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Research Article

Applications of the Novel (G'/G)-Expansion Method for a Time Fractional Simplified Modified Camassa-Holm (MCH) Equation

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We use the fractional derivatives in modified Riemann-Liouville derivative sense to construct exact solutions of time fractional simplified modified Camassa-Holm (MCH) equation. A generalized fractional complex transform is properly used to convert this equation to ordinary differential equation and, as a result, many exact analytical solutions are obtained with more free parameters. When these free parameters are taken as particular values, the traveling wave solutions are expressed by the hyperbolic functions, the trigonometric functions, and the rational functions. Moreover, the numerical presentations of some of the solutions have been demonstrated with the aid of commercial software Maple. The recital of the method is trustworthy and useful and gives more new general exact solutions.

1. Introduction

The class of fractional calculus is one of the most convenient classes of fractional differential equations which were viewed as generalized differential equations [1]. In the sense that much of the theory and, hence, applications of differential equations can be extended smoothly to fractional differential equations with the same flavor and spirit of the realm of differential equation, the seeds of fractional calculus were planted over three hundred years ago from a gracious idea of L'Hopital, who wrote a letter to Leibniz on 1695, asking about a rigorous description of the derivative of order $n = 0.5$. Fractional calculus is the theory of differentiation and integration of noninteger order and embodies the generality of the conventional differential and integral calculus. Therefore, some of the properties of the fractional integral and derivatives differ from the conventional ones in order to allow its implementation in a broader assortment of cases, which cannot be appropriately illustrated by the conventional integer-order calculus. Fractional calculus is painstaking to be a very authoritative tool to help scientists to unearth the concealed properties of the dynamics of multifaceted systems in all fields of sciences and engineering. In recent years, fractional calculus played an imperative role of a proficient, expedient, and elementary theoretical structure for more adequate modeling of multifaceted dynamic processes. Therefore, mounting applications of fractional calculus can be seen in modeling, signal processing, electromagnetism, mechanics, physics, biology, medicine, chemistry, bioengineering, biological systems, and in many other areas [2, 3]. Recently, it has turned out that those differential equations are involving derivatives of noninteger [4]. For example, the nonlinear oscillation of earthquakes can be modeled with fractional derivatives [5]. More recently, applications have included classes of nonlinear equation with multiorder fractional derivatives. We apply a generalized fractional complex transform [6–9] to convert fractional order differential equation to ordinary differential equation. Many important phenomena in electromagnetic, viscoelasticity, electrochemistry, and material science are well described by differential equations of fractional order [10–14]. A physical interpretation of the fractional calculus was given in [15–19]. With the development of symbolic computation software, like Maple, many numerical and analytical methods to search for exact solutions of NLEEs have attracted more attention. As a result, the researchers developed and established many methods, for example, the Cole-Hopf transformation [20], the Tanh-function method [21–24], the inverse scattering transform method [25], the variational iteration method [26, 27], Exp-function method [28–31], and F-expansion method [32, 33] that are used for searching the exact solutions.

Recently, a straightforward and concise method, called (G'/G)-expansion method, was introduced by Wang et al. [34] and demonstrated that it is a powerful method for seeking analytic solutions of NLEEs. (G'/G)-expansion is a reliable technique, which gives various types of the solitary wave solutions including the hyperbolic functions, the trigonometric functions, and the rational functions. It is also evident from the literature that such solutions always satisfy the given nonlinear differential equations. For additional references, see the articles [35–40]. In order to establish the efficiency and assiduousness of (G'/G)-expansion method and to extend the range of applicability, further research has been carried out by several researchers. For instance, Zhang et al. [41] made a generalization of (G'/G)-expansion method for the evolution equations with variable coefficients. Zhang et al. [42] also presented an improved (G'/G)-expansion method to seek more general traveling wave solutions. Zayed [43] presented a new approach of (G'/G)-expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation, $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0$, where e_2, e_1, e_0 are arbitrary constants and obtained new exact solutions. Zayed [44] again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = AG(\xi) + BG^2(\xi)$, where A and B are arbitrary constants.

In this paper, we will apply novel (G'/G)-expansion method introduced by Alam et al. [45] to solve the time fractional simplified modified Camassa-Holm (MCH) equation in the sense of modified Riemann-Liouville derivative by Jumarie [46] and abundant new families of exact solutions are found. The Jumarie modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \times \dfrac{d}{dt}\displaystyle\int_0^t (t-\xi)^{-\alpha}\left(f(\xi) - f(0)\right) d\xi, & 0 < \alpha < 1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, & n \le \alpha < n+1, \\ & n \ge 1. \end{cases} \quad (1)$$

Some important properties of Jumarie's derivative are

$$D_t^\alpha f(t) = \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-\alpha)} t^{\tau-\alpha}, \quad (2)$$

$$D_t^\alpha \left(f(t)\, g(t)\right) = g(t)\, D_t^\alpha f(t) + f(t)\, D_t^\alpha g(t), \quad (3)$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)]\, D_t^\alpha g(t) = D_g^\alpha f[g(t)] \left(g'(t)\right)^\alpha. \quad (4)$$

2. Description of the Method

Suppose that a fractional partial differential equation in the independent variables, say t, is given by

$$S\left(u, u_x, u_t, D_t^\alpha u, \ldots\right) = 0, \quad 0 < \alpha \le 1, \quad (5)$$

where $D_t^\alpha u$ is Jumarie's modified Riemann-Liouville derivatives of $u, u(x,t)$ is an unknown function, S is a polynomial in u, and its various partial derivatives including fractional derivatives in which the highest order derivatives and nonlinear terms are involved.

The main steps of the method are as follows.

Step 1. Li and He [7] proposed a fractional complex transformation to convert fractional partial differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$u(x,t) = u(\xi), \qquad \xi = Lx + V\frac{t^\alpha}{\Gamma(1+\alpha)}, \quad (6)$$

where L, V are arbitrary constants with $L, V \neq 0$, permits us to convert (5) into an ordinary differential equation of integer order in the form

$$P\left(u, u', u'', u''', \ldots\right) = 0, \quad (7)$$

where the superscripts stand for the ordinary derivatives with respect to ξ.

Step 2. Integrating (7) term by term one or more times if possible yields constant(s) of integration which can be calculated later on.

Step 3. Assume that the solution of (7) can be represented as

$$u(\xi) = \sum_{i=-m}^{m} \alpha_i (k + \Phi(\xi))^i, \quad (8)$$

where

$$\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}, \quad (9)$$

where both α_{-m} and α_m cannot be zero simultaneously. α_i $(i = 0, \pm1, \pm2, \ldots, \pm m)$ and k are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ordinary differential equation as an auxiliary equation

$$GG'' = AGG' + BG^2 + C\left(G'\right)^2, \quad (10)$$

where A, B, and C are real constants.

Equation (10) can be reduced to the following Riccati equation by making use of the Cole-Hopf transformation $\Phi(\xi) = \ln(G(\xi))_\xi = G'(\xi)/G(\xi)$ as

$$\Phi'(\xi) = B + A\Phi(\xi) + (C-1)\Phi^2(\xi). \quad (11)$$

Equation (11) has twenty five solutions [47].

Step 4. The positive integer m can be determined by balancing the highest order linear term with the nonlinear term of the highest order come out in (7).

Step 5. Substituting (8) together with (9) and (10) into (7), we obtain polynomials in $(k+(G'/G))^{i}$ and $(k+(G'/G))^{-i}$ $(i = 0,1,2,\ldots,m)$. Collecting each coefficient of the resulted polynomials to zero yields an overdetermined set of algebraic equations for α_i $(i=0,\pm1,\pm2,\ldots,\pm m)$, k, L, and V.

Step 6. The values of the arbitrary constants can be obtained by solving the algebraic equations obtained in Step 4. The obtained values of the arbitrary constants and the solutions of (10) yield abundant exact traveling wave solutions of the nonlinear evolution equation (5).

3. Application of the Method to the Time Fractional Simplified (MCH) Equation

Now, consider the following time fractional simplified modified Camassa-Holm (MCH) equation:

$$\begin{gathered} D_t^{\alpha}u + 2\delta u_x - u_{xxt} + \gamma u^2 u_x = 0, \\ \text{where } \delta \in \Re, \quad \gamma > 0, \quad 0 < \alpha \leq 1, \end{gathered} \tag{12}$$

which is the variation of the equation

$$\begin{gathered} u_t + 2\delta u_x - u_{xxt} + \gamma u^2 u_x = 0, \\ \text{where } \delta \in \Re, \quad \gamma > 0. \end{gathered} \tag{13}$$

Many researchers investigated the simplified MCH equation by using different methods to establish exact solutions. For example, Liu et al. [48] were concerned about the (G'/G)-expansion method to solve the simplified MCH equation, whereas the second order linear ordinary differential equation (LODE) is considered as an auxiliary equation. Wazwaz [49] studied this equation by using the sine-cosine algorithm. Zaman and Sultana [50] used the (G'/G)-expansion method together with the generalized Riccati equation to MCH equation to find the exact solutions. Alam and Akbar [51] applied the generalized (G'/G)-expansion method to look for the exact solutions via the simplified MCH equation. Further details of MCH equation can be found in references [52, 53].

By the use of (4), (12) is converted into an ordinary differential equation of integer order and after integrating once, we obtain

$$(V+2\delta L)\,u - VL^2u'' + \gamma L\frac{u^3}{3} + C_1 = 0, \tag{14}$$

where C_1 is an integral constant which is to be determined later.

Considering the homogeneous balance between u'' and u^3 in (14), we obtain $3m = m+2$; that is, $m=2$. Therefore, the trial solution formula (8) becomes

$$u(\xi) = \alpha_{-1}(k+\Phi(\xi))^{-1} + \alpha_0 + \alpha_1(k+\Phi(\xi)). \tag{15}$$

Using (15) into (14), left hand side is converted into polynomials in $(k+(G'/G))^{i}$ and $(k+(G'/G))^{-i}$ $(i = 0,1,2,\ldots,m)$. Equating the coefficients of same power of the resulted polynomials to zero, we obtain a system of algebraic equations for α_0, α_1, α_{-1}, k, C_1, L, and V (which are omitted for the sake of simplicity). Solving the overdetermined set of algebraic equations by using the symbolic computation software, such as Maple 13, we obtain the following four solution sets.

Set 1. Consider

$$\begin{gathered} \alpha_0 = \pm i\frac{\sqrt{6}\delta L\,(A+2k-2Ck)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}}, \\ \alpha_1 = \pm i\frac{2\sqrt{6}\delta L\,(C-1)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}}, \\ V = -\frac{4\delta L}{L^2\left(A^2-4BC+4B\right)+2}, \\ L=L, \quad k=k, \quad \alpha_{-1}=0, \quad C_1=0, \end{gathered} \tag{16}$$

where k, L, A, B, and C are arbitrary constants.

Set 2. Consider

$$\begin{gathered} \alpha_0 = \mp i\frac{\sqrt{6}\delta L\,(A+2k-2Ck)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}}, \\ \alpha_{-1} = \pm i\frac{2\sqrt{6}\delta L\left(kA+k^2-Ck^2-B\right)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}}, \\ V = -\frac{4\delta L}{L^2\left(A^2-4BC+4B\right)+2}, \\ L=L, \quad k=k, \quad \alpha_1=0, \quad C_1=0, \end{gathered} \tag{17}$$

where k, L, A, B, and C are arbitrary constants.

Set 3. Consider

$$\begin{gathered} \alpha_1 = \pm 2i\frac{\sqrt{3}\delta L\,(C-1)}{\sqrt{\gamma\left(2L^2\left(A^2-4BC+4B\right)+1\right)}}, \\ \alpha_{-1} = \pm i\frac{\sqrt{3}\delta L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma\left(2L^2\left(A^2-4BC+4B\right)+1\right)}\,(C-1)}, \\ V = -\frac{2\delta L}{2L^2\left(A^2-4BC+4B\right)+1}, \\ k = \frac{A}{2(C-1)}, \quad L=L, \quad \alpha_0=0, \quad C_1=0, \end{gathered} \tag{18}$$

where L, A, B, and C are arbitrary constants.

Set 4. Consider

$$\alpha_{-1} = \pm i\frac{\sqrt{6}\delta L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}\,(C-1)},$$
$$V = -\frac{4\delta L}{L^2\left(A^2-4BC+4B\right)+2}, \tag{19}$$
$$k = \frac{A}{2(C-1)}, \quad L = L, \quad \alpha_0 = 0, \quad \alpha_1 = 0, \quad C_1 = 0,$$

where L, A, B, and C are arbitrary constants.

Substituting (16)–(19) into (15), we obtain

$$\begin{aligned} u_1(\xi) = & \pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \times\left(k+\left(\frac{G'}{G}\right)\right), \end{aligned} \tag{20}$$

where

$$\begin{aligned} \xi &= Lx-\left(\frac{4\delta L}{L^2\left(A^2-4BC+4B\right)+2}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ u_2(\xi) = & \mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \pm i\frac{2\sqrt{6}\delta L\left(kA+k^2-Ck^2-B\right)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \times\left(k+\left(\frac{G'}{G}\right)\right)^{-1}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} \xi &= Lx-\left(\frac{4\delta L}{L^2\left(A^2-4BC+4B\right)+2}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ u_3(\xi) = & \pm 2i\frac{\sqrt{3}\delta L(C-1)}{\sqrt{\gamma\left(2L^2\left(A^2-4BC+4B\right)+1\right)}} \\ & \times\left(\frac{A}{2(C-1)}+\left(\frac{G'}{G}\right)\right) \\ & \pm i\frac{\sqrt{3}\delta L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma\left(2L^2\left(A^2-4BC+4B\right)+1\right)}\,(C-1)} \\ & \times\left(\frac{A}{2(C-1)}+\left(\frac{G'}{G}\right)\right)^{-1}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \xi &= Lx-\left(\frac{2\delta L}{2L^2\left(A^2-4BC+4B\right)+1}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}, \\ u_4(\xi) = & \pm i\frac{\sqrt{6}\delta L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}\,(C-1)} \\ & \times\left(\frac{A}{2(C-1)}+\left(\frac{G'}{G}\right)\right)^{-1}, \end{aligned} \tag{23}$$

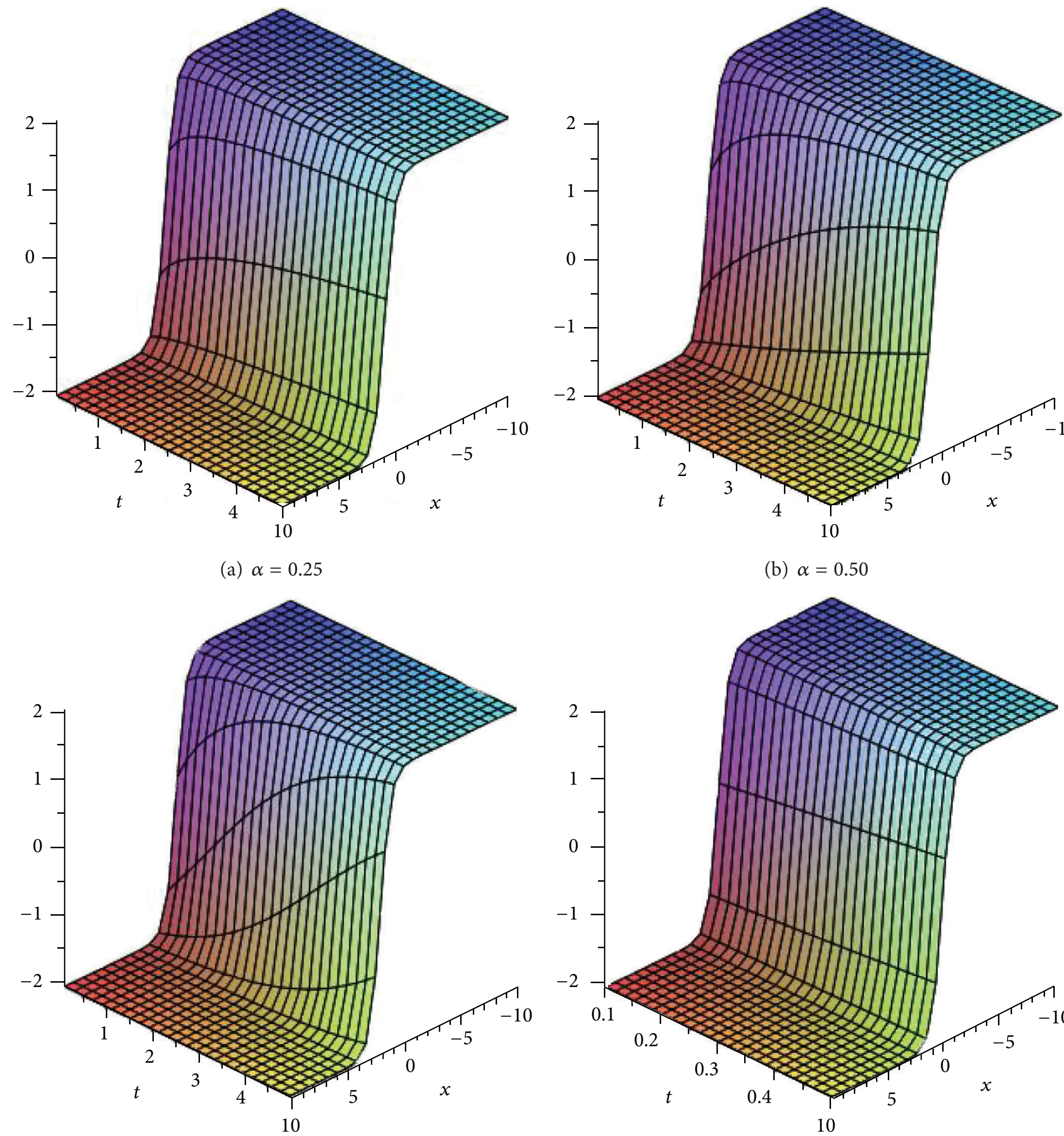
where

$$\xi = Lx-\left(\frac{4\delta L}{L^2\left(A^2-4BC+4B\right)+2}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{24}$$

Substituting the solutions $G(\xi)$ of (10) into (20) and simplifying, we obtain the following solutions.

When $\Delta = A^2-4BC+4B > 0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$) (Figure 1),

$$\begin{aligned} u_1^1(\xi) = & \pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}, \\ u_1^2(\xi) = & \pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}, \\ u_1^3(\xi) = & \pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2-4BC+4B\right)+2\right)}} \\ & \times\left\{k-\frac{1}{2(C-1)}\right. \\ & \left.\times\left(A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}, \end{aligned}$$



(a) $\alpha = 0.25$

(b) $\alpha = 0.50$

(c) $\alpha = 0.75$

(d) $\alpha = 1$

Figure 1: (a)–(d) show the kink solution for u_1^1 for different values of parameters.

$$
\begin{aligned}
u_1^4(\xi) = {} & \pm i \frac{\sqrt{6\delta} L (A + 2k - 2Ck)}{\sqrt{\gamma \left(L^2 \left(A^2 - 4BC + 4B\right) + 2\right)}} \\
& \pm i \frac{2\sqrt{6\delta} L (C - 1)}{\sqrt{\gamma \left(L^2 \left(A^2 - 4BC + 4B\right) + 2\right)}} \\
& \times \left\{ k - \frac{1}{2(C-1)} \right. \\
& \quad \left. \times \left(A + \sqrt{\Delta} \left(\coth\left(\sqrt{\Delta}\xi\right) \pm \operatorname{csch}\left(\sqrt{\Delta}\xi\right) \right) \right) \right\},
\end{aligned}
$$

$$
\begin{aligned}
u_1^5(\xi) = {} & \pm i \frac{\sqrt{6\delta} L (A + 2k - 2Ck)}{\sqrt{\gamma \left(L^2 \left(A^2 - 4BC + 4B\right) + 2\right)}} \\
& \pm i \frac{2\sqrt{6\delta} L (C - 1)}{\sqrt{\gamma \left(L^2 \left(A^2 - 4BC + 4B\right) + 2\right)}} \\
& \times \left\{ k - \frac{1}{4(C-1)} \right. \\
& \quad \times \left(2A + \sqrt{\Delta} \right.
\end{aligned}
$$

$$\times\left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\bigg)\bigg\},$$

$$u_1^6(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\left(\pm\sqrt{\Delta(F^2+H^2)}-F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\right)\times\left(F\sinh\left(\sqrt{\Delta}\xi\right)+B\right)^{-1}\right\}\right],$$

$$u_1^7(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\left(\pm\sqrt{\Delta(F^2+H^2)}+F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\right)\times\left(F\sinh\left(\sqrt{\Delta}\xi\right)+B\right)^{-1}\right\}\right], \tag{25}$$

where F and H are real constants (Figure 2). Consider

$$u_1^8(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi/2\right)-A\cosh\left(\sqrt{\Delta}\xi/2\right)}\right\},$$

$$u_1^9(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi/2\right)-A\sinh\left(\sqrt{\Delta}\xi/2\right)}\right\},$$

$$u_1^{10}(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)-A\cosh\left(\sqrt{\Delta}\xi\right)\pm i\sqrt{\Delta}}\right\},$$

$$u_1^{11}(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)-A\sinh\left(\sqrt{\Delta}\xi\right)\pm\sqrt{\Delta}}\right\}. \tag{26}$$

When $\Delta = A^2-4BC+4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$u_1^{12}(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{1}{2(C-1)}\times\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\},$$

$$u_1^{13}(\xi)=\pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k-\frac{1}{2(C-1)}\times\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\},$$

$$u_1^{14}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times\left\{k-\frac{1}{2(C-1)} \times\left(-A+\sqrt{-\Delta} \times\left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\},$$

$$u_1^{15}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times\left\{k-\frac{1}{2(C-1)} \times\left(A+\sqrt{-\Delta} \times\left(\cot\left(\sqrt{-\Delta}\xi\right)\pm\operatorname{csch}\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\},$$

$$u_1^{16}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times\left[k+\frac{1}{2(C-1)} \times\left\{-A+\left(\pm\sqrt{-\Delta(F^2-H^2)} - F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right) \times\left(F\sin\left(\sqrt{-\Delta}\xi\right)+B\right)^{-1}\right\}\right],$$

$$u_1^{17}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times\left[k+\frac{1}{2(C-1)} \times\left\{-A+\left(\pm\sqrt{-\Delta(F^2-H^2)} - F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right) \times\left(F\sin\left(\sqrt{-\Delta}\xi\right)+B\right)^{-1}\right\}\right],$$

$$u_1^{18}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times\left[k+\frac{1}{2(C-1)} \times\left\{-A+\left(\pm\sqrt{-\Delta(F^2-H^2)} + F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right) \times\left(F\sin\left(\sqrt{-\Delta}\xi\right)+B\right)^{-1}\right\}\right], \tag{27}$$

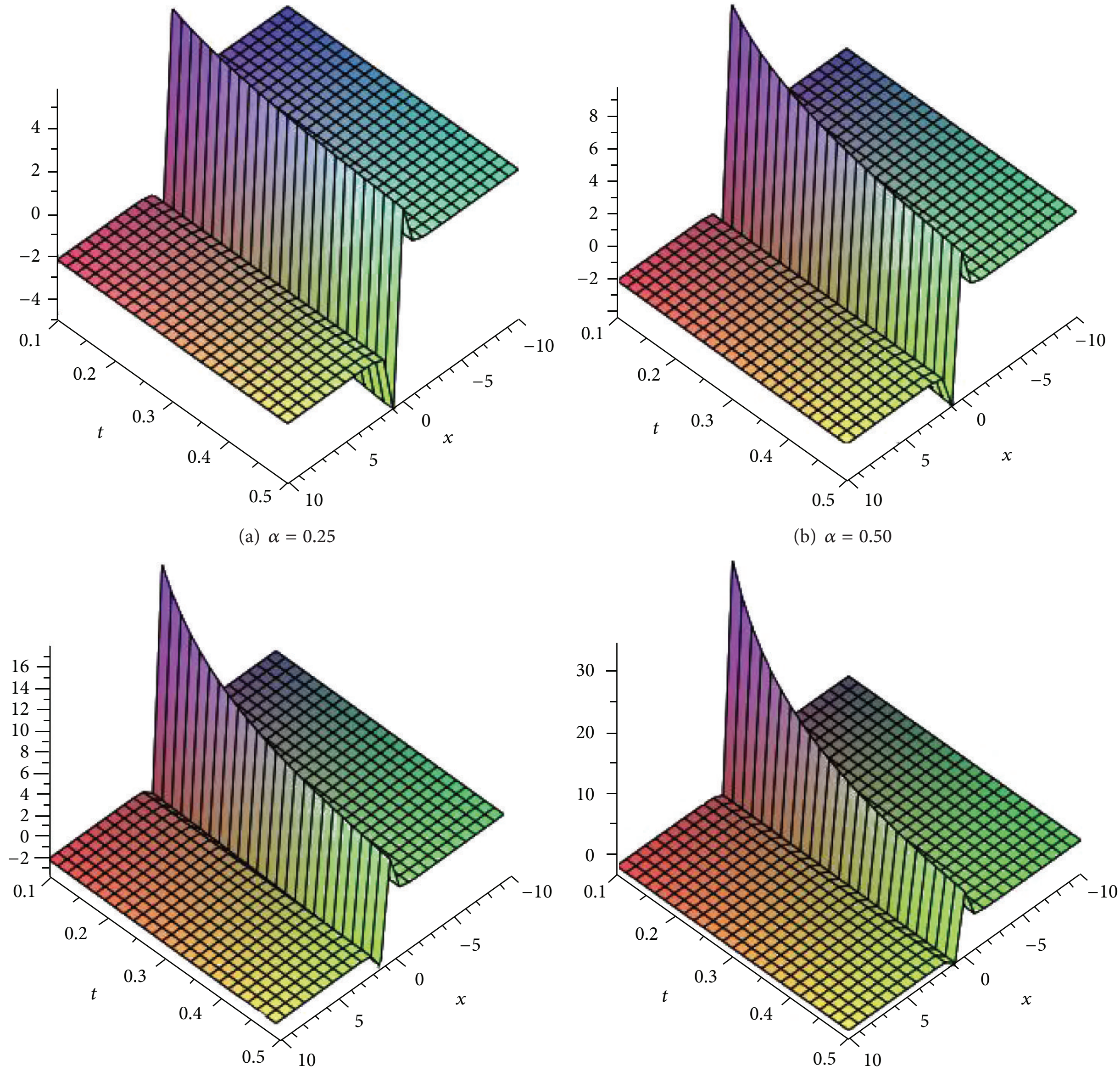
where F and H are real constants such that $F^2 - H^2 > 0$.

Consider

$$u_1^{19}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times\left\{k-\frac{2B\cos\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi/2\right)+A\cos\left(\sqrt{-\Delta}\xi/2\right)}\right\},$$

$$u_1^{20}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times\left\{k+\frac{2B\sin\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi/2\right)-A\sin\left(\sqrt{-\Delta}\xi/2\right)}\right\},$$

$$u_1^{21}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$ (c) $\alpha = 0.75$ (d) $\alpha = 1$

Figure 2: (a)–(d) show the singular solution for u_1^2 for different values of parameters.

$$\times \left\{k - \left(2B\cos\left(\sqrt{-\Delta}\xi\right)\right) \times \left(\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + A\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}\right)^{-1}\right\},$$

$$u_1^{22}(\xi) = \pm i\frac{\sqrt{6}\delta L\left(A + 2k - 2Ck\right)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i\frac{2\sqrt{6}\delta L\left(C - 1\right)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}}$$

$$\times \left\{k + \left(2B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right) \times \left(\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) \pm \sqrt{-\Delta}\right)^{-1}\right\}. \tag{28}$$

When $B = 0$ and $A(C-1) \neq 0$,

$$u_1^{23}(\xi) = \pm i\frac{\sqrt{6}\delta L\left(A + 2k - 2Ck\right)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\times\left\{k-\frac{Ac_1}{(C-1)\{c_1+\cosh(A\xi)-\sinh(A\xi)\}}\right\},$$

$$u_1^{24}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\times\left\{k-\frac{A(\cosh(A\xi)+\sinh(A\xi))}{(C-1)\{c_1+\cosh(A\xi)+\sinh(A\xi)\}}\right\}, \quad (29)$$

where c_1 is an arbitrary constant.

When $A=B=0$ and $(C-1)\neq 0$, the solution of (12) is

$$u_1^{25}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \quad (30)$$

$$\times\left\{k-\frac{1}{(C-1)\xi+c_2}\right\},$$

where c_2 is an arbitrary constant.

Substituting the solutions $G(\xi)$ of (10) in (21) and simplifying, we obtain the following solutions.

When $\Delta = A^2-4BC+4B>0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$),

$$u_2^{1}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(kA+k^2-Ck^2-B)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_2^{2}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(kA+k^2-Ck^2-B)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_2^{3}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(kA+k^2-Ck^2-B)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\times\left\{k-\frac{1}{2(C-1)}\right.$$

$$\left.\times\left(A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}^{-1}. \quad (31)$$

The other families of exact solutions of (12) are omitted for convenience.

When $\Delta = A^2-4BC+4B<0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$) (Figure 3),

$$u_2^{12}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(kA+k^2-Ck^2-B)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\times\left\{k+\frac{1}{2(C-1)}\right.$$

$$\left.\times\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_2^{13}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

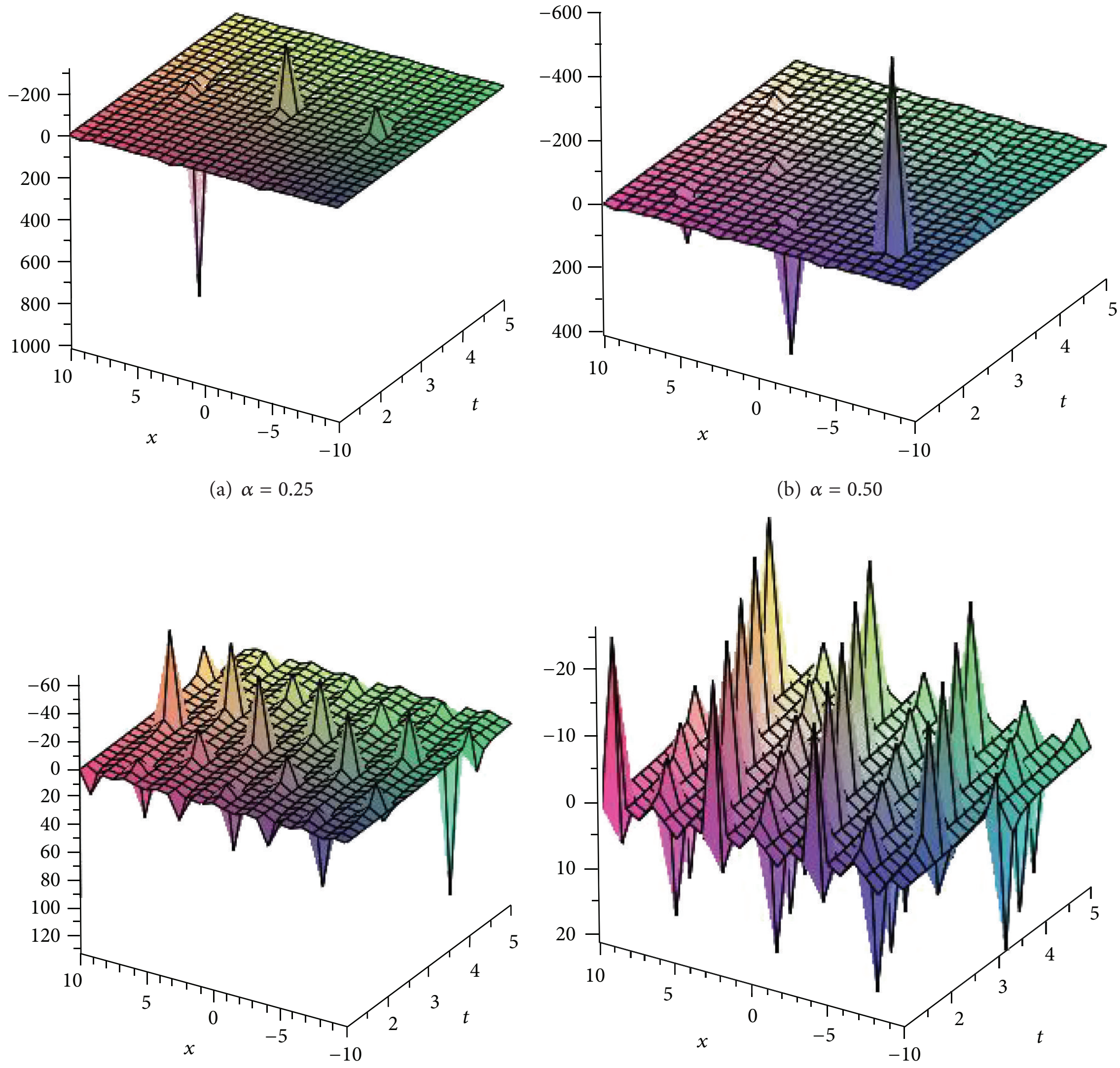
$$\pm i\frac{2\sqrt{6}\delta L(kA+k^2-Ck^2-B)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\times\left\{k-\frac{1}{2(C-1)}\right.$$

$$\left.\times\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_2^{14}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(kA+k^2-Ck^2-B)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$



(a) $\alpha = 0.25$

(b) $\alpha = 0.50$

(c) $\alpha = 0.75$

(d) $\alpha = 1$

Figure 3: (a)–(d) show the periodic solution for u_2^{12} for different values of parameters.

$$\times \left\{k + \frac{1}{2(C-1)} \times \left(-A + \sqrt{-\Delta} \times \left(\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}^{-1}. \quad (32)$$

When $A = B = 0$ and $(C-1) \neq 0$, the solution of (12) is

$$u_2^{25}(\xi) = u_2(\xi) = \mp i \frac{\sqrt{6\delta} L (A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i \frac{2\sqrt{6\delta} L \left(kA + k^2 - Ck^2 - B\right)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left\{k - \frac{1}{(C-1)\xi + c_2}\right\}^{-1}, \quad (33)$$

where c_2 is an arbitrary constant.

We can write down the other families of exact solutions of (12) which are omitted for practicality.

Similarly, by substituting the solutions $G(\xi)$ of (10) into (22) and simplifying, we obtain the following solutions.

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$
\begin{aligned}
u_3^1(\xi) = {} & \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right) \\
& \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1}, \\
u_3^2(\xi) = {} & \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right) \\
& \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1}, \\
u_3^3(\xi) = {} & \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \\
& \times\left(\frac{1}{2(C-1)}\right. \\
& \quad \left.\times\left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right) \\
& \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \\
& \times\left(\frac{1}{2(C-1)}\right. \\
& \quad \left.\times\left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right)^{-1}.
\end{aligned} \tag{34}
$$

Others families of exact solutions are omitted for the sake of simplicity.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$) (Figure 4),

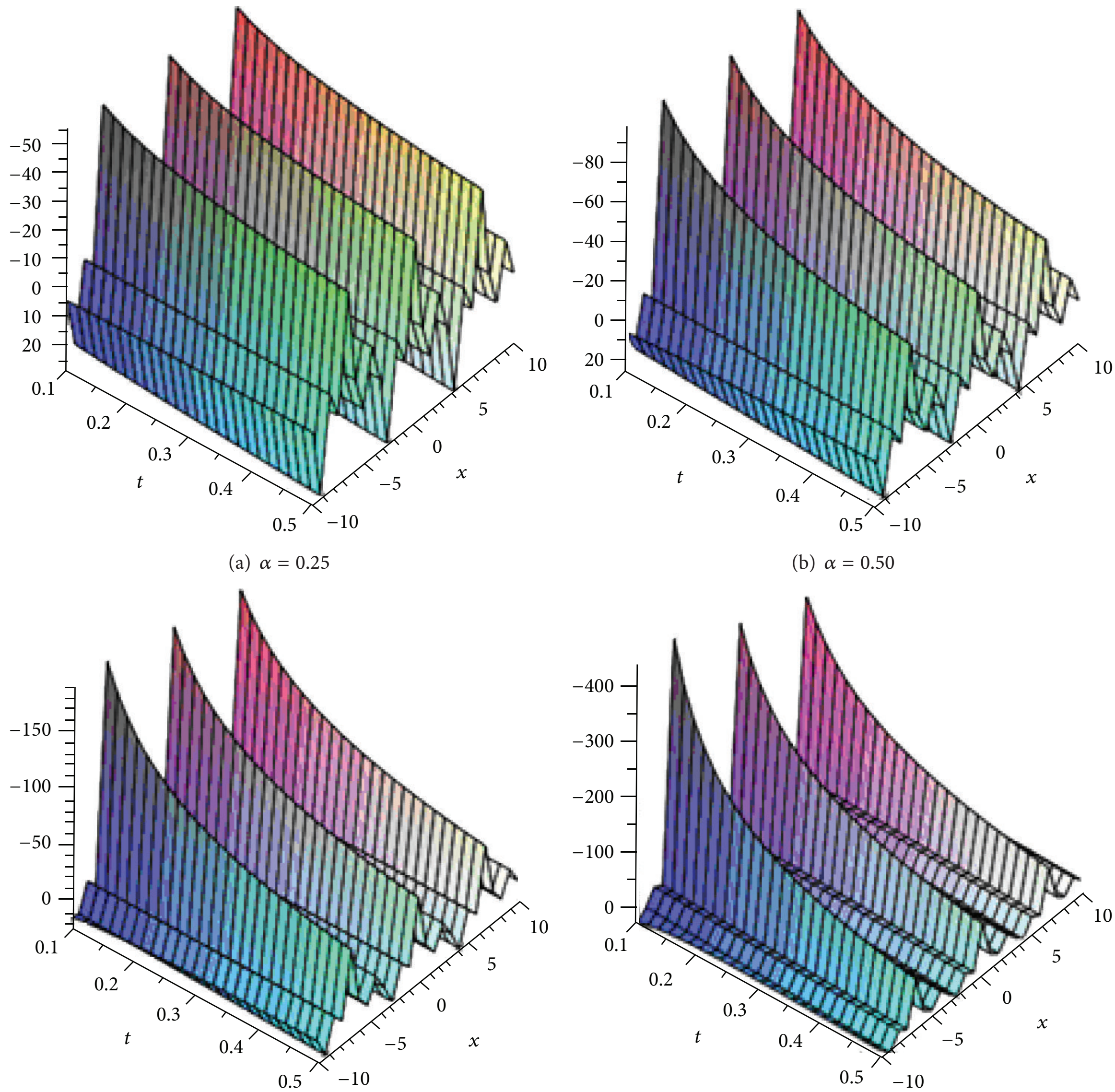
$$
\begin{aligned}
u_3^{12}(\xi) = {} & \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right) \\
& \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1}, \\
u_3^{13}(\xi) = {} & \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right) \\
& \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \\
& \times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1}, \\
u_3^{14}(\xi) = {} & \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \\
& \times\left(\frac{1}{2(C-1)}\right. \\
& \quad \left.\times\left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right\}\right) \\
& \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \\
& \times\left(\frac{1}{2(C-1)}\right. \\
& \quad \left.\times\left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right\}\right)^{-1}.
\end{aligned} \tag{35}
$$

When $(C-1)\neq 0$ and $A = B = 0$, the solution of (12) is

$$
\begin{aligned}
u_3^{25}(\xi) = {} & \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \\
& \times\left(\frac{A}{2(C-1)}-\frac{1}{(C-1)\xi+c_2}\right) \\
& \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \\
& \times\left(\frac{A}{2(C-1)}-\frac{1}{(C-1)\xi+c_2}\right)^{-1},
\end{aligned} \tag{36}
$$

where c_2 is an arbitrary constant.

Other exact solutions of (12) are omitted here for convenience.



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

Figure 4: (a)–(d) show singular kink solution for u_3^{12} for different values of parameters.

Finally, by substituting the solutions $G(\xi)$ of (10) into (23) and simplifying, we obtain the following solutions.

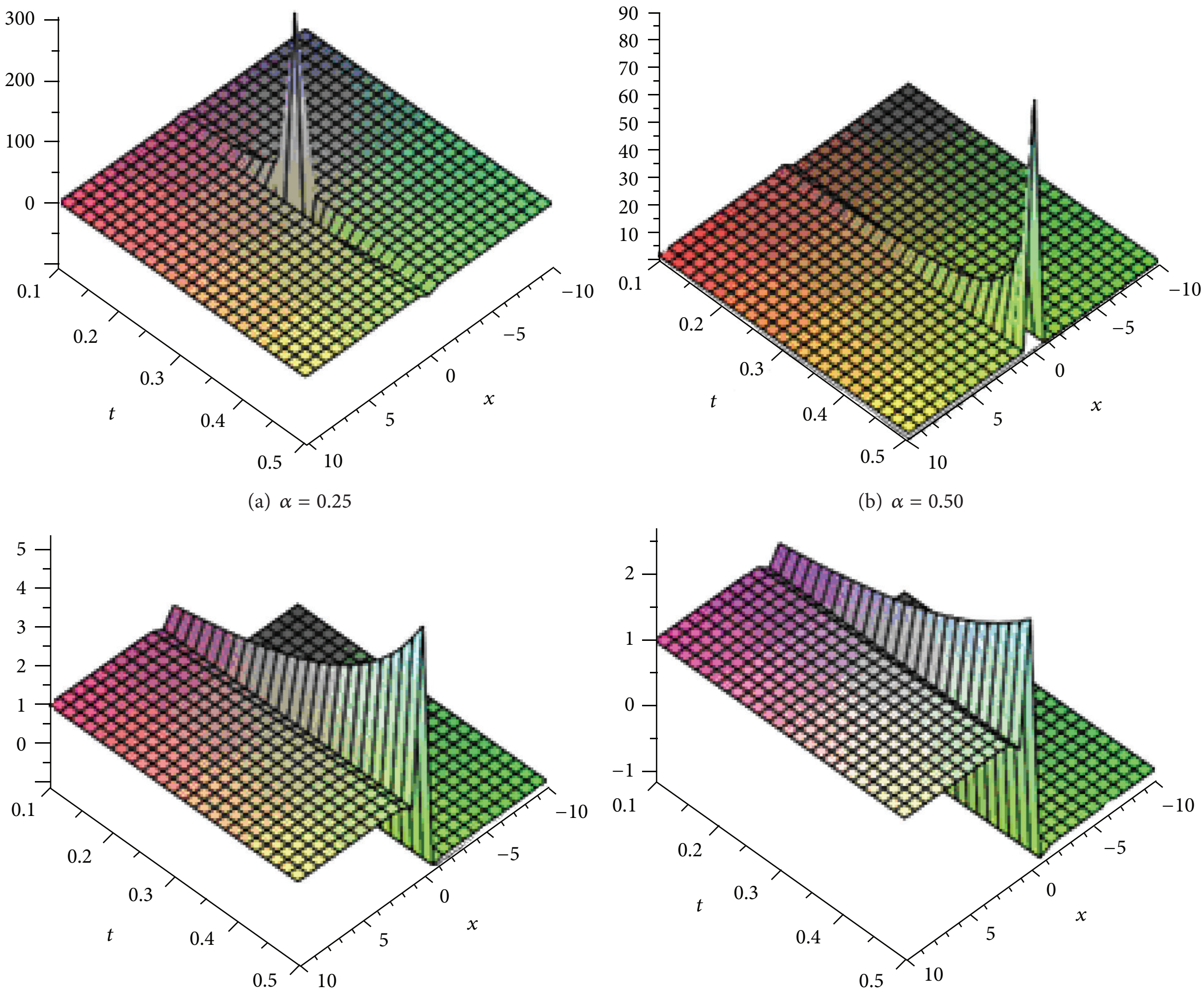
When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$) (Figure 5),

$$u_4^1(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_4^2(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_4^3(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)} \times \left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right) \pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right)^{-1}. \tag{37}$$

Others families of exact solutions are omitted for the sake of ease.



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

FIGURE 5: (a)–(d) show traveling wave solution for u_4^3 for different values of parameters.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$u_4^{12}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_4^{13}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_4^{14}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)} \times \left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right\}\right)^{-1}. \tag{38}$$

When $(C-1) \neq 0$ and $A = B = 0$, the solution of (12) is

$$u_4^{25}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{A}{2(C-1)} - \frac{1}{(C-1)\xi + c_2}\right)^{-1}, \tag{39}$$

where c_2 is an arbitrary constant.

Table 1: Comparison between our solutions and Liu et al. [48] solutions.

Obtained solutions	Liu et al. [48] solutions
(i) If $L=1, A=2, B=0, C=2, \delta=-1, \gamma=1, k=0, \alpha=1$, and $u_1^1(\xi)=u_{1,2}(x,t)$, then the solution is $u_{1,2}(x,t)=\pm 2\tanh\left(x+\frac{2}{3}t\right)$.	(i) If $C_1=1, C_2=0, \lambda=2, \mu=0, a=1$, and $k=1$, then the solution is $u_{1,2}(x,t)=\pm 2\tanh\left(x+\frac{2}{3}t\right)$.
(ii) If $L=1, A=2, B=1, C=3, \delta=-1, \gamma=1, k=0, \alpha=1$, and $u_1^{12}(\xi)=u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t)=\pm 2\sqrt{3}\tan(x+2\ t)$.	(ii) If $C_1=1, C_2=0, \lambda^2-4\ \mu=-4, a=1$, and $k=1$, then the solution is $u_{3,4}(x,t)=\pm 2\sqrt{3}\tan(x+2\ t)$.
(iii) If $L=1, A=0, B=0, C=2, \delta=-1, \gamma=1, k=0$, $\alpha=1, c_2=0$, and $u_1^{25}(\xi)=u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t)=\pm 2\sqrt{3}\ \frac{1}{x+2\ t}$.	(iii) If $C_1=1, C_2=1, \lambda=2, \mu=1, a=1$, and $k=-1$, then the solution is $u_{3,4}(x,t)=\pm\ 2\sqrt{3}\ \frac{1}{x+2\ t}$.
(iv) If $L=1, A=2, B=0, C=2, \delta=1, \gamma=1, k=0\ , \alpha=1$, and $u_1^1(\xi)=u_{1,2}(x,t)$, then the solution is $u_{3,4}(x,t)=\pm 2i\tanh\left(x-\frac{2}{3}t\right)$.	(iv) If $C_1=1, C_2=0, \lambda=2, \mu=0, a=1$, and $k=1$, then the solution is $u_{3,4}(x,t)=\pm 2i\tanh\left(x-\frac{2}{3}t\right)$.
(v) If $L=1, A=1, B=\frac{1}{2}, C=3, \delta=-1, \gamma=1, k=0, \alpha=1$, and $u_1^{12}(\xi)=u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t)=\pm\ \sqrt{6}i\tan\frac{1}{2}(x-4\ t)$.	(v) If $C_1=1, C_2=0, \lambda=0, \mu=\frac{1}{4}, a=1$, and $k=1$, then the solution is $u_{3,4}(x,t)=\pm\ \sqrt{6}\ i\tan\frac{1}{2}(x-4\ t)$.
(vi) If $L=1, A=0, B=0, C=2, \delta=1, \gamma=1, k=0$, $\alpha=1, c_2=0$, and $u_1^{25}(\xi)=u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t)=\pm i2\sqrt{3}\ \frac{1}{x-2\ t}$.	(vi) If $C_1=1, C_2=1, \lambda=2, \mu=1, a=1$, and $k=1$, then the solution is $u_{3,4}(x,t)=\pm i2\sqrt{3}\ \frac{1}{x-2\ t}$.

Other exact solutions of (12) are omitted here for expediency.

4. Conclusions

A novel (G'/G)-expansion method is applied to fractional partial differential equation successfully. As applications, abundant new exact solutions for the time fractional simplified modified Camassa-Holm (MCH) equation have been successfully obtained. The nonlinear fractional complex transformation for ξ is very important, which ensures that a certain fractional partial differential equation can be converted into another ordinary differential equation of integer order. The obtained solutions are more general with more parameters. Also comparison has been made in the form of table (Table 1), which shows that some of our solutions are in full agreement with the results obtained previously. Thus, novel (G'/G)-expansion method would be a powerful mathematical tool for solving nonlinear evolution equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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