}\right| = c\,|w(z)|. \tag{11}$$

From Lemma 3, we deduce that the image of the unit disk D under the mapping $f(z)/zf'(z)$ is the disk whose center is a and whose radius is ρ, where

$$a = \left[1 - c^2|w(z)|^2 \cdot \frac{2\alpha - 1 - i\tan\beta}{1 - i\tan\beta}\right]\frac{1}{1 - c^2|w(z)|^2}$$
$$= \frac{1}{1 - c^2|w(z)|^2}\left\{1 - c^2|w(z)|^2\left[2\alpha\cos^2\beta - \cos(2\beta) + i(\alpha - 1)\sin(2\beta)\right]\right\},$$
$$\rho = \frac{c\,|w(z)| \cdot 2(1-\alpha)\cos\beta}{1 - c^2|w(z)|^2}. \tag{12}$$

So we have

$$\left|\frac{f(z)}{zf'(z)} - a\right| \le \rho. \tag{13}$$

Then

$$|a| - \rho \le \left|\frac{f(z)}{zf'(z)}\right| \le |a| + \rho. \tag{14}$$

On the one hand, in view of (14), we have

$$\begin{aligned}\left|\frac{f(z)}{zf'(z)}\right| \le{}& \frac{1}{1-c^2|w(z)|^2} \\ &\times\Big\{\left|1-c^2|w(z)|^2\left(2\alpha\cos^2\beta-\cos(2\beta)\right)\right| \\ &\quad + c^2|w(z)|^2(1-\alpha)\left|\sin(2\beta)\right| \\ &\quad + c|w(z)|\cdot 2(1-\alpha)\cos\beta\Big\}.\end{aligned} \tag{15}$$

Observing that

$$\begin{aligned}2\alpha\cos^2\beta-\cos(2\beta) &= 2\alpha\cos^2\beta - 2\cos^2\beta + 1 \\ &= 2(\alpha-1)\cos^2\beta + 1 \\ &= 1 - 2(1-\alpha)\cos^2\beta\end{aligned} \tag{16}$$

and $1-2(1-\alpha)\cos^2\beta < 1$ for $\alpha \in [1/2, 1)$ and $\beta \in (-\pi/2, \pi/2)$, we get

$$1 - c^2|w(z)|^2\left(2\alpha\cos^2\beta - \cos(2\beta)\right) > 0 \tag{17}$$

for $c \in (0,1)$ and $|w(z)| < 1$. Thus, in view of (15), (16), and (17), we obtain

$$\begin{aligned}\left|\frac{f(z)}{zf'(z)}\right| \le{}& \frac{1}{1-c^2|w(z)|^2} \\ &\times\Big\{1-c^2|w(z)|^2\left(2\alpha\cos^2\beta-\cos(2\beta)\right) \\ &\quad + c^2|w(z)|^2(1-\alpha)\left|\sin(2\beta)\right| \\ &\quad + c|w(z)|\cdot 2(1-\alpha)\cos\beta\Big\} \\ ={}& \frac{1}{1-c^2|w(z)|^2} \\ &\times\Big\{1+c^2|w(z)|^2 \\ &\quad\times\Big[(1-\alpha)\left|\sin(2\beta)\right| \\ &\qquad -\left(2\alpha\cos^2\beta-\cos(2\beta)\right)\Big] \\ &\quad + c|w(z)|\cdot 2(1-\alpha)\cos\beta\Big\} \\ ={}& \frac{1}{1-c^2|w(z)|^2} \\ &\times\Big\{1+c^2|w(z)|^2 \\ &\quad\times\Big[2(1-\alpha)\left|\sin\beta\right|\cos\beta \\ &\qquad -\left(1-2(1-\alpha)\cos^2\beta\right)\Big] \\ &\quad + c|w(z)|\cdot 2(1-\alpha)\cos\beta\Big\} \\ ={}& \frac{1}{1-c^2|w(z)|^2} \\ &\times\Big\{1+c^2|w(z)|^2 \\ &\quad\times\left[2(1-\alpha)\cos\beta\left(\left|\sin\beta\right|+\cos\beta\right)-1\right] \\ &\quad + c|w(z)|\cdot 2(1-\alpha)\cos\beta\Big\}.\end{aligned} \tag{18}$$

Let

$$\begin{gathered}c^2\left[2(1-\alpha)\cos\beta\left(\left|\sin\beta\right|+\cos\beta\right)-1\right] = m_1, \\ 2c(1-\alpha)\cos\beta = n.\end{gathered} \tag{19}$$

Then we have

$$\left|\frac{f(z)}{zf'(z)}\right| \le \frac{1+m_1|w(z)|^2+n|w(z)|}{1-c^2|w(z)|^2}. \tag{20}$$

This means that

$$\left|\frac{zf'(z)}{f(z)}\right| \ge \frac{1-c^2|w(z)|^2}{1+m_1|w(z)|^2+n|w(z)|}. \tag{21}$$

Let

$$|w(z)| = x, \qquad \frac{1-c^2x^2}{1+m_1x^2+nx} = g(x). \tag{22}$$

Obviously, we have

$$g'(x) = -\frac{nc^2x^2+2\left(m_1+c^2\right)x+n}{\left(1+m_1x^2+nx\right)^2}. \tag{23}$$

Observing that

$$m_1 + c^2 = c^2\cdot 2(1-\alpha)\cos\beta\left(\left|\sin\beta\right|+\cos\beta\right) > 0 \tag{24}$$

and $n > 0$, $x = |w(z)| \ge 0$, we deduce that $g'(x) < 0$. So $g(x)$ is a monotone decreasing function for $x \in [0,1)$. Also we have $|w(z)| \le |z|$ from Lemma 4. Then

$$\left|\frac{zf'(z)}{f(z)}\right| \ge \frac{1-c^2|w(z)|^2}{1+m_1|w(z)|^2+n|w(z)|} \ge \frac{1-c^2|z|^2}{1+m_1|z|^2+n|z|}. \tag{25}$$

On the other hand, by direct computations, we have

$$\begin{aligned} |a|^2 &= \frac{1}{\left(1-c^2|w(z)|^2\right)^2} \\ &\quad\times\left\{\left[1-c^2|w(z)|^2\left(1+2(\alpha-1)\cos^2\beta\right)\right]^2\right. \\ &\quad\left.+c^4|w(z)|^4[2(\alpha-1)\cos\beta\sin\beta]^2\right\} \\ &= \frac{1}{\left(1-c^2|w(z)|^2\right)^2} \\ &\quad\times\left\{1-2c^2|w(z)|^2\left[1+2(\alpha-1)\cos^2\beta\right]\right. \\ &\quad\left.+c^4|w(z)|^4\left[1+4\alpha(\alpha-1)\cos^2\beta\right]\right\}, \\ |\rho|^2 &= \frac{1}{\left(1-c^2|w(z)|^2\right)^2}\cdot 4c^2|w(z)|^2(1-\alpha)^2\cos^2\beta. \end{aligned} \tag{26}$$

It follows that

$$\begin{aligned} &\left(|a|^2-\rho^2\right)\left(1-c^2|w(z)|^2\right)^2 \\ &\quad= 1-2c^2|w(z)|^2\left[1+2(\alpha-1)\cos^2\beta+2(1-\alpha)^2\cos^2\beta\right] \\ &\qquad+c^4|w(z)|^4\left[1+4\alpha(\alpha-1)\cos^2\beta\right] \\ &\quad= 1-2c^2|w(z)|^2\left[1+2\alpha(\alpha-1)\cos^2\beta\right] \\ &\qquad+c^4|w(z)|^4\left[1+4\alpha(\alpha-1)\cos^2\beta\right] \\ &\quad= \left[1-2c^2|w(z)|^2+c^4|w(z)|^4\right]+c^4|w(z)|^4 \\ &\qquad\cdot 4\alpha(\alpha-1)\cos^2\beta-2c^2|w(z)|^2\cdot 2\alpha(\alpha-1)\cos^2\beta \\ &\quad= \left(1-c^2|w(z)|^2\right)^2+4\alpha(\alpha-1)\cos^2\beta \\ &\qquad\cdot c^2|w(z)|^2\left(c^2|w(z)|^2-1\right)>0. \end{aligned} \tag{27}$$

This means that $|a|>\rho$. By (14) we know that

$$\left|\frac{f(z)}{zf'(z)}\right|\geq|a|-\rho. \tag{28}$$

In view of (15) and (19), we have

$$\begin{aligned} &\left|\frac{zf'(z)}{f(z)}\right| \\ &\quad\leq\frac{1}{|a|-\rho}=\frac{|a|+\rho}{|a|^2-\rho^2} \\ &\quad=\left(\left(1-c^2|w(z)|^2\right)^2(|a|+\rho)\right) \\ &\qquad\times\left(\left(1-c^2|w(z)|^2\right)^2+4\alpha(\alpha-1)\cos^2\beta\right. \\ &\qquad\left.\cdot c^2|w(z)|^2\left(c^2|w(z)|^2-1\right)\right)^{-1} \\ &\quad\leq\frac{1+m_1|w(z)|^2+n|w(z)|}{1-c^2|w(z)|^2+4\alpha(1-\alpha)\cos^2\beta\cdot c^2|w(z)|^2} \\ &\quad=\frac{1+m_1|w(z)|^2+n|w(z)|}{1+\left[4\alpha(1-\alpha)\cos^2\beta-1\right]c^2|w(z)|^2}. \end{aligned} \tag{29}$$

Let

$$c^2\left[1-4\alpha(1-\alpha)\cos^2\beta\right]=m_2. \tag{30}$$

Then

$$\left|\frac{zf'(z)}{f(z)}\right|\leq\frac{1+m_1|w(z)|^2+n|w(z)|}{1-m_2|w(z)|^2}. \tag{31}$$

Let

$$|w(z)|=x,\qquad\frac{1+m_1x^2+nx}{1-m_2x^2}=h(x). \tag{32}$$

Immediately, we have

$$\begin{aligned} &h'(x) \\ &\quad=\frac{(2m_1x+n)\left(1-m_2x^2\right)+\left(1+m_1x^2+nx\right)\cdot 2m_2x}{\left(1-m_2x^2\right)^2} \\ &\quad=\frac{nm_2x^2+2(m_1+m_2)x+n}{\left(1-m_2x^2\right)^2} \\ &\quad=\frac{nm_2}{\left(1-m_2x^2\right)^2} \\ &\qquad\times\left[\left(x+\frac{m_1+m_2}{nm_2}\right)^2+\frac{n^2m_2-(m_1+m_2)^2}{n^2m_2^2}\right]. \end{aligned} \tag{33}$$

Also, we can get

$$\begin{aligned} &n^2m_2-(m_1+m_2)^2 \\ &\quad=-4c^2(1-\alpha)^2\cos^2\beta\cdot 2(1-2\alpha)\left|\sin\beta\right|\cos\beta\geq 0 \end{aligned} \tag{34}$$

for $\alpha\in[1/2,1)$, $\beta\in(-\pi/2,\pi/2)$. Moreover, it is obvious that $m_2>0$ and $n>0$. So we obtain $h'(x)>0$. Therefore $h(x)$ is a monotone increasing function for $x\in[0,1)$. In addition, we have $|w(z)|\leq|z|$ from Lemma 4. Hence

$$\left|\frac{zf'(z)}{f(z)}\right|\leq\frac{1+m_1|w(z)|^2+n|w(z)|}{1-m_2|w(z)|^2}\leq\frac{1+m_1|z|^2+n|z|}{1-m_2|z|^2}. \tag{35}$$

From the above results, we obtain

$$\frac{1-c^2|z|^2}{1+m_1|z|^2+n|z|} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+m_1|z|^2+n|z|}{1-m_2|z|^2}. \quad (36)$$

This completes the proof. □

Theorem 7. *Suppose that $f(z)$ is a strongly almost starlike function of order α on D and $\alpha \in [0,1)$, $c \in (0,1)$. Then*

$$\begin{aligned}&\frac{1-c^2|z|^2}{1+c^2(1-2\alpha)|z|^2+2c(1-\alpha)|z|}\\ &\quad\le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+c^2(1-2\alpha)|z|^2+2c(1-\alpha)|z|}{1-c^2(1-2\alpha)^2|z|^2}.\end{aligned} \quad (37)$$

Proof. Let $\beta=0$ and $\alpha \in [0,1)$ in Theorem 6. Then (34) holds, so we can obtain the same result; that is,

$$\frac{1-c^2|z|^2}{1+m_1|z|^2+n|z|} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+m_1|z|^2+n|z|}{1-m_2|z|^2}, \quad (38)$$

where

$$\begin{aligned}m_1 &= c^2(1-2\alpha), \qquad n = 2c(1-2\alpha),\\ m_2 &= c^2(1-2\alpha)^2.\end{aligned} \quad (39)$$

Therefore we get the conclusion. □

Let $\alpha=0$ in Theorem 7; we can get the following result for strongly starlike functions.

Corollary 8. *Let $f(z)$ be a strongly starlike function on D and $c \in (0,1)$. Then*

$$\frac{1-c|z|}{1+c|z|} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+c|z|}{1-c|z|}. \quad (40)$$

Theorem 9. *Let $f(z)$ be a strongly almost spirallike function of type β and order α on D and $\alpha \in (1/2,1)$, $\beta \in (-\pi/2,\pi/2)$, $c \in (0,1)$. Then*

$$\begin{aligned}|f(z)| &\le |z|\left(1+\sqrt{m_2}|z|\right)^{(m_1+m_2-n\sqrt{m_2})/-2m_2}\\ &\quad\cdot\left(1-\sqrt{m_2}|z|\right)^{(m_1+m_2+n\sqrt{m_2})/-2m_2},\end{aligned} \quad (41)$$

where

$$\begin{aligned}m_1 &= c^2\left[2(1-\alpha)\cos\beta\left(|\sin\beta|+\cos\beta\right)-1\right],\\ n &= 2c(1-\alpha)\cos\beta,\\ m_2 &= c^2\left[1-4\alpha(1-\alpha)\cos^2\beta\right].\end{aligned} \quad (42)$$

Proof. From Theorem 6, we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+m_1|z|^2+n|z|}{1-m_2|z|^2}. \quad (43)$$

Let $z=re^{i\theta}$. Since

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = r\frac{\partial \ln|f(z)|}{\partial r}, \quad (44)$$

we get

$$r\frac{\partial \ln|f(z)|}{\partial r} \le \frac{1+m_1|z|^2+n|z|}{1-m_2|z|^2}. \quad (45)$$

Thus

$$\int_{\varepsilon}^{|z|}\frac{\partial \ln|f(z)|}{\partial r}dr \le \int_{\varepsilon}^{|z|}\frac{1+m_1r^2+nr}{(1-m_2r^2)r}dr. \quad (46)$$

Furthermore,

$$\begin{aligned}&\int_{\varepsilon}^{|z|}\frac{1+m_1r^2+nr}{(1-m_2r^2)r}dr\\ &\quad=(m_1+m_2)\int_{\varepsilon}^{|z|}\frac{r}{1-m_2r^2}dr\\ &\qquad+n\int_{\varepsilon}^{|z|}\frac{dr}{1-m_2r^2}+\int_{\varepsilon}^{|z|}\frac{dr}{r}\\ &\quad=\frac{m_1+m_2}{-2m_2}\ln\left|1-m_2r^2\right|\Big|_{r=\varepsilon}^{r=|z|}\\ &\qquad+\frac{n}{2\sqrt{m_2}}\ln\left|\frac{-2m_2r-2\sqrt{m_2}}{-2m_2r+2\sqrt{m_2}}\right|\Big|_{r=\varepsilon}^{r=|z|}+\ln r\Big|_{r=\varepsilon}^{r=|z|}.\end{aligned} \quad (47)$$

It follows that

$$\begin{aligned}&\ln\left|f\left(re^{i\theta}\right)\right|\Big|_{r=\varepsilon}^{r=|z|}\\ &\quad\le\frac{m_1+m_2}{-2m_2}\ln\left|1-m_2r^2\right|\Big|_{r=\varepsilon}^{r=|z|}\\ &\qquad+\frac{n}{2\sqrt{m_2}}\ln\left|\frac{\sqrt{m_2}r+1}{\sqrt{m_2}r-1}\right|\Big|_{r=\varepsilon}^{r=|z|}+\ln r\Big|_{r=\varepsilon}^{r=|z|}.\end{aligned} \quad (48)$$

Let $\varepsilon \to 0$; we have

$$\begin{aligned}\ln|f(z)| &\le \frac{m_1+m_2}{-2m_2}\ln\left|1-m_2|z|^2\right|\\ &\quad+\frac{n}{2\sqrt{m_2}}\ln\left|\frac{\sqrt{m_2}|z|+1}{\sqrt{m_2}|z|-1}\right|+\ln|z|.\end{aligned} \quad (49)$$

Consequently,

$$\begin{aligned}|f(z)| &\le |z|\cdot\left|1-m_2|z|^2\right|^{(m_1+m_2)/-2m_2}\\ &\quad\cdot\left|\frac{\sqrt{m_2}|z|+1}{\sqrt{m_2}|z|-1}\right|^{n/2\sqrt{m_2}}.\end{aligned} \quad (50)$$

Observing that $m_2 < 1$, we have

$$\begin{aligned}
&\left|f(z)\right| \\
&\leq |z|\cdot\left(1-m_2|z|^2\right)^{(m_1+m_2)/-2m_2} \\
&\quad\cdot\left(\frac{1+\sqrt{m_2}\,|z|}{1-\sqrt{m_2}\,|z|}\right)^{n/2\sqrt{m_2}} \\
&=|z|\left(1+\sqrt{m_2}\,|z|\right)^{((m_1+m_2)/-2m_2)+(n/2\sqrt{m_2})} \\
&\quad\cdot\left(1-\sqrt{m_2}\,|z|\right)^{((m_1+m_2)/-2m_2)-(n/2\sqrt{m_2})} \\
&=|z|\left(1+\sqrt{m_2}\,|z|\right)^{(m_1+m_2-n\sqrt{m_2})/-2m_2} \\
&\quad\cdot\left(1-\sqrt{m_2}\,|z|\right)^{(m_1+m_2+n\sqrt{m_2})/-2m_2}.
\end{aligned} \tag{51}$$

This completes the proof. □

Similar to Theorem 9, by Theorem 7, we can get the following results.

Theorem 10. *Let $f(z)$ be a strongly almost starlike function of order $1/2$ on D and $c\in(0,1)$. Then*

$$\left|f(z)\right| \leq e^{c}\,|z|. \tag{52}$$

Theorem 11. *Let $f(z)$ be a strongly almost starlike function of order α on D and $\alpha\in[0,1)\setminus\{1/2\}, c\in(0,1)$. Then*

$$\begin{aligned}
\left|f(z)\right| \leq\; & |z|\cdot\left[1+c\,|1-2\alpha|\,|z|\right]^{((1-\alpha)/(2\alpha-1))+((1-\alpha)/|1-2\alpha|)} \\
&\cdot\left[1-c\,|1-2\alpha|\,|z|\right]^{((1-\alpha)/(2\alpha-1))-((1-\alpha)/|1-2\alpha|)}.
\end{aligned} \tag{53}$$

Remark 12. Let $1/2<\alpha<1$ in Theorem 11. Then we have

$$\left|f(z)\right| \leq |z|\cdot\left[1+c\,(2\alpha-1)\,|z|\right]^{(2(1-\alpha))/(2\alpha-1)}. \tag{54}$$

Let $0<\alpha<1/2$ in Theorem 10. Then we have

$$\left|f(z)\right| \leq |z|\cdot\left[1-c\,(1-2\alpha)\,|z|\right]^{(2(1-\alpha))/(2\alpha-1)}. \tag{55}$$

Let $\alpha=0$ in Theorem 11; we can get the following result.

Corollary 13. *Let $f(z)$ be a strongly starlike function on D and $c\in(0,1)$. Then*

$$\left|f(z)\right| \leq \frac{|z|}{(1-c\,|z|)^2}. \tag{56}$$

Proof. According to Corollary 8, we obtain

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+c\,|z|}{1-c\,|z|}. \tag{57}$$

Let $z=re^{i\theta}$. Since

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = r\frac{\partial\ln\left|f(z)\right|}{\partial r}, \tag{58}$$

we have

$$r\frac{\partial\ln\left|f(z)\right|}{\partial r} \leq \frac{1+c\,|z|}{1-c\,|z|}. \tag{59}$$

Thus

$$\begin{aligned}
\int_{\varepsilon}^{|z|}\frac{\partial\ln\left|f(z)\right|}{\partial r}dr &\leq \int_{\varepsilon}^{|z|}\frac{1+cr}{(1-cr)\,r}dr \\
&= \int_{\varepsilon}^{|z|}\frac{2c}{1-cr}dr+\int_{\varepsilon}^{|z|}\frac{dr}{r}.
\end{aligned} \tag{60}$$

So we get

$$\ln\left|f\left(re^{i\theta}\right)\right|\Big|_{r=\varepsilon}^{r=|z|} \leq 2c\frac{\ln(1-cr)}{-c}\Big|_{r=\varepsilon}^{r=|z|}+\ln|r|\Big|_{r=\varepsilon}^{r=|z|}. \tag{61}$$

Letting $\varepsilon\to 0$, it follows that

$$\ln\left|f(z)\right| \leq -2\ln(1-c\,|z|)+\ln|z|. \tag{62}$$

Therefore we obtain

$$\left|f(z)\right| \leq \frac{|z|}{(1-c\,|z|)^2}. \tag{63}$$

Also, we can get the conclusion by letting $\alpha=0$ in Theorem 11. This completes the proof. □

Theorem 14. *Suppose that $f(z)$ is a strongly starlike function on D and $c\in(0,1)$; then*

$$e^{(4(2c^2\,|z|^2-1))/(1+c|z|)^2}\cdot\frac{|z|}{(1+c\,|z|)^2} < \left|f(z)\right| \leq \frac{|z|}{(1-c\,|z|)^2}. \tag{64}$$

Proof. On the one hand, from Corollary 13, we obtain $|f(z)|\leq(|z|)/(1-c|z|)^2$.

On the other hand, by a and ρ in the proof of Theorem 6, we can obtain

$$\frac{\operatorname{Re}a-\rho}{(|a|+\rho)^2} = \frac{(1-c\,|w(z)|)^3}{(1+c\,|w(z)|)^3} \tag{65}$$

for $\alpha=\beta=0$. Let $\lambda(x)=(1-cx)^3/(1+cx)^3$. Then we have

$$\lambda'(x) = \frac{-6c(1-cx)^2}{(1+cx)^4} < 0. \tag{66}$$

Therefore $(1-c|w(z)|)^3/(1+c|w(z)|)^3$ is a monotone increasing function with respect to $|w(z)|$. Also we can know that $|w(z)|\leq|z|$ from Lemma 4. Hence

$$\frac{\operatorname{Re}a-\rho}{(|a|+\rho)^2} = \frac{(1-c\,|w(z)|)^3}{(1+c\,|w(z)|)^3} > \frac{(1-c\,|z|)^3}{(1+c\,|z|)^3}. \tag{67}$$

By (14) we obtain

$$\operatorname{Re}\frac{f(z)}{zf'(z)} \geq \operatorname{Re}a-\rho. \tag{68}$$

Furthermore, $|f(z)/zf'(z)| \le |a| + \rho$, so

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \frac{\operatorname{Re}\left(f(z)/zf'(z)\right)}{\left|f(z)/zf'(z)\right|^2} > \frac{\operatorname{Re} a - \rho}{\left(|a| + \rho\right)^2}. \tag{69}$$

Let $z = re^{i\theta}$. Since $\operatorname{Re}(zf'(z)/f(z)) = r(\partial \ln|f(z)|)/\partial r$, we have

$$r\frac{\partial \ln\left|f(z)\right|}{\partial r} > \frac{(1-c|z|)^3}{(1+c|z|)^3}. \tag{70}$$

Therefore we obtain

$$\int_{\varepsilon}^{|z|} \frac{\partial \ln\left|f(z)\right|}{\partial r} dr > \int_{\varepsilon}^{|z|} \frac{(1-cr)^3}{(1+cr)^3} \cdot \frac{dr}{r}. \tag{71}$$

Then we have

$$\begin{aligned}
&\ln\left|f(z)\right| \\
&\quad > -\frac{4}{c|z|} \cdot \frac{1}{(1+c|z|)^2} \\
&\qquad + \frac{4}{c}\left[\frac{1}{|z|} + \frac{c}{1+c|z|} - c + 2c\ln|z| - 2c\ln(1+c|z|)\right].
\end{aligned} \tag{72}$$

So

$$\left|f(z)\right| > e^{(4(2c^2|z|^2-1))/(1+c|z|)^2} \cdot \frac{|z|}{(1+c|z|)^2}. \tag{73}$$

Therefore we obtain

$$\begin{aligned}
&e^{(4(2c^2|z|^2-1))/(1+c|z|)^2} \cdot \frac{|z|}{(1+c|z|)^2} \\
&\quad < \left|f(z)\right| \le \frac{|z|}{(1-c|z|)^2}.
\end{aligned} \tag{74}$$

This completes the proof. □

From Theorems 6 and 9, we can get the following result.

Theorem 15. *Let $f(z)$ be a strongly almost spirallike function of type β and order α on D and $\alpha \in (1/2, 1)$, $\beta \in (-\pi/2, \pi/2)$, $c \in (0, 1)$. Then*

$$\begin{aligned}
&\left|f'(z)\right| \\
&\quad \le \left(1 + m_1|z|^2 + n|z|\right)\left(1 + \sqrt{m_2}|z|\right)^{(m_1+3m_2-n\sqrt{m_2})/-2m_2} \\
&\qquad \cdot \left(1 - \sqrt{m_2}|z|\right)^{(m_1+3m_2+n\sqrt{m_2})/-2m_2},
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
m_1 &= c^2\left[2(1-\alpha)\cos\beta\left(\left|\sin\beta\right| + \cos\beta\right) - 1\right], \\
m_2 &= c^2\left[1 - 4\alpha(1-\alpha)\cos^2\beta\right], \\
n &= 2c(1-\alpha)\cos\beta.
\end{aligned} \tag{76}$$

From Theorems 7 and 11, we can get the following result.

Theorem 16. *Let $f(z)$ be a strongly almost starlike function of order α on D and $\alpha \in [0,1) \setminus \{1/2\}$, $c \in (0,1)$. Then*

$$\begin{aligned}
\left|f'(z)\right| &\le \left[1 + c^2(1-2\alpha)|z|^2 + 2c(1-\alpha)|z|\right] \\
&\quad \times \left[1 + c|1-2\alpha||z|\right]^{((2-3\alpha)/(2\alpha-1))+((1-\alpha)/|1-2\alpha|)} \\
&\quad \cdot \left[1 - c|1-2\alpha||z|\right]^{((2-3\alpha)/(2\alpha-1))-((1-\alpha)/|1-2\alpha|)}.
\end{aligned} \tag{77}$$

Let $\alpha = 0$ in Theorem 16; we can get the following result.

Corollary 17. *Let $f(z)$ be a strongly starlike function on D and $c \in (0,1)$. Then*

$$\left|f'(z)\right| \le \frac{1+c|z|}{(1-c|z|)^3}. \tag{78}$$

Conflict of Interests

The authors declare that they have no conflict of interests.

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