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Research Article

Conditional Stability for an Inverse Problem of Determining a Space-Dependent Source Coefficient in the Advection-Dispersion Equation with Robin's Boundary Condition

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This paper deals with an inverse problem of determining the space-dependent source coefficient in one-dimensional advection-dispersion equation with Robin's boundary condition. Data compatibility for the inverse problem is analyzed by which an admissible set for the unknown is set forth. Furthermore, with the help of an integral identity, a conditional Lipschitz stability is established by suitably controlling the solution of an adjoint problem.

1. Introduction

The process of solute transport and transformation in the soils and groundwater is always involving some complicated physical and/or chemical reactions. By the mass conservation law, the process can often be described by mathematical models of advection-dispersion and/or reaction diffusion equations with source/sink terms. In many cases for the solute transport model, the dispersion/diffusion coefficient, the source/sink term, and other physical quantities are often unknown and cannot be measured easily. So, the method of inverse problem and parameter identification has had wide applications in the research of soil and groundwater problems.

Denoting $\Omega_T = \{(x,t) : 0 < x < l, 0 < t < T\}$, the one-dimensional advection-dispersion equation usually utilized is given as

$$R\frac{\partial c}{\partial t} - \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right) + v\frac{\partial c}{\partial x} + f(x,t;c) = 0, \quad (x,t) \in \Omega_T, \tag{1}$$

where $R \geq 1$ is the retardation factor; $c = c(x,t)$ is the volume-averaged concentration at point x and time t; $D = D(x)$ is the dispersion coefficient; $v > 0$ is the average flow velocity; and $f = f(x,t;c)$ represents a general source/sink term in the solute transportation.

Based on (1), a well-posed forward problem can be obtained together with suitable initial boundary value conditions. As for inverse problems of identifying the source term in the advection-dispersion equation, there are quite a few researches on the uniqueness and existence, but conditional stability seems to be paid less attention to our knowledge, especially for construction of Lipschitz stability. The reason maybe comes from the methodology suitable for the stability analysis. The variational integral identity method, also known as monotonicity method or the adjoint method, see, for example, [1–5], has been applied to parameter identification problems in the parabolic equations, by which uniqueness results can be obtained in a general way. Nevertheless, the integral identity method can also be utilized to construct stability for the inverse problems based on (1). In [6], conditional stability has ever been discussed for the inverse problem of determining the source coefficient $q(x)$ in (1) in the case of $f = q(x)c$ with Dirichlet boundary conditions. However, there is a conceptional fault in the proof of the stability in [6] when defining the norm of the unknown. The norm could have no meanings when using an unsuitable bilinear form to give its definition. Another difference of this paper with the work of [6] lies in using different boundary conditions. We will consider the corresponding inverse problem for (1) with Robin's boundary condition at the left-hand side of the domain and the homogeneous Neumann boundary condition at the right-hand side of the domain.

Therefore, we have reason to consider the inverse problem of determining the source coefficient $q = q(x)$ in (1) also in the case of $f = q(x)c$ but with the boundary condition of Robin's type. By data compatibility analysis and using the integral identity method, the solution of the forward problem can be positive and monotone on the time variable, and an admissible set for the unknown source coefficient is given. Most important of all, a new bilinear form is put forward with the aid of the solution of the adjoint problem, and the L^2 norm for the unknown is well-defined by which a conditional Lipschitz stability for the inverse problem is established. The rest of this paper is organized as follows.

In Section 2, the forward problem and the inverse problem of determining the space-dependent source coefficient with final observations are introduced. In Section 3, data compatibility is analyzed based on an integral identity by which the solution of the forward problem can be monotone and positive, and an admissible set for the unknown is set forth. In Section 4, an integral identity combining variations of the known functions with the changes of the unknown is established with the aid of an adjoint problem, and a suitable bilinear form and the corresponding norm for the unknown are defined by which Lipschitz stability for the inverse problem is constructed.

2. The Forward Problem and the Inverse Problem

Let us begin with the forward problem before discussing the inverse problem. In this paper for (1), suppose that the retardation factor $R = 1$ and the dispersal coefficient D is a positive constant and the source term has the form of $f = q(x)c$. For the constant dispersion coefficient D, it always has the representation of $D = a_L v$, where $a_L > 0$ is the longitude dispersion coefficient and v is also the average flow velocity. Then the model we are to deal with is given as [7]

$$\frac{\partial c}{\partial t} - a_L v \frac{\partial^2 c}{\partial x^2} + v \frac{\partial c}{\partial x} + q(x) c = 0, \quad (x,t) \in \Omega_T. \tag{2}$$

The initial value condition is given as

$$c(x,0) = c_0(x), \quad 0 \le x \le l. \tag{3}$$

The left boundary at $x = 0$ is given with Robin's condition:

$$vc(0,t) - Dc_x(0,t) = g(t), \quad 0 \le t \le T, \tag{4}$$

where the function $g = g(t)$ represents the solute flux through the left boundary because of the difference of concentration at the two sides of the boundary. At the right boundary $x = l$, we assume it is an outflow boundary or the boundary is imperceivable; that is, we have the condition:

$$c_x(l,t) = 0, \quad 0 \le t \le T. \tag{5}$$

By (2), together with the initial boundary value conditions (3)–(5), we get the so-called forward problem. Furthermore, under suitable conditions for the initial boundary value functions, for example, $c_0(x) \in C([0,l])$, $g(t) \in C([0,T])$, and $q(x)$ is bounded measurable, we know from [8] that the above forward problem has unique solution $c(x,t) \in C^{2,1}(\Omega_T)$. On the other hand, if the source coefficient function $q = q(x)$ in (2) is unknown, we have to determine it by some additional information of the solution. The additional data we have are the final observations at one final time $t = T$ given as [9–12]

$$c(x,T) = \theta(x), \quad 0 \le x \le l. \tag{6}$$

As a result, an inverse problem of determining the source coefficient $q(x)$ is formulated by (2)–(5) together with the additional information (6). To the best of our knowledge, conditional stability for the above inverse source problem for (2) with Robin's boundary condition has not been studied in the known literature. In what follows, we will firstly give data compatibility analysis for the inverse problem and then put forward an integral identity with which a conditional stability is constructed by suitably controlling an adjoint problem.

3. Data Compatibility Analysis

Consider the inverse problem (2)–(6). Noting the reality of real problems, the initial value function $c_0(x)$, the flux function $g(t)$, the additional function $\theta(x)$, and the hydraulic parameters a_L and v are all claimed to be nonnegative on $(x,t) \in \Omega_T$ throughout the paper if there is no specification. In addition, by $\|\cdot\|_2$ we denote the L^2-norm in the corresponding space.

Suppose that $c_0(x)$, $g(t)$, and $\theta(x)$ satisfy condition (A):

(A) $c_0(x) \in C^2[0,l]$, $g(t) \in C^1[0,T]$, and $\theta(x)$ is piecewise continuous differential on $x \in [0,l]$, and there are positive constants M_0 and ε such that $\|c_0\|_2 \le M_0$ and $\|\theta\|_2 \ge \varepsilon$.

Moreover, the data should satisfy consistency condition (B):

(B) $vc_0(0) - Dc_0'(0) = g(0)$; $c_0'(l) = \theta'(l) = 0$.

Denote $c = c(x,t;q)$ as the unique solution of the forward problem (2)–(5) for any prescribed source coefficient $q = q(x)$ belonging to suitable spaces. With a similar method as used in [4, 6, 13, 14], we can prove the following theorem to reveal the monotonicity and positivity of the solution to the forward problem.

Theorem 1. *Suppose that the conditions (A) and (B) are satisfied and $c = c(x,t;q)$ is a priori bounded; then the following statements hold.*

(1) *If the solute flux $g(t)$ is monotone decreasing on $t \in [0,T]$ and $q = q(x)$ has the property (P1):*

(P1) $q(x)c_0(x) - vc_0'(x) + Dc_0''(x) \ge 0, 0 \le x \le l,$

then for each given $x \in [0, l]$, *it follows that*

$$0 \le \theta(x) = c(x,T) \le c(x,t) \le c(x,0) = c_0(x), \quad 0 < t < T. \tag{7}$$

(2) *If the solute flux* $g(t)$ *is monotone increasing on* $t \in [0,T]$ *and* $q = q(x)$ *has the property (P2):*

(P2) $q(x)c_0(x) - vc_0'(x) + Dc_0''(x) \le 0$, $0 \le x \le l$,

then for each given $x \in [0, l]$, *it follows that*

$$0 \le c_0(x) = c(x,0) \le c(x,t) \le c(x,T) = \theta(x), \quad 0 < t < T. \tag{8}$$

Proof. We only prove the assertion (1), and (2) can be proved similarly. For $c = c(x,t;q)$ and any smooth test function $\varphi(x,t)$, we have

$$\int_{\Omega_T} (c_t - Dc_{xx} + vc_x + q(x)c)\varphi_t dx\, dt = 0. \tag{9}$$

Integration by parts leads to

$$\begin{aligned}\int_{\Omega_T} c_t(\varphi_t + D\varphi_{xx} + v\varphi_x - q(x)\varphi)\, dx\, dt &= \int_0^T [D(c_x\varphi_t - c\varphi_{xt}) - vc\varphi_t]_{x=0}^{x=l} dt \\ &\quad + \int_0^l [Dc\varphi_{xx} + vc\varphi_x - q(x)c\varphi]_{t=0}^{t=T} dx.\end{aligned} \tag{10}$$

Denote $h = v/D = 1/a_L$. If there is

$$\varphi_x(l,t) + h\varphi(l,t) = 0, \tag{11}$$

then

$$\varphi_{xt}(l,t) + h\varphi_t(l,t) = 0, \tag{12}$$

and we get

$$D\varphi_{xt}(l,t) + v\varphi_t(l,t) = 0. \tag{13}$$

Let $G = G(x,t)$ be any arbitrary nonnegative function on $(x,t) \in \Omega_T$ and $\varphi = \varphi(x,t)$ solve the following adjoint problem:

$$\begin{aligned}&\varphi_t + D\varphi_{xx} + v\varphi_x - q(x)\varphi = G(x,t), \quad (x,t) \in \Omega_T, \\ &\varphi_x(0,t) = 0; \quad \varphi_x(l,t) + h\varphi(l,t) = 0, \quad 0 \le t \le T, \\ &\varphi(x,T) = 0, \quad 0 \le x \le l.\end{aligned} \tag{14}$$

Then, together with the initial boundary value conditions given by (3), (4), and (5), the equality (10) is reduced to

$$\begin{aligned}\int_{\Omega_T} c_t G(x,t)\, dx\, dt &= \int_0^l c(x,0)[-D\varphi_{xx}(x,0) - v\varphi_x(x,0) \\ &\quad + q(x)\varphi(x,0)]\, dx \\ &\quad + \int_0^T g(t)\varphi_t(0,t)\, dt := I_1 + I_2,\end{aligned} \tag{15}$$

where

$$\begin{aligned}I_1 &= \int_0^l c_0(x) \\ &\quad \times [q(x)\varphi(x,0) - v\varphi_x(x,0) - D\varphi_{xx}(x,0)]\, dx, \\ I_2 &= \int_0^T g(t)\varphi_t(0,t)\, dt.\end{aligned} \tag{16}$$

Firstly by using integration by parts for I_1, we have

$$\begin{aligned}I_1 &= -D[c_0(x)\varphi_x(x,0) - c_0'(x)\varphi(x,0)]_0^l \\ &\quad - v[c_0(x)\varphi(x,0)]_0^l \\ &\quad + \int_0^l [q(x)c_0(x) + vc_0'(x) - Dc_0''(x)] \\ &\quad \times \varphi(x,0)\, dx.\end{aligned} \tag{17}$$

Thanks to the consistency condition (B) and the boundary value conditions of the adjoint problem (14), we can get

$$\begin{aligned}I_1 &= g(0)\varphi(0,0) \\ &\quad + \int_0^l [q(x)c_0(x) + vc_0'(x) - Dc_0''(x)]\varphi(x,0)\, dx.\end{aligned} \tag{18}$$

Also by integration by parts for I_2 and noting $\varphi(0,T) = 0$, we have

$$I_2 = -g(0)\varphi(0,0) - \int_0^T g'(t)\varphi(0,t)\, dt. \tag{19}$$

So, we get

$$\begin{aligned}\int_{\Omega_T} c_t G(x,t)\, dx\, dt &= \int_0^l [q(x)c_0(x) + vc_0'(x) - Dc_0''(x)]\varphi(x,0)\, dx \\ &\quad + \int_0^T [-g'(t)]\varphi(0,t)\, dt.\end{aligned} \tag{20}$$

By applying the maximum principle of parabolic type of PDE, see, for example [15], to the adjoint problem (14), we conclude that if $G(x,t)$ is nonnegative but is otherwise arbitrary, then $\varphi(x,t)$ is negative on Ω_T, and so does for $\varphi(0,t)$ and $\varphi(x,0)$. Then together with the property (P1), we know that the sign of the first term of the right-hand side of (20) is nonpositive. In addition, by virtue of the monotone decreasing property of the boundary flux $g(t)$, the sign of the second term of the right-hand side of (20) is also nonpositive. Hence, the sign of expression (20) or (15) is nonpositive; that is, we have

$$\int_{\Omega_T} c_t G(x,t)\, dx\, dt \le 0. \tag{21}$$

Since $G = G(x,t)$ is nonnegative but otherwise arbitrary, it follows that if there is any positive measure subset of Ω_T where $c_t(x,t)$ is positive, then a contradiction of (21) can be achieved by choosing the support $G = G(x,t)$ in this positive measure set. This proves that for each $x \in [0,l]$, there is $c_t(x,t) \le 0$ for $0 \le t \le T$, and the assertion (1) is valid. The proof is completed. □

Remark 2. By setting $v = 0$ and $f(x,t;c) = 0$ in (1), we get the model ever discussed in [4] where Neumann's boundary conditions are employed and the initial value is zero. Thanks to $c_0(x) = 0$, we have by equality (15)

$$\int_{\Omega_T} c_t G(x,t)\,dx\,dt = \int_0^T g(t)\,\varphi_t(0,t)\,dt. \tag{22}$$

Since $\varphi(0,t) \le \varphi(0,T)$ for $0 < t < T$, there is $\varphi_t(0,t) \ge 0$. Hence we deduce that assertion (1) is also valid if $g(t) < 0$ and assertion (2) is valid for $g(t) > 0$. However, if considering nonzero initial condition, the property (P1)/(P2) is needed, and the assumption of monotonicity for $g(t)$ can be replaced by the properties of keeping its positive/negative sign for $t \in (0,T)$ and $g(0) = 0$ such that the assertions of this theorem can be proved too.

According to Theorem 1, we can get two sufficient conditions under which the inverse problem (2)–(6) is of data compatibility. That is, the unknown source coefficient $q(x)$ should have one of the following conditions with the known data functions $c_0(x)$ and $g(t)$:

(C1) $g'(t) \le 0$ for $t \in [0,T]$, and $q(x)$ has property (P1);

(C2) $g'(t) \ge 0, t \in [0,T]$, and $q(x)$ has property (P2).

In summary, if the conditions (A), (B), and (C1) are satisfied, the solution $c = c(x,t;q)$ is monotone decreasing on $t \in [0,T]$ for each $x \in [0,l]$. If the conditions (A), (B), and (C2) are satisfied, then the solution $c = c(x,t;q)$ is monotone increasing on $t \in [0,T]$ for each $x \in [0,l]$.

In the following discussions, we assume that the condition (C1) is satisfied and the unknown source coefficient $q = q(x)$ is continuous and has property (P1). Moreover, let the function $q = q(x)$ take nonnegative values for $x \in [0,l]$. Thus, an admissible set for the unknown source coefficients is defined as follows:

$$\begin{aligned} S_{\mathrm{ad}} = \{ q : q \text{ is continuous}, q \ge 0 \text{ for } x \in [0,l], \\ \text{and has property (P1)} \}. \end{aligned} \tag{23}$$

In the next section, we will establish a conditional stability for the inverse problem of determining $q \in S_{\mathrm{ad}}$ with the help of an integral identity by suitably controlling an adjoint problem.

4. Conditional Stability of the Inverse Problem

It is more difficult to prove stability than to prove existence and uniqueness for an inverse problem, and it always needs additional conditions and suitable topology to construct a stability which is called conditional stability for the inverse problem.

4.1. Construction of Integral Identity. In this section, a variational integral identity with the aid of an adjoint problem is established, which reflects a corresponding relation of the unknown source coefficient with those of the initial boundary values and the additional data.

Theorem 3. *Let $c_i = c(x,t;q_i)$ $(i = 1,2)$ be two solutions corresponding to the initial boundary value functions $c_0^i(x)$ and $g_i(t)$ $(i = 1,2)$ and let $\theta_i(x)$ $(i = 1,2)$ be the corresponding additional observations. Then it follows that*

$$\begin{aligned} &\int_{\Omega_T} c_2 (q_1 - q_2)\,\varphi(w)(x,t)\,dx\,dt \\ &\quad = \int_0^l (\theta_2 - \theta_1)\,w(x)\,dx + \int_0^l \left(c_0^1 - c_0^2\right)\varphi(x,0)\,dx \\ &\quad + \int_0^T (g_1 - g_2)\,\varphi(0,t)\,dt, \end{aligned} \tag{24}$$

where $\varphi = \varphi(w)(x,t)$ is the solution of a suitable adjoint problem with input data $w = w(x)$ given in the proof of this theorem.

Proof. Denoting $C = c_1 - c_2$ and noting that c_1 and c_2 both satisfy (2) and (3)–(6), we have

$$C_t - DC_{xx} + vC_x + q_1 C = (q_2 - q_1)\,c_2, \tag{25}$$

$$C(x,0) = c_0^1 - c_0^2, \tag{26}$$

$$vC(0,t) - DC_x(0,t) = g_1 - g_2, \quad C_x(l,t) = 0, \tag{27}$$

$$C(x,T) = \theta_1 - \theta_2. \tag{28}$$

By smooth test function $\varphi = \varphi(x,t)$ and by multiplying two sides of (25) and integrating on Ω_T, respectively, we have

$$\begin{aligned} &\int_{\Omega_T} \left[C_t - DC_{xx} + vC_x + q_1 C\right]\varphi(x,t)\,dx\,dt \\ &\quad = \int_{\Omega_T} c_2 \left[q_2 - q_1\right]\varphi(x,t)\,dx\,dt. \end{aligned} \tag{29}$$

Integration by parts for the left-hand side of equality (29) and with a similar method as used in the proof of expression (15), we get the identity (24) as long as choosing $\varphi(x,t)$ as the solution of the adjoint problem:

$$\begin{gathered} \varphi_t + D\varphi_{xx} + v\varphi_x - q_1(x)\,\varphi = 0, \quad (x,t) \in \Omega_T, \\ \varphi_x(0,t) = 0, \quad \varphi_x(l,t) + h\varphi(l,t) = 0, \quad t \in [0,T], \\ \varphi(x,T) = w(x), \quad x \in [0,l], \end{gathered} \tag{30}$$

where $h = v/D = 1/a_L$ and $w = w(x)$ is called a controllable input. Since the adjoint problem is completely controlled by the unique input $w = w(x)$, we denote the solution of the adjoint problem (30) by $\varphi = \varphi(w)(x,t)$. The proof is over. □

Obviously, by setting $\tau = T - t$ and $x = x$ for the adjoint problem (30) and also denoting τ as t, it is transformed to a normal initial boundary value problem given as follows:

$$\begin{gathered} \varphi_t - D\varphi_{xx} - v\varphi_x + q_1(x)\,\varphi = 0, \quad (x,t) \in \Omega_T, \\ \varphi_x(0,t) = 0, \quad \varphi_x(l,t) + h\varphi(l,t) = 0, \quad t \in [0,T], \\ \varphi(x,0) = w(x), \quad x \in [0,l]. \end{gathered} \tag{31}$$

In what follows, we cope with problem (31) instead of problem (30). For this adjoint problem, we can see below that its solution is really determined by the unique input $w = w(x)$.

4.2. Lipschitz Stability. Firstly, by applying variable separation method and Sturm-Liouville eigenvalue theory [16], we can get an explicit expression of the solution for the adjoint problem (31).

Lemma 4. *Suppose that the source coefficient $q(x) \in S_{ad}$ and the hydraulic parameters D and v are positive; then there is*

$$\varphi(w)(x,t) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n D t) X_n(x), \tag{32}$$

where λ_n, $X_n(x)$ $(n = 1, 2, \ldots)$ are eigenvalues and corresponding eigenfunctions of the following Sturm-Liouville problem, respectively,

$$\begin{gathered} X'' + hX' - \frac{q_1(x)}{D} X + \lambda X = 0, \\ X'(0) = 0, \qquad X'(l) + hX(l) = 0, \end{gathered} \tag{33}$$

$$a_n = \frac{(w, X_n)}{(X_n, X_n)} = \frac{\int_0^l \rho(x) w(x) X_n(x)\, dx}{\|X_n\|_2^2}, \tag{34}$$

where $\rho(x) = \exp(hx)$ is the weighted function of the eigenvalue problem (33) and the eigenfunctions series $\{X_n(x)\}$ are orthogonal and complete on $L^2(0,l)$ with the weighted function $\rho(x)$.

Remark 5. By multiplying $\rho(x) = \exp(hx)$ at the two sides of the differential equation in (33), we have

$$-\frac{d}{dx}\left(\rho(x) X'\right) + Q(x) X = \lambda \rho(x) X, \tag{35}$$

where $Q(x) = \rho(x) q_1(x)/D \geq 0$ for $x \in [0,l]$. Combing with the homogeneous boundary conditions $X'(0) = 0$ and $X'(l) + hX(l) = 0$, we deduce that the operator $-(d/dx)[\rho(x)(d/dx)] + Q(x)$ is self-adjoint and the problem (33) is a regular Sturm-Liouville eigenvalue problem, and so the eigenfunctions $X_n(x)$, $n = 1, 2, \ldots$ are orthogonal with the weighted function $\rho(x)$ for $x \in [0,l]$.

With the aid of Lemma 4 and identity (24), we define a bilinear form $\mathscr{B}(q,w) : L^2(0,1) \times L^2(0,1) \to R$ by

$$\mathscr{B}(q,w) = \int_{\Omega_T} c_2 q(x) \varphi(w)(x,t)\, dx\, dt, \tag{36}$$

where $\varphi(w)$ is the solution of the adjoint problem (31) for $w \in L^2(0,l)$ and $c_2 = c(x,t;q_2)$ is a definite solution of the forward problem (2)–(5) for any given $q_2 \in S_{\text{ad}}$ and c_2 is nonnegative and bounded for $(x,t) \in \Omega_T$ which can be regarded as a weighted function of the bilinear form. It is noticeable that $\mathscr{B}$ is defined with $q(x)$ and $w(x)$ which is different from those definitions given in [6, 17].

By using maximum-minimum principle in the adjoint problem (31), we know that the solution $\varphi = \varphi(w)(x,t)$ takes nonnegative values as long as $w = w(x) \in L^2(0,l)$ taking nonnegative values on $x \in [0,l]$. In addition, by general theory of parabolic equation [8], there exists positive constant M_1 such that

$$\|\varphi\|_{L^2(\Omega_T)} \leq M_1 \|w\|_{L^2(0,l)}. \tag{37}$$

Then we get the following lemma.

Lemma 6. *The functional $\mathscr{B}$ defined by (36) is a bounded bilinear form.*

Proof. In fact, $\mathscr{B}$ is linear on q from (36) and together with the expressions (32) and (34) it follows that $\varphi(w)$ is linear on w, and then $\mathscr{B}$ is linear on w too. Next, using Hölder inequality for the right-hand side of (36), we have

$$\begin{aligned} |\mathscr{B}(q,w)| &\leq M_1 \left(\int_{\Omega_T} [q(x)]^2 dx\, dt\right)^{1/2} \\ &\quad \times \left(\int_{\Omega_T} [\varphi(w)]^2 dx\, dt\right)^{1/2} \\ &= M_2 \sqrt{T} \|q\|_{L^2(0,l)} \|\varphi\|_{L^2(\Omega_T)} \\ &\leq M_3 \|q\|_{L^2(0,l)} \|w\|_{L^2(0,l)}, \end{aligned} \tag{38}$$

where $M_3 = M_1 M_2 \sqrt{T}$ and M_2 here is a positive constant depending on the domain and the prescribed solution c_2; then $\mathscr{B}$ is bounded on q and w. The proof is over. □

By definition (36) and Lemmas 4 and 6, we define an admissible space for the inputs as $W = \{w \in L^2(0,l) : w \geq 0, \|w\|_2 \leq E\}$ and a norm for the source coefficient via

$$\|q\|_2 = \sup_{w \in W,\, w \neq 0} \frac{|\mathscr{B}(q,w)|}{\|c_2\|_2 \|w\|_2}, \tag{39}$$

where $E > 0$ is a constant and $\|c_2\|_2 = (\int_{\Omega_T} [c(x,t;q_2)]^2 dx\, dt)^{1/2}$.

It is not difficult to testify that the above definition is well-defined, by which we can construct a conditional stability for the inversion of the source coefficient based on the integral identity (24).

Theorem 7. *Suppose that the assumptions (A), (B), and (C1) are satisfied and (c_i, q_i) $(i = 1, 2)$ are two pairs of solutions to the inverse problem (2)–(6) corresponding to the data c_0^i and g_i $(i = 1, 2)$ and θ_i $(i = 1, 2)$ are the additional observations. Then for $q \in S_{ad}$ and $w \in W$, there exists constant $M = M(D, v, \Omega_T, \|\theta\|_2)$ such that*

$$\begin{aligned} &\|q_1 - q_2\|_2 \\ &\quad \leq M\left(\|\theta_1 - \theta_2\|_2 + \left\|c_0^1 - c_0^2\right\|_2 + \|g_1 - g_2\|_2\right). \end{aligned} \tag{40}$$

Proof. Firstly by assertion (1) of Theorem 1, there is

$$[\theta(x)]^2 \leq [c(x,t)]^2 \leq [c_0(x)]^2, \quad 0 < x < l, \ 0 < t < T, \tag{41}$$

for any solution $c(x,t)$ of the forward problem (2)–(5) with the final function $\theta(x) = c(x,T)$ and the initial function $c_0(x) = c(x,0)$. Then noting condition (A), we have, for the solution $c_2 = c(x,t;q_2)$,

$$\frac{1}{M_0\sqrt{T}} \leq \frac{1}{\|c_0\|_2\sqrt{T}} \leq \frac{1}{\|c_2\|_2} \leq \frac{1}{\|\theta\|_2\sqrt{T}} \leq \frac{1}{\varepsilon\sqrt{T}}. \tag{42}$$

Next by definition (39) and utilizing the integral identity (24) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}\|q_1 - q_2\|_2 \leq{}& \frac{1}{\|\theta\|_2\sqrt{T}} \\ &\cdot \left\{ \|\theta_1 - \theta_2\|_2 \sup_{w\in W,\, w\neq 0} \frac{\|w\|_2}{\|w\|_2} \right. \\ &+ \left\| c_0^1 - c_0^2 \right\|_2 \sup_{w\in W,\, w\neq 0} \frac{\|\varphi(x,0)\|_2}{\|w\|_2} \\ &\left. + \|g_1 - g_2\|_2 \sup_{w\in W,\, w\neq 0} \frac{\|\varphi(0,t)\|_2}{\|w\|_2} \right\}.\end{aligned} \tag{43}$$

Also by the general theory of 1D parabolic equation, we know that there exists a constant $\overline{M} > 0$ depending on D, v and the domain Ω_T such that

$$\|\varphi(x,0)\|_2, \|\varphi(0,t)\|_2 \leq \overline{M}\|w\|_2. \tag{44}$$

Therefore, by setting

$$M = \frac{1}{\|\theta\|_2\sqrt{T}} \max\left\{1, \overline{M}\right\} \tag{45}$$

and noting (43) it follows that (40) is valid. The proof is over. □

Remark 8. This theorem shows a conditional Lipschitz stability for the inverse problem (2)–(6). Furthermore, by this theorem we can see that the uniqueness in the meaning of L^2 for the inverse problem can be easily deduced by the stability estimate (40).

Conflict of Interests

The authors declare that there is no conflict of interests with any commercial identities regarding the publication of this paper.

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