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Research Article

On Solution of Integrodifferential Equation with Delay Parameter by Sinc Basis Functions

I. Najafi Khalilsaraye and K. Maleknejad

Department of Mathematics, College of Basic Sciences, Karaj Branch, Islamic Azad University, Alborz, Iran

Correspondence should be addressed to K. Maleknejad; maleknejad@iust.ac.ir

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We want to find a numerical solution for an integrodifferential equation with an integral boundary condition and delay parameter. This type of problems arises in mathematical physics, mechanics, population growth, and other fields of physics and mathematical chemistry. So, convergence of this approach is discussed by presenting a theorem which gives exponential type convergence rate and guarantees the accuracy of that. Finally, by some numerical examples, we show the efficiency and accuracy of this numerical method.

1. Introduction

Discussing integrodifferential equations with integral boundary condition consisting of delay parameter is a worthy and significant branch of nonlinear applied mathematics. It is important that integrodifferential equation with delay parameter is generated often in investigations connected with chemical engineering, mathematical physics, underground water flow, engineering, and so on (see [1, 2]). Note that the problems with integral boundary conditions have various applications in applied fields such as population growth problems and blood flow problems. For a detailed description of the integrodifferential equations with delay parameter and problems with integral boundary conditions, the reader can refer to references of [3–7].

In this paper we discuss the following problem which shows a first-order integrodifferential equation with integral boundary condition consisting of delay parameter. The existence and uniqueness of solution for this problem are proved in [8]. But analytic solving and reaching an exact solution are impossible. Then in this paper we approximate the exact solution by numerical method

$$\frac{dy}{dx} = g\left(x, y\left(x\right), y\left(\gamma_1\left(x\right)\right), Uy\left(x\right), Vy\left(x\right)\right) \equiv Gy\left(x\right),$$
$$y\left(0\right) = \eta_1 y\left(\kappa\right) + \eta_2 \int_0^b \omega\left(r, y\left(r\right)\right) dr + a, \tag{1}$$

where $x \in I = [0, b]$ $(b > 0)$, $g \in C(I \times \mathbb{R}^4, \mathbb{R})$, where C is family of all continuous functions, and $\gamma_1 \in C(I, I)$, $\kappa \in (0, b]$, $\omega \in C(I \times \mathbb{R}, \mathbb{R})$, $\eta_1, \eta_2, a \in \mathbb{R}$.

And

$$\left(Uy\right)\left(x\right) = \int_0^{\gamma_2(x)} k\left(x, r\right) y\left(B\left(r\right)\right) dr,$$
$$\left(Vy\right)\left(x\right) = \int_0^b h\left(x, r\right) y\left(D\left(r\right)\right) dr. \tag{2}$$

Here $\gamma_2, B, D \in C(I, I)$, $k(x, r) \in C[A, \mathbb{R}^+]$, and $h(x, r) \in C[A_0, \mathbb{R}^+]$ that

$$A = \left\{(x, r) \in \mathbb{R}^2 \mid 0 \le r \le \gamma_2\left(x\right), x \in I\right\},$$
$$A_0 = \left\{(x, r) \in \mathbb{R}^2 \mid 0 \le r \le x, x \in I\right\}. \tag{3}$$

Here if $\eta_1 = 1$, $\eta_2 = a = 0$, and $\kappa = b$, then we have a problem with boundary condition of kind periodic, and if $\eta_1 = 0$, then we have a problem with integral boundary condition, and if $\eta_1 = \eta_2 = 0$, we have a problem with an initial condition. So, problem (1) is general type of these cases.

Now, the following functional integral equation can be easily concluded from (1):

$$y(x) = \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\,dr + a + \int_0^x g\left(t, y(t), y(\gamma_1(t)), \int_0^{\gamma_2(t)} k(t,r)\, y(B(r))\,dr, \int_0^b h(t,r)\, y(D(r))\,dr\right) dt \quad (4)$$

and in this paper we want to discuss solution of (1) in point of equivalent integral equation (4). But we know the fact that we cannot solve this integral equation to give an exact solution, so numerical approaches are used to reach an approximated solution.

Numerical approaches for estimated solution of integrodifferential and integral equation and search for existence and uniqueness of solution for some problems have been researched by many authors and reader can see these methods in [9–15]. In these references authors use methods on an estimate by basis function such as wavelets, polynomials, and so forth or use some quadrature formulas. But these technics usually have convergence rate of polynomial order with respect to M where M represents the cardinal of terms of sum in the expansion or the cardinal of points of the quadrature formula. In [16] author showed that if we employ the Sinc approach, the convergence rate is exponential order such as $O(\exp(-cM^{1/2}))$ with some $c > 0$. We know the exponential rate is much faster than that of polynomial rate. So, in this paper, we employ the Sinc function instead of base function and an iterative technique to estimate exact solution of (4) in marked points. Our approach dose not contain of changing the solution of (4) to a system of algebraic equations by expanding $y(x)$ as basis function with unknown coefficients, so this technique has computations less than other methods and exponential rate in accuracy. Also in this present paper, we prove a theorem to guarantee the convergence of numerical technique.

2. Main Results

In this section, we introduce basic requirements and theorem to prove existence and uniqueness of solution for first-order integrodifferential equation with integral boundary condition consisting of delay parameter (1). For detailed descriptions, we refer the reader to [17, 18].

Definition 1. A function as $y \in C^1(I, I)$ is called a lower solution of (1) if

$$\frac{dy}{dx} \le Gy(x), \quad x \in I, \qquad y(0) \le \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\,dr + a \quad (5)$$

and it is an upper solution of (1) if

$$\frac{dy}{dx} \ge Gy(x), \quad x \in I, \qquad y(0) \ge \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\,dr + a. \quad (6)$$

Theorem 2. *Let $u_0, v_0 \in C^1(I, \mathbb{R})$ be lower and upper solutions of (1), respectively, and $u_0(x) \le v_0(x)$, $x \in I$.*

In addition consider the following.

(L_1): *$g \in C(I \times \mathbb{R}^4, \mathbb{R})$, $\gamma_1, \gamma_2, B, D \in C(I, I)$, $\gamma_1(x), \gamma_2(x), B(x), D(x) \le x$, $\forall x \in I$, $\kappa \in (0, b)$, $\omega \in C(I \times \mathbb{R}, \mathbb{R})$, and $\eta_1, \eta_2 \ge 0$.*

(L_2): *there are nonnegative bounded integrable functions $M_1(x)$, $M_2(x)$, $M_3(x)$, $M_4(x)$ on I that*

$$\int_0^b \left[M_1(t) + M_2(x) + M_3(x) \int_0^{\gamma_2(x)} k(x,r)\,dr + M_4(x) \int_0^b h(x,r)\,dr\right] dx \le 1 \quad (7)$$

such that

$$g(x, \phi_1, \phi_2, U\phi_1, V\phi_1) - g(x, \psi_1, \psi_2, U\psi_1, V\psi_1) \ge -M_1(x)(\phi_1 - \psi_1) - M_2(x)(\phi_2 - \psi_2) - M_3(x)\,U(\phi_1 - \psi_1) - M_4(x)\,V(\phi_1 - \psi_1) \quad (8)$$

if $u_0 \le \psi_1 \le \phi_1 \le v_0$, $u_0(\gamma_1(x)) \le \psi_2 \le \phi_2 \le v_0(\gamma_1(x))$.

(L_3): *there is $\theta(x) \in C(I, \mathbb{R}^+)$ such that $\omega(x, \psi) - \omega(x, \psi^-) \ge \theta(x)(\psi - \psi^-)$, and if $u_0(t) \le \psi^- \le \psi \le v_0(x)$. then problem (1) has extremal solutions $u, v \in [u_0, v_0]$. In addition, there are monotone sequences $u_n(x), v_n(x) \subset [u_0, v_0]$ such that $u_n \to u$, $v_n \to v$ for $n \to +\infty$ and these are convergent uniformly on $x \in I$, where $u_n(x)$, $v_n(x)$ are defined as*

$$\begin{aligned} u_n(x) = &\int_0^x e^{-\int_r^x M_1(s)ds} \times \left[g(r, u_{n-1}(r), u_{n-1}(\gamma_1(r)), Uu_{n-1}(r), Vu_{n-1}(r)) + M_1(r)\,u_{n-1}(r) - M_2(r)(u_n - u_{n-1})(\gamma_1(r)) - M_3(r)\,U(u_n - u_{n-1})(r) - M_4(r)\,V(u_n - u_{n-1})(r)\right] dr \\ &+ e^{-\int_0^x M_1(s)ds} \left[\eta_1 u_{n-1}(\kappa) + \eta_2 \int_0^b \omega(r, u_{n-1}(r))\,dr + a\right]; \\ &\forall x \in I, \quad n = 1, 2, 3, \ldots, \end{aligned}$$

$$\begin{aligned} v_n(x) = & \int_0^x e^{-\int_r^x M_1(s)ds} \\ & \times [g(r, v_{n-1}(r), v_{n-1}(\gamma_1(r)), \\ & \quad Uv_{n-1}(r), Vv_{n-1}(r)) + M_1(r) z_{n-1}(r) \\ & \quad - M_2(r)(v_n - v_{n-1})(\gamma_2(r)) \\ & \quad - M_3(r) U(v_n - v_{n-1})(r) \\ & \quad - M_4(r) V(v_n - v_{n-1})(r)] \, dr \\ & + e^{-\int_0^x M_1(s)ds} \left[\eta_1 v_{n-1}(\kappa) \right. \\ & \left. + \eta_2 \int_0^b \omega(r, v_{n-1}(r)) \, dr + a \right]; \\ & \forall x \in I, \quad n = 1, 2, 3, \ldots, \\ u_0 \le & u_1 \le \cdots \le u_n \le \cdots \le u \le v \le \cdots \le v_n \\ & \le \cdots \le v_1 \le v_0. \end{aligned} \tag{9}$$

Proof. See [18]. □

Theorem 3. *Consider that assumptions of Theorem 2 hold. Moreover, consider the following.*

(L_4): *there are nonnegative bounded functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\alpha_4(x)$ on I, such that*

$$\begin{aligned} & g(x, \phi_1, \phi_2, U\phi_1, V\phi_1) - g(x, \psi_1, \psi_2, U\psi_1, V\psi_1) \\ & \quad \le \alpha_1(x)(\phi_1 - \psi_1) + \alpha_2(x)(\phi_2 - \psi_2) \\ & \qquad + \alpha_3(x) U(\phi_1 - \psi_1) + \alpha_4(x) V(\phi_1 - \psi_1) \end{aligned} \tag{10}$$

if $u_0(x) \le \psi_1 \le \phi_1 \le v_0(x)$, $u_0(\gamma_1(x)) \le \psi_2 \le \phi_2 \le v_0(\gamma_1(x))$.

(L_5): *there is $\beta(x) \in \mathscr{C}(I, \mathbb{R}^+)$ such that $\omega(x,\psi) - \omega(x,\psi^-) \le \beta(x)(\psi - \psi^-)$ if $u_0(t) \le \psi^- \le \psi \le v_0(x)$.*

Then problem (1) *has a unique solution $u^- \in [u_0, v_0]$. In addition, there are sequences $u_n(x), v_n(x) \subset [u_0, v_0]$ that these are monotone and $u_n \to u^-$, $v_n \to u^-$ for $n \to +\infty$. This convergence is uniformly on $x \in I$, where $u_n(x)$, $v_n(x)$ are defined as* (9) *such that $\|u_n - u^-\|_c \le L^n \|v_0 - u_0\|_c$, $n \in \mathbb{N}$, where*

$$\begin{aligned} L = \eta_1 + \int_0^b & \left[\eta_2 \beta(x) + \alpha_1(x) + M_1(x) + \alpha_2(x) + M_2(x) \right. \\ & + (\alpha_3(x) + M_3(x)) \int_0^{\gamma_2(x)} K(x, r) \, dr \\ & \left. + (\alpha_4(x) + M_4(x)) \int_0^b h(x, r) \, dr \right] dx < 1. \end{aligned} \tag{11}$$

Proof. See [18]. □

3. Sinc Function

In this section, we recall the basis function and some of its applicabilities. In here, definition of sinc(x) function is followed by

$$\operatorname{Sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}; & x \ne 0, \\ 1; & x = 0. \end{cases} \tag{12}$$

Now, for $h > 0$ and integer j, we define jth Sinc function with step size h by

$$S(j,h)(x) = \frac{\sin(\pi(x - jh)/h)}{\pi(x - jh)/h}. \tag{13}$$

3.1. Sinc Estimation on $[a,b]$. Let $x = \varphi(w)$ be a transformation that denotes a conformal transformation which transfers the simply connected domain A onto a strip region A_d such that

$$\varphi((a,b)) = (-\infty, \infty), \qquad \lim_{x \to a} \varphi(x) = -\infty, \qquad \lim_{x \to b} \varphi(x) = \infty. \tag{14}$$

In here ∂A is boundary of A and in order to have the Sinc estimation on (a,b) conformal transformation is applied as follows:

$$\varphi(t) = \ln\left(\frac{t - a}{b - t}\right). \tag{15}$$

This function transfers the eye-shaped complex region

$$\left\{ w = x + iy : \left| \arg\left(\frac{w - a}{b - w}\right) \right| < d \le \frac{\pi}{2} \right\} \tag{16}$$

onto A_d that it is a strip region:

$$A_d = \left\{ \sigma = \alpha + \beta i : |\beta| < d < \frac{\pi}{2} \right\}. \tag{17}$$

The basis functions on finite interval (a,b) are given by

$$S(j,h) \circ \varphi(x) = \frac{\sin(\pi(\varphi(x) - jh)/h)}{\pi(\varphi(x) - jh)/h}, \tag{18}$$

and also, Sinc function for interpolation points $x_j = jh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1; & j = k, \\ 0; & j \ne k. \end{cases} \tag{19}$$

Then, $S(j,h) \circ \varphi(x)$ shows behavior of Kronecker delta function on the network points

$$x_j = \varphi^{-1}(jh) = \frac{a + be^{jh}}{1 + e^{jh}}. \tag{20}$$

The approximation of $g(x)$ by interpolation and quadrature formulas for $\int_a^b g(x)dx$ is

$$g(x) \approx \sum_{j=-M}^{M} g(x_j) S(j,h) \circ \varphi(x),$$
$$\int_a^b g(x)\,dx \approx h \sum_{j=-M}^{M} \frac{g(x_j)}{\varphi'(x_j)}. \tag{21}$$

Theorem 4. *Consider that, for a map $w = \varphi^{-1}(\xi)$, the map $g(\varphi^{-1}(\xi))$ satisfies*

(1) *$g \in H^1(D_d)$, for $d > 0$,*

(2) *g decays exponentially on the real line such that*

$$|g(x)| \le \alpha \exp(-\beta |x|), \quad \forall x \in \mathbb{R},\ \alpha, \beta > 0 \tag{22}$$

with some α, β, and d. Then one has

$$\sup_{a<x<b} \left| g(x) - \sum_{j=-M}^{M} g(\varphi^{-1}(jh)) S(j,h) \circ \varphi(x) \right| \le C\sqrt{M} \exp\left(-\sqrt{\pi d \beta M}\right). \tag{23}$$

That there is some C and h is $h = \sqrt{\pi d/\beta M}$.

Proof. See [16]. □

Definition 5. Let $L_\alpha(A)$ be the set of all analytic functions g, for which there exists a constant C, such that

$$|g(z)| \le C \frac{|e^{\varphi(z)}|^\alpha}{(1+|e^{\varphi(z)}|)^{2\alpha}}; \quad z \in A,\ 0 < \alpha \le 1. \tag{24}$$

Theorem 6. *Let $g/\varphi' \in L_\alpha(A)$, with $0 < \alpha \le 1$ and $0 < d \le \pi$; also let $h = \sqrt{\pi d/\alpha M}$. Then there exists a constant C_1, which is independent of M, such that*

$$\left| \int_a^{x_j} g(t)\,dt - h \sum_{k=-M}^{M} \delta_{jk}^{(-1)} \frac{g(x_j)}{\varphi'(x_k)} \right| \le C_1 e^{-\sqrt{\pi d \alpha M}}, \tag{25}$$

where

$$\delta_{jk}^{(-1)} = \frac{1}{2} + \int_0^{j-k} \frac{\sin(\pi t)}{\pi t} dt, \tag{26}$$

and $\varphi(x)$, x_k are defined as above.

Proof. See [16]. □

3.2. Sinc-Qudrature Method. In this section, for solving equation

$$y(x) = \lambda_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\,dr + a + \int_0^x g\Bigg(s, y(s), y(\gamma_1(s)), \int_0^{\gamma_2(s)} k(s,r)\, y(B(r))\,dr, \int_0^b h(s,r)\, y(D(s))\,dr\Bigg) ds, \tag{27}$$

we try to discrete integral equation by quadrature formula as

$$\int_a^b g(s)\,ds = h \sum_{j=-M}^{M} \frac{g(s_j)}{\varphi'(s_j)} + O\left(\exp\left(-\frac{2\pi d M}{\log(2\pi d M/\beta)}\right)\right)$$
$$\int_a^s f(x)\,dx = h \sum_{j=-M}^{M} \frac{g(x_j)}{\varphi'(x_j)} \eta_{h,j}(s) + O\left(\frac{\log M}{M} \exp\left(-\frac{\pi d M}{\log(\pi d M/\beta)}\right)\right), \tag{28}$$

where

$$\eta_{h,j}(s) = \frac{1}{2} + \frac{1}{\pi} r_i\left(\pi \frac{r - jh}{h}\right); \quad r_i = \int_0^x \frac{\sin(t)}{t} dt \tag{29}$$

with $x_j = s_j = (a + be^{jh})/(1 + jh)$, $j = -M, \ldots, M$ and $h = (1/M)\log(\pi d M/\beta)$ (see [16]).

Now, by substituting quadrature formulas in the integral equation (4), we have

$$y_M(x) = \eta_1 y(\kappa) + \eta_2 h \sum_{j=-M}^{M} \frac{\omega(s_j, y(s_j))}{\varphi'(s_j)} + a + h \sum_{j=-M}^{M} g\Bigg(s_j, y(s_j), y(\alpha(s_j)), h \sum_{i=-M}^{M} \frac{K(s_j, r_i)\, y(B(r_i))}{\varphi'(r_i)} \eta_{h,i}(t), h \sum_{i=-M}^{M} \frac{h(s_j, r_i)\, y(D(r_i))}{\varphi'(r_i)}\Bigg) \times (\varphi'(s_j))^{-1} \eta_{h,j}(x). \tag{30}$$

Now, for $n = 1, 2, 3, \ldots$, let

$$\begin{aligned} y_{1,M}(x) &= \eta_1 y(\kappa), \\ y_{n+1,M}(x) &= \eta_1 y_{n,M}(\kappa) + \eta_2 h \sum_{j=-M}^{M} \frac{\omega(s_j, y_{n,M}(s_j))}{\varphi'(s_j)} + a \\ &\quad + h \sum_{j=-M}^{M} g\Bigg(s_j, y_{n,M}(s_j), y_{n,M}(\alpha(s_j)), \\ &\qquad h \sum_{i=-M}^{M} \frac{a(s_j, r_i)\, y_{n,M}(B(r_i))}{\varphi'(r_i)} \eta_{h,i}(x), \\ &\qquad h \sum_{i=-M}^{M} \frac{h(s_j, r_i)\, y_{n,M}(\delta(r_i))}{\varphi'(r_i)}\Bigg) \\ &\quad \times (\varphi'(s_j))^{-1} \eta_{h,j}(x). \end{aligned} \tag{31}$$

4. Convergence of Method

In this section, we present a theorem that shows a bound for $y(x) - y_n(x)$ with the real norm where $y(x)$ is the exact solution of problem (4) and $y_n(x)$ is an estimation for $y(x)$ by using Sinc function in interpolaion and quadrature formula. The result is shown as follows.

Theorem 7. *Under the assumptions* (L_1)–(L_5), *iterative approximation approach* (31) *is convergent to exact solution if* $y_1(x)$ *is closed enough to the it and*

$$N_2 \le \frac{1 - (\eta_1 + \eta_2 b N_1)}{b(2 + N_3 + N_4)}, \tag{32}$$

where $N_1 = \sup\{\beta(x); x \in I\}$, $N_2 = \max\{\alpha_i(x); i = 1, 2, 3, 4, x \in I\}$, $N_3 = \sup\{K(x, r); x, r \in I\}$, *and* $N_4 = \sup\{h(x, r); x, r \in I\}$.

Proof. For a fixed M let

$$\begin{aligned} y_{n+1,M}(x) &= \eta_1 x_{n,M}(\tau) + \eta_2 \int_0^b \omega(r, y_{n,M}(r))\, dr + a \\ &\quad + \int_0^x g(s, y_{n,M}(s), y_{n,M}(\gamma_1(s)), U y_{n,M}(s), \\ &\qquad V y_{n,M}(s))\, ds, \\ y(x) &= \eta_1 y(\kappa) + \eta_2 \int_0^b \omega(r, y(r))\, dr + a \\ &\quad + \int_0^x g(s, x(s), y(\gamma_1(s)), U y(s), V y(s))\, ds. \end{aligned} \tag{33}$$

Now

$$\begin{aligned} &|y_{n+1,M}(x) - y(x)| \\ &\le \eta_1 |y_n(\kappa) - y(\kappa)| \\ &\quad + \eta_2 \left| \int_0^b (\omega(r, y_n(r)) - \omega(r, y(r)))\, dr \right| \\ &\quad + \Bigg| \int_0^x \Bigg[g\Bigg(s, y_n(s), y_n(\gamma_1(s)), \\ &\qquad \int_0^{\gamma_2(s)} K(s, r)\, y_n(B(r))\, dr, \\ &\qquad \int_0^b h(s, r)\, y_n(D(r))\, dr\Bigg) \\ &\qquad - f\Bigg(s, y(s), y(\gamma_1(s)), \\ &\qquad \int_0^{\gamma_2(s)} K(s, r)\, y(B(r))\, dr, \\ &\qquad \int_0^b h(s, r)\, y(D(r))\, dr\Bigg)\Bigg] ds \Bigg| \\ &\le \eta_1 |y_n(\kappa) - y(\kappa)| + \eta_2 b N_1 |y_n(x) - y(x)| \\ &\quad + b N_2 |y_n(x) - y(x)| + b N_2 |y_n(\gamma_1(x)) - y(\gamma_1(x))| \\ &\quad + b N_2 N_3 |x_n(B(r)) - y(B(r))| \\ &\quad + b N_2 M_4 |y_n(D(r)) - y(D(r))| \\ &\le (\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4)) |y_{n,M}(x) - y(x)|. \end{aligned} \tag{34}$$

Then

$$\begin{aligned} |y_{n+1,M}(x) - y(x)| &\le (\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4))^n \\ &\quad \times |y_{1,M}(x) - y(x)|. \end{aligned} \tag{35}$$

Now, by assumption in theorem, we have $\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4) < 1$.

Because $\eta_1 + \eta_2 b N_1 + b N_2 (2 + N_3 + N_4) < \eta_1 + \eta_2 b N_1 + b((1 - (\eta_1 + \eta_2 b N_1))/b(2 + N_3 + N_4))(2 + N_3 + N_4) = 1$ and then $\lim_{n \to +\infty} y_{n+1,M}(x) = y(x)$, $0 \le x \le b$. □

5. Illustrative Examples

In this section, for showing efficiency of the iterative scheme and in order to show the facts of the exact solution, we give some examples below. All routines have been written in Mathematica 7 and a Dual-Core CPU 2.00 GHz is used to run the programs. Also, about efficiency and accuracy of the proposed numerical method, we present absolute errors for different examples. Therefore numerical results are shown in figures to illustrate the efficiency of this scheme.

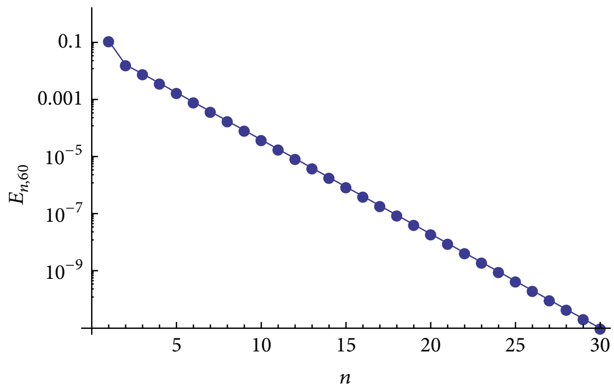

FIGURE 1: Absolute errors of Example 1. For different values of iteration n.

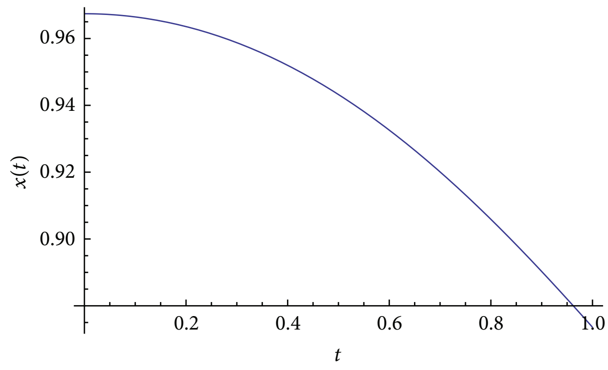

FIGURE 2: Approximate solution for Example 1.

Example 1. Consider the following problem:

$$\frac{dy}{dx} = \frac{1}{15}x^3\left[x - y(x)\right] - \frac{1}{5}xy\left(\gamma_1(x)\right) + \frac{1}{100}x^2y^2\left(\gamma_1(x)\right)$$
$$+ \frac{1}{500}x\left[x^3 - \int_0^{(1/2)x} 8xry\left(r^2\right)dr\right]^5$$
$$- \frac{1}{600}x^2\left[x^3 - \int_0^{(1/2)x} 8xry(r^2)dr\right]^6$$
$$+ \frac{1}{700}x^2\left[x^2 - \int_0^1 2x^2ry\left(r^2\right)dr\right]^7$$
$$- \frac{1}{800}x^3\left[x^2 - \int_0^1 2x^2ry(r^2)dr\right]^8 \equiv Gy(x),$$
$$x \in I = [0,1],$$
$$y(0) = \frac{1}{4}y(\kappa) + \frac{1}{4}\int_0^1 (2r + y(r))\,dr + \frac{1}{4}, \quad (36)$$

where $\kappa \in (0,1]$, $\gamma_1 \in C(I,I)$, and $\gamma_1(x) \le x$ on I.

Now, we want to show that assumptions (L_1)–(L_5) and convergence criterion which was prepared in Theorem 7 are satisfied for integral equation (4).

Noting that $\gamma_2(x) = (1/2)x$, $B(x) = D(x) = x^2$, $\forall x \in I$, then (L_1)–(L_5) is true. For details see [6].

In addition, we have $\beta(x) = 1$, $\alpha_1(x) = 0$, $\alpha_2(x) = (1/50)x^2$, $\alpha_3(x) = \alpha_4(x) = (1/100)x^{17}$, $K(x,r) = 8xr$, $h(x,r) = 2x^2r$, and $\eta_1 = \eta_2 = 1/4$; then $N_1 = 1$, $N_2 = 1/50$, $N_3 = 8$, and $N_4 = 2$, and then

$$N_2 = \frac{1}{50} \le \frac{1 - (\eta_1 + \eta_2 bN_1)}{b(2 + N_3 + N_4)} = \frac{1}{24}. \quad (37)$$

Therefore, the iterative method (31) converges to the exact solution of this equation. Now, based on Sinc quadrature scheme we can have some successive approximations for solution of this example for $\kappa = 1/2$. Absolute error for nth approximation and N points quadrature method is defined by

$$e_{n,M} = \max_{x\in[0,1]}\left\{\left|y_{n+1,M}(x) - y_{n,M}(x)\right|\right\}; \quad n = 1,2,3,\ldots. \quad (38)$$

For different values of n absolute errors are depicted in Figure 1 and approximate solution for $n = 30$ is depicted in Figure 2.

Example 2. In Example 1 of [19], authors considered the integrodifferential equation

$$\frac{dy}{dx} = 1 - \frac{1}{3}x^3 + \int_0^1 x^3y^2(z)\,dz,$$
$$y(0) = 0. \quad (39)$$

The exact solution is $y(x) = x$. Maximum absolute error for each iteration and different values of quadrature points are depicted in Figures 3 and 4. By comparing these results with the numerical results given in [19] in Table 2, efficiency and accuracy of current approach are guaranteed.

6. Conclusion

In this paper, we apply a numerical approach by Sinc function for reaching the estimated solution of integrodifferential equation with enteral boundary condition and with delay parameter. To reach this aim we change this problem to a functional enteral equation. The Sinc estimation has exponential convergence rate such as $O(-ce^{M^{1/2}})$ that this property is an advantage. so we applied it to solve our problem by using collocation method. Finally, some examples are solved by this numerical method to show the efficiency and accuracy of Sinc estimation. It is worthy to note that this method can

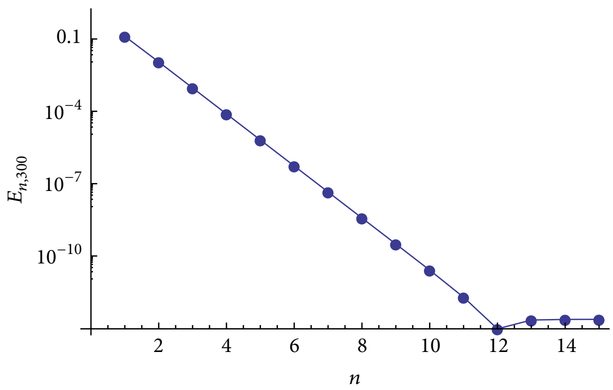

Figure 3: Maximum absolute error related to each iteration in Example 2.

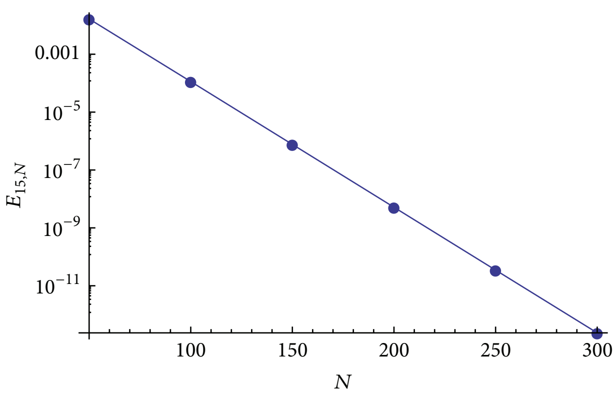

Figure 4: Maximum error related to different quadrature points in Example 2.

be used for solving integrodifferential equations with integral boundary conditions with deviating arguments arising in all sciences such as chemistry, physics, and other fields of applied mathematics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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