Research Article

On the Weak Solution to a Fractional Nonlinear Schrödinger Equation

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We obtain the existence of a global weak solution to a fractional nonlinear Schrödinger equation by the Galerkin method. Its uniqueness is also discussed. In our proof, we use harmonic analysis techniques and compactness arguments.

1. Introduction

This paper is concerned with the following fractional partial differential equations in the n dimensional torus $\mathbb{T}^n$:

$$\frac{\partial u}{\partial t} - i\Lambda^{2\alpha} u + |u|^{\rho} u = f, \tag{1}$$

where $\alpha \in (0, 1)$ and $\rho \in (0, \infty)$ are real numbers and $i = \sqrt{-1}$. $f(x, t)$ is given and $u(x, t)$ is a complex-valued function. Here, $\Lambda^{2\alpha} = (-\Delta)^{\alpha}$, the fractional Laplacian with respect to x.

The fractional Laplacian operator appears in a wide class of physical systems and engineering problems, including Lévy flights, viscoelasticity, electrochemistry, control, porous media, electromagnetic, stochastic interfaces, and anomalous diffusion problems, and attracts the interests of many mathematicians; see [1–5], for example. The quasigeostrophic equation with fractional dissipation has been also extensively studied; see Constantin et al. [6–10], for example. In mathematical physics, the fractional Laplacian is often applied to describe many complicated phenomena via partial differential equations.

The Schrödinger equation is a fundamental equation in physics, which describes nonrelativistic quantum mechanical behavior. It is well known that Feynman and Hibbs [11] used path integrals over Brownian paths to derive the standard Schrödinger equation ($\alpha = 1$ in (1)). Recently, Laskin [12, 13]

showed that the path integral over the Lévy-like quantum mechanical paths allows us to generalize the classical quantum mechanics. Namely, if the path integral over Brownian trajectories leads to the well-known Schrödinger equation, then the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation includes the space derivative of order 2α instead of second-order space derivative in the standard Schrödinger equation. Laskin [14] showed the hermiticity of the fractional Hamilton operator and established the parity conservation law. Guo et al. [15] obtained the existence of a unique global smooth solution to the periodical boundary value problem for the fractional nonlinear Schrödinger equation.

Interestingly enough, there are also some other models involving the damping term $|u|^{\rho} u$; see [16, 17].

In studying (1), there exist some essential difficulties. First, since the fractional differential operator is defined by Fourier series and is nonlocal except when $\alpha \in \mathbb{N}$, which means that $\Lambda^{2\alpha} u(x)$ depends not only on $u(y)$ for y near x, but also on $u(y)$ for all y. Moreover, integration by part for nonlinear term is not valid. These bring new difficulties when doing energy estimate. And thus new harmonic analysis methods must be introduced to overcome these difficulties. Second, there are some difficulties in the convergence of the approximate solutions because of the nonlinear term. A compactness device should be given to treat this case.

We now collect the notations in this paper. The fractional Laplacian $\Lambda^{2\alpha} f$, for $\alpha \in \mathbb{R}$, can be defined as

$$\widehat{\Lambda^{2\alpha} f}_k = |k|^{2\alpha} \widehat{f}_k \cdot, \quad k \in \mathbb{Z}^n, \tag{2}$$

where $\widehat{f}_k$ is the Fourier coefficients of f:

$$\widehat{f}_k = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(x) e^{-ik \cdot x} dx, \quad k \in \mathbb{Z}^n. \tag{3}$$

We shall also invoke the notion of inhomogeneous Sobolev space $\dot{H}^s(\mathbb{T}^n)$ $(s \in \mathbb{R})$, which comprises all tempered distributions f on $\mathbb{R}^n$ such that

$$\|f\|_{H^s(\mathbb{T}^n)} = \left(\sum_{k \in \mathbb{Z}^n \backslash \{0\}} \left(1 + |k|^2\right)^s |\widehat{u}_k|^2 \right)^{1/2}. \tag{4}$$

We now end this introduction by outlining the rest of this paper. In Section 2, we prove the existence of a weak solution to (1); see Theorem 8. The uniqueness of such weak solutions is discussed in Section 3; see Theorem 9.

2. Existence of a Weak Solution

In this section, we prove the existence of a weak solution to the following system:

$$\frac{\partial u}{\partial t} - i\Lambda^{2\alpha} u + |u|^\rho u = f, \quad (x, t) \in \mathbb{T}^n \times (0, T) \tag{5}$$

$$u(0) = u_0, \quad x \in \mathbb{T}^n. \tag{6}$$

Let us first recall and prove some fundamental Lemmas.

Lemma 1 (see [18]). *Let X be a Banach space; consider*

$$f \in L^p(0, T; X), \quad \frac{\partial f}{\partial t} \in L^p(0, T; X), \tag{7}$$

for some $1 \le p \le \infty$; then f is continuous map from $[0, T]$ to X.

Lemma 2 (see [18]). *Let $\mathscr{D}$ be a bounded domain in $\mathbb{R}^n \times \mathbb{R}_+$, and g_k, g are in $L^q(\mathscr{D})(1 < q < \infty)$ with*

$$\|g_k\|_{L^q(\mathscr{D})} \le C, \quad g_k \longrightarrow g, \text{ a.e. } \mathscr{D}; \tag{8}$$

then $g_k \to g$ weakly in $L^q(\mathscr{D})$.

Lemma 3 (see [18]). *Let $T > 0$, $1 < p_1, p_2 < \infty$, and B_0, B, B_1 be three Banach spaces satisfying $B_0 \hookrightarrow B \hookrightarrow B_1$, where B_0, B_1 are reflexive and the embedding $B_0 \hookrightarrow B$ is compact. Endow the space*

$$W = \left\{ w; w \in L^{p_0}(0, T; B_0), \frac{\partial w}{\partial t} \in L^{p_1}(0, T; B_1) \right\} \tag{9}$$

with the norm

$$\|w\|_W = \|w\|_{L^{p_0}(0,T;B_0)} + \left\| \frac{\partial w}{\partial t} \right\|_{L^{p_1}(0,T;B_1)}, \quad \forall w \in W. \tag{10}$$

Then the embedding $W \hookrightarrow L^{p_0}(0, T; B)$ is compact.

In the following developments, we modify the methods in [19].

Proposition 4. *Let $0 < \alpha < 1$, $x \in \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, and $u \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\Lambda^{2\alpha} u(x) = C_\alpha \sum_{\nu \in \mathbb{Z}^n} PV \int_{\mathbb{T}^n} \frac{u(x) - u(y)}{|x - y - \nu|^{n+2\alpha}} dy, \tag{11}$$

with $C_\alpha > 0$.

Proof. First, we have

$$\Lambda^{2\alpha} u(x) = \sum_{|\nu| > 0} |\nu|^{2\alpha} \widehat{u}(\nu) e^{i\nu \cdot x}$$

$$= -\sum_{|\nu| > 0} |\nu|^{2\alpha - 2} \widehat{\Delta u}(\nu) e^{i\nu \cdot x}. \tag{12}$$

Let $\Phi_\varepsilon(x) = (|x|^{2\alpha - 2})_\varepsilon * \varphi_\varepsilon(x)$, where $(|x|^{2\alpha - 2})_\varepsilon = |x|^{2\alpha - 2} \cdot \chi(|x|/\varepsilon)$ with $\chi \in C^\infty([0, \infty))$,

$$\chi(x) = \begin{cases} 0, & \text{if } |x| \le 1, \\ 1, & \text{if } |x| \ge 2, \end{cases} \tag{13}$$

and $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$ is a standard approximation of identity, $0 \le \varphi \in C^\infty(\mathbb{R}^n)$, supp $\varphi \subset B_1$, and $\int \varphi = 1$. Now we can write

$$\Lambda^{2\alpha} u(x) = -\lim_{\varepsilon \to 0} \sum_{\nu \in \mathbb{Z}^n} \Phi_\varepsilon(\nu) \widehat{\Delta u}(\nu) e^{i\nu \cdot x}$$

$$= -\lim_{\varepsilon \to 0} \left(\sum_{\nu \in \mathbb{Z}^n} \Phi_\varepsilon(\nu) e^{i\nu \cdot x} \right) * \left(\sum_{\nu \in \mathbb{Z}^n} \widehat{\Delta u}(\nu) e^{i\nu \cdot x} \right). \tag{14}$$

Poisson's summation then yields

$$\Lambda^{2\alpha} u(x) = -\lim_{\varepsilon \to 0} \left(\sum_{\nu \in \mathbb{Z}^n} \widehat{\Phi_\varepsilon}(x - \nu) \right) * \Delta u(x)$$

$$= \lim_{\varepsilon \to 0} \sum_{\nu \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \widehat{\Phi_\varepsilon}(x - y - \nu) \Delta [u(x) - u(y)] dy$$

$$= \lim_{\varepsilon \to 0} \sum_{\nu \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \Delta(\widehat{\Phi_\varepsilon})(x - y - \nu) [u(x) - u(y)] dy. \tag{15}$$

Due to the fact that

$$\widehat{\Phi_\varepsilon}(\eta) = \left(\widehat{|x|^{2\alpha - 2}}\right)_\varepsilon \cdot \widehat{\varphi_\varepsilon}(\eta) = \left(\widehat{|x|^{2\alpha - 2}}\right)_\varepsilon \cdot \widehat{\varphi}(\varepsilon\eta),$$

$$\Delta \widehat{\Phi_\varepsilon}(\eta) = \Delta\left(\left(\widehat{|x|^{2\alpha - 2}}\right)_\varepsilon\right)(\eta) \cdot \widehat{\varphi}(\varepsilon\eta) + O(\varepsilon),$$

$$\left(\widehat{|x|^{2\alpha-2}}\right)_\varepsilon(y) = \frac{c_\alpha}{|y|^{n+2\alpha-2}}$$

$$- \int_{\mathbb{T}^n} e^{-iy\cdot x}|x|^{2\alpha-2}\left(1 - \chi\left(\frac{|x|}{\varepsilon}\right)\right)dx,$$

$$\Delta\left(\widehat{|x|^{2\alpha-2}}\right)_\varepsilon(y) = \frac{\tilde{c}_\alpha}{|y|^{n+2\alpha}}$$

$$- \int_{\mathbb{T}^n} e^{-iy\cdot x}|x|^{2\alpha}\left(1 - \chi\left(\frac{|x|}{\varepsilon}\right)\right)dx,$$

$$(16)$$

we obtain

$$\sum_{\nu\in\mathbb{Z}^n} \Delta\left(\widehat{\Phi_\varepsilon}\right)(y-\nu) = \tilde{c}_\alpha \sum_{\nu\in\mathbb{Z}^n} \frac{1}{|y-\nu|^{n+2\alpha}}$$

$$+ O\left(\sum_{\nu\in\mathbb{Z}^n} \frac{1}{|y-\nu|^{n+\delta}} O\left(\varepsilon^\delta\right)\right),$$

$$(17)$$

for some $\delta > 0$.

Therefore,

$$\Lambda^{2\alpha} u(x)$$

$$= \lim_{\varepsilon\to 0} \sum_{\nu\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \Delta\left(\widehat{\Phi_\varepsilon}\right)(x-y-\nu)\left[u(x)-u(y)\right]dy \qquad (18)$$

$$= C_\alpha \sum_{\nu\in\mathbb{Z}^n} PV \int_{\mathbb{T}^n} \frac{u(x)-u(y)}{|x-y-\nu|^{n+2\alpha}}dy.$$

$$\square$$

Lemma 5. *Let* $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, *and let* u *be a complex-valued function satisfying* $u, u^{2\alpha} \in L^p(\mathbb{R}^n)$ *with* $0 < \alpha < 1$ *and* $1 \le p < \infty$. *Then*

$$\mathrm{Re}\left(\int_{\mathbb{T}^n} |u|^{p-2} u\overline{\Lambda^{2\alpha}u}\; dx\right) \ge 0. \qquad (19)$$

Proof. For $0 < \alpha < 1$, by invoking Proposition 4 and changing of variables, we have

$$\mathrm{Re}\left(\int_{\mathbb{T}^n} |u|^{p-2} u\overline{\Lambda^{2\alpha}u}\; dx\right)$$

$$= C_\alpha \lim_{\varepsilon\to 0} \mathrm{Re}\left(\iint_{|y-x|\ge\varepsilon} |u(x)|^{p-2} u(x) \frac{\overline{u(x)-u(y)}}{|x-y|^{n+2\alpha}} dy\, dx\right)$$

$$= -C_\alpha \lim_{\varepsilon\to 0} \mathrm{Re}\left(\int_{|y-x|\ge\varepsilon} |u(y)|^{p-2} u(y) \frac{\overline{u(x)-u(y)}}{|x-y|^{n+2\alpha}} dy\, dx\right)$$

$$= \frac{C_\alpha}{2} \lim_{\varepsilon\to 0} \mathrm{Re}\left(\iint_{|y-x|\ge\varepsilon} \left[|u(x)|^{p-2} u(x) - |u(y)|^{p-2} u(y)\right]\right.$$

$$\left.\times \frac{\overline{u(x)-u(y)}}{|x-y|^{n+2\alpha}} dy\, dx\right)$$

$$\ge 0.$$

$$(20)$$

$$\square$$

In the following Lemma, we give a characterization of the eigenvalues and eigenvectors of the pseudodifferential operators $\Lambda^{2\alpha}$.

Lemma 6. *Let* $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. *Then there exist a sequence of real numbers* $\{\lambda_j\}_{j=1}^\infty$ *and a sequence of periodical functions* $\{\omega_j\}_{j=1}^\infty \subset H^{2\alpha}(\mathbb{T}^n)$, *such that*

$$\Lambda^{2\alpha}\omega_j = \lambda_j\omega_j, \quad j = 1, 2, 3, \ldots. \qquad (21)$$

Moreover, $\{\omega_j\}_{j=1}^\infty$ *is a basis of* $H^{2\alpha}(\mathbb{T}^n)$ *satisfying*

$$\left(\omega_j, \omega_k\right) \equiv \int_{\mathbb{T}^n} \omega_j\overline{\omega_k}dx = \delta_{jk}, \qquad (22)$$

$$\left(\Lambda^{2\alpha}\omega_j, \omega_k\right) = \lambda_j\delta_{jk}, \qquad (23)$$

for $j, k = 1, 2, 3, \ldots.$

Notice that the lemma is a direct consequence of elliptic regularity and functional analysis; see [20], for example.

Let us now give the weak formulation of (5)-(6).

Definition 7. Let $T > 0$, $f \in L^2(0, T; H^\alpha(\mathbb{T}^n))$, $\partial f/\partial t \in L^2(0, T; L^2(\mathbb{T}^n))$, and $u_0 \in H^\alpha(\mathbb{T}^n)\cap L^{2(\rho+1)}(\mathbb{T}^n)$. A measurable complex-valued function $u(x, t)$ is said to be a weak solution to (5)-(6) on $[0, T]$, provided the following:

(1) $u \in L^\infty(0, T; H^\alpha(\mathbb{T}^n)) \cap L^p(0, T; L^p(\mathbb{T}^n))$ and $\partial u/\partial t \in L^\infty(0, T; L^2(\mathbb{T}^n))$;

(2) u satisfies (5) in the sense of distributions;

(3) $u(x, 0) = u_0(x)$ a.e. $\mathbb{T}^n$.

Here and hereafter, $p = \rho + 2$.

Now, we state our existence results in the following theorem.

Theorem 8. *Let* $T > 0$ *be a given time, and*

$$u_0 \in H^\alpha(\mathbb{T}^n) \cap L^{2(\rho+1)}(\mathbb{T}^n),$$

$$f \in L^2(0, T; H^\alpha(\mathbb{T}^n)), \quad \frac{\partial f}{\partial t} \in L^2(0, T; L^2(\mathbb{T}^n)). \qquad (24)$$

Then there exists at least one weak solution u *to* (5)-(6) *on* $[0, T]$, *taking* u_0 *as initial data.*

Proof. We use Galerkin method.

Step 1. Construction of approximate solution.

Let $\{\omega_j\}_{j=1}^\infty$ be given as in Lemma 6. We consider the approximate solution which has the form

$$u_m(t) = \sum_{j=1}^m c_{jm}(t)\omega_j, \qquad (25)$$

where c_{jm} satisfy the following ordinary differential system:

$$\left(\frac{\partial u_m}{\partial t}(t), \omega_j\right) - i\left(\Lambda^{2\alpha} u_m(t), \omega_j\right)$$

$$+ \left(|u_m(t)|^\rho u_m(t), \omega_j\right) \tag{26}$$

$$= \left(f(t), \omega_j\right), \quad 1 \le j \le m,$$

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{jm}\omega_j \longrightarrow u_0 \quad \text{in } H^\alpha(\mathbb{T}^n) \cap L^{2(\rho+1)}(\mathbb{T}^n),$$

$$\text{as } m \longrightarrow \infty. \tag{27}$$

Here and hereafter, $(u, v) = \int_{\mathbb{T}^n} u\bar{v}\, dx$ is the inner product in $L^2(\mathbb{T}^n)$.

The system (26)-(27) is nonsingular because $\{\omega_j\}$ are linear independent. Thus we may apply standard theory of ordinary differential equations to obtain the existence of a local solution to equations (26)-(27) on $[0, t_m]$, for some $t_m > 0$. We shall then, in the next step, establish some a priori estimates of the obtained solutions. This will ensure that $t_m = T$.

Step 2. A priori estimates.

By multiplying (26) by $\overline{c_{jm}}(t)$ and summing with $1 \le j \le m$, we have

$$\left(\frac{\partial u_m}{\partial t}(t), u_m(t)\right) + i\left(\Lambda^\alpha u_m(t), \Lambda^\alpha u_m(t)\right)$$

$$+ \left(|u_m(t)|^\rho u_m(t), u_m(t)\right) = \left(f(t), u_m(t)\right). \tag{28}$$

Taking real part of (28) and invoking Hölder inequality then yields

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_{L^2(\mathbb{T}^n)}^2 + \int_{\mathbb{T}^n}|u_m(t)|^\rho dx$$

$$= \text{Re}\left(f(t), u_m(t)\right) \tag{29}$$

$$\le \|u_m(t)\|_{L^2(\mathbb{T}^n)}^2 + \|f(t)\|_{L^2(\mathbb{T}^n)}^2.$$

By Gronwall's inequality, we have

$$u_m \text{ is bounded in } L^\infty\left(0, T; L^2(\mathbb{T}^n)\right), \quad L^\rho\left(0, T; L^\rho(\mathbb{T}^n)\right). \tag{30}$$

By multiplying (26) by $\lambda_j \overline{c_{jm}}(t)$, summing with $1 \le j \le m$, and noticing (23), we get

$$\left(\frac{\partial \Lambda^\alpha u_m}{\partial t}(t), \Lambda^\alpha u_m\right) - i\|\Lambda^{2\alpha} u_m(t)\|_{L^2(\mathbb{T}^n)}^2$$

$$+ \left(|u_m(t)|^\rho u_m(t), \Lambda^{2\alpha} u_m(t)\right) = \left(\Lambda^\alpha f(t), \Lambda^\alpha u_m(t)\right). \tag{31}$$

Thanks to Lemma 5, we may consider the real part of (31) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha u_m\|_{L^2(\mathbb{T}^n)} \le \text{Re}\left(\Lambda^\alpha f(t), \Lambda^\alpha u_m\right). \tag{32}$$

By Gronwall's inequality again, we have

$$u_m \text{ is bounded in } L^\infty\left(0, T; H^\alpha(\mathbb{T}^n)\right). \tag{33}$$

By multiplying (26) by $\overline{c'_{jm}}(t)$, summing with $1 \le j \le m$, and taking $t = 0$, we have

$$\left\|\frac{\partial u_m}{\partial t}(0)\right\|_{L^2(\mathbb{T}^n)}^2 - \frac{i}{2}\frac{d}{dt}\bigg|_{t=0}\|\Lambda^\alpha u_m(t)\|_{L^2(\mathbb{T}^n)}$$

$$+ \left(|u_m(0)|^\rho u_m(0), \frac{\partial u_m}{\partial t}(0)\right) = \left(f(0), \frac{\partial u_m}{\partial t}(0)\right). \tag{34}$$

Looking into the real part of (34), we see

$$\left\|\frac{\partial u_m}{\partial t}(0)\right\|_{L^2(\mathbb{T}^n)}^2$$

$$\le \left(\|u_m(0)\|_{L^{2(\rho+1)}(\mathbb{T}^n)} + \|f(0)\|_{L^2(\mathbb{T}^n)}\right)\left\|\frac{\partial u_m}{\partial t}(0)\right\|_{L^2(\mathbb{T}^n)}. \tag{35}$$

Thus,

$$\left\|\frac{\partial u_m}{\partial t}(0)\right\|_{L^2(\mathbb{T}^n)} \le C. \tag{36}$$

Now we can obtain the estimate for $u'_m(t)$. By differentiating (26) with respect to t, we get

$$\left(\frac{\partial^2 u_m}{\partial t^2}(t), \omega_j\right) + i\left(\frac{\partial \Lambda^{2\alpha} u_m}{\partial t}(t), \omega_j\right)$$

$$+ \left(\frac{\partial}{\partial t}\left(|u_m(t)|^\rho u_m(t)\right), \omega_j\right) \tag{37}$$

$$= \left(\frac{\partial f}{\partial t}(t), \omega_j\right), \quad 1 \le j \le m.$$

By multiplying (37) by $\overline{c'_{jm}}(t)$ and summing with $1 \le j \le m$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m}{\partial t}(t)\right\|_{L^2(\mathbb{T}^n)}^2 - \frac{i}{2}\left\|\frac{\partial \Lambda^\alpha u_m}{\partial t}(t)\right\|_{L^2(\mathbb{T}^n)}$$

$$+ \left(\frac{\partial}{\partial t}\left(|u_m(t)|^\rho u_m(t)\right), \frac{\partial u_m}{\partial t}(t)\right) = \left(\frac{\partial f}{\partial t}(t), \frac{\partial u_m}{\partial t}(t)\right). \tag{38}$$

Notice that

$$\left(\frac{\partial}{\partial t}\left(|u_m(t)|^\rho u_m(t)\right), \frac{\partial u_m}{\partial t}(t)\right)$$

$$= \int_{\mathbb{T}^n}|u_m(t)|^\rho\left|\frac{\partial u_m}{\partial t}(t)\right|^2 dx$$

$$+ \frac{\rho}{2}\int_{\mathbb{T}^n}|u_m(t)|^{\rho-2}$$

$$\times \left(\left|u_m(t)\frac{\partial \overline{u_m}}{\partial t}(t)\right|^2 + |u_m(t)|^2\left|\frac{\partial u_m}{\partial t}\right|^2\right) dx$$

$$\ge 0. \tag{39}$$

Taking the real part of (38) then yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial u_m}{\partial t}(t)\right\|_{L^2(\mathbb{T}^n)}^2$$

$$\leq 2\operatorname{Re}\left(\frac{\partial f}{\partial t}(t), \frac{\partial u_m}{\partial t}(t)\right) \tag{40}$$

$$\leq \left\|\frac{\partial f}{\partial t}(t)\right\|_{L^2(\mathbb{T}^n)}^2 + \left\|\frac{\partial u_m}{\partial t}(t)\right\|_{L^2(\mathbb{T}^n)}^2.$$

By Gronwall's inequality, we deduce

$$\frac{\partial u_m}{\partial t} \text{ is bounded in } L^\infty\left(0,T;L^2\left(\mathbb{T}^n\right)\right). \tag{41}$$

Step 3. Convergence process.

By (30), (33), and (41), we have, up to a subsequence, still denoted by u_m, that

$$u_m \rightharpoonup u \text{ weakly} * \text{ in } L^\infty\left(0,T;H^\alpha\left(\mathbb{T}^n\right)\right), \quad \text{as } m \longrightarrow \infty, \tag{42}$$

$$u_m \rightharpoonup u \text{ weakly in } L^p\left(0,T;L^p\left(\mathbb{T}^n\right)\right), \quad \text{as } m \longrightarrow \infty, \tag{43}$$

$$\frac{\partial u_m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } L^\infty\left(0,T;L^2\left(\mathbb{T}^n\right)\right), \quad \text{as } m \longrightarrow \infty. \tag{44}$$

Thus by Lemma 3, we find

$$u_m \longrightarrow u \text{ strongly in } L^\infty\left(0,T;L^2\left(\mathbb{T}^n\right)\right), \text{ a.e. } \mathbb{T}^n, \tag{45}$$
$$\text{as } m \longrightarrow \infty.$$

By (44), there exists a function $w = w(x,t)$ such that

$$|u_m|^\rho u_m \rightharpoonup w(x,t) \text{ weakly in } L^q\left(0,T;L^q\left(\mathbb{T}^n\right)\right), \tag{46}$$
$$\text{as } m \longrightarrow \infty,$$

with $(1/p) + (1/q) = 1$. By Lemma 2 and the fact that

$$|u_m|^\rho u_m \longrightarrow |u|^\rho u, \text{ a.e. } \mathbb{T}^n, \tag{47}$$

we see

$$w = |u|^\rho u. \tag{48}$$

Fix j; we now pass to limit $m \to \infty$ in (26) to deduce

$$\left(\frac{\partial u}{\partial t}(t), \omega_j\right) - i\left(\Lambda^\alpha u(t), \Lambda^\alpha \omega_j\right) + \left(|u(t)|^\rho u(t), \omega_j\right) \\ = \left(f(t), \omega_j\right). \tag{49}$$

A simple density argument then shows

$$\left(\frac{\partial u}{\partial t}(t), v\right) + i\left(\Lambda^\alpha u(t), \Lambda^\alpha v\right) + \left(|u(t)|^\rho u(t), v\right) \\ = \left(f(t), v\right), \tag{50}$$

for all $v \in H^\alpha(\mathbb{T}^n) \cap L^p(\mathbb{T}^n)$.

The proof of Theorem 8 is completed. □

3. Uniqueness of Weak Solutions

In this section, we will discuss the uniqueness of weak solutions of (5)-(6). More precisely, we have the following theorem.

Theorem 9. *Assume as in Theorem 8. Then there exists a unique weak solution of* (5)-(6).

Proof. Let u and v be two weak solutions of (5)-(6), given in Theorem 8, with the same datum. Then $w = u - v$ satisfies

$$\frac{\partial w}{\partial t} + i\Lambda^{2\alpha} w = |v|^\rho v - |u|^\rho u, \tag{51}$$

$$w(0) = 0. \tag{52}$$

Also, we have

$$w \in L^\infty\left(0,T;H^\alpha\left(\mathbb{T}^n\right)\right) \cap L^p\left(0,T;L^p\left(\mathbb{T}^n\right)\right), \\ \frac{\partial w}{\partial t} \in L^\infty\left(0,T;L^2\left(\mathbb{T}^n\right)\right). \tag{53}$$

By taking the inner product of (51) with w in $L^2(\mathbb{T}^n)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2(\mathbb{T}^n)}^2 - i\|w\|_{H^\alpha(\mathbb{T}^n)}^2 \\ = \int_{\mathbb{T}^n}\left(|v|^\rho v - |u|^\rho u\right)\overline{(u-v)}\mathrm{d}x. \tag{54}$$

By taking the real part of (54) and noticing

$$\operatorname{Re}\left(\int_{\mathbb{T}^n}\left(|v|^\rho v - |u|^\rho u\right)\overline{(u-v)}\mathrm{d}x\right) \leq 0, \tag{55}$$

we have

$$\|w\|_{L^2(\mathbb{T}^n)}^2 \leq \|w(0)\|_{L^2(\mathbb{T}^n)}^2 = 0. \tag{56}$$

Thus $w = 0$ and $u \equiv v$ a.e. in $\mathbb{T}^n$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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