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Research Article

Extended Auxiliary Equation Method and Its Applications to Three Generalized NLS Equations

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The auxiliary equation method proposed by Sirendaoreji is extended to construct new types of elliptic function solutions of nonlinear evolution equations. The effectiveness of the extended method is demonstrated by applications to the RKL model, the generalized derivative NLS equation and the Kundu-Eckhaus equation. Not only are the Jacobian elliptic function solutions are derived, but also the solitary wave solutions and trigonometric function solutions are obtained in a unified way.

1. Introduction

Partial differential equations describe various nonlinear phenomena in natural and applied sciences such as fluid dynamics, plasma physics, solid state physics, optical fibers, acoustics, biology, and mathematical finance. It is of significant importance to solve nonlinear partial differential equations (NLPDEs) from both theoretical and practical points of view. In the past decades, many powerful methods for solving NLPDEs have been developed, such as the inverse scattering method [1], Bäcklund and Darboux transform [2, 3], Hirota's bilinear method [4], Painlevé analysis method [5], variable separation method [6–8], tanh-function method [9–11], variational iteration method [12], homotopy perturbation method [13], Jacobian elliptic function expansion [14–20], F-expansion method [21, 22], and Fan subequation method [23, 24] and its various extensions [25–29].

Sirendaoreji [30, 31] presented a new auxiliary equation method by introducing a new first-order nonlinear ordinary differential equation:

$$\left(\frac{d\phi}{d\xi}\right)^2 = c_2\phi^2(\xi) + c_4\phi^4(\xi) + c_6\phi^6(\xi), \quad \phi = \phi(\xi), \tag{1}$$

where c_j $(j = 2, 4, 6)$ are real parameters to be determined. The key idea of the auxiliary equation method is to use the solutions of (1) instead of $\tanh\xi$ in tanh-function method and extended tanh-function method [23]. Zhang and his coworker [32], Huang et al. [33], and Yomba [34] have improved the Sirendaoreji method in different manners.

The structure of this paper is organized as follows. In Section 2, an extended auxiliary equation method is described to construct exact solutions of NLPDEs. In Section 3, we apply this improved method to three generalized NLS equations with cubic-quintic terms. Some conclusions are given in Section 4.

2. An Extended Auxiliary Equation Method

For the sake of simplicity, we assume JacobiSN$(\xi, m) = \mathrm{sn}(\xi)$, JacobiCN$(\xi, m) = \mathrm{cn}(\xi)$, and JacobiDN$(\xi, m) = \mathrm{dn}(\xi)$. The derivatives of the above three kinds of Jacobian elliptic functions satisfy [35]:

$$\mathrm{sn}'(\xi) = \mathrm{cn}(\xi)\,\mathrm{dn}(\xi), \qquad \mathrm{cn}'(\xi) = -\mathrm{sn}(\xi)\,\mathrm{dn}(\xi), \qquad \mathrm{dn}'(\xi) = -m^2\mathrm{sn}(\xi)\,\mathrm{cn}(\xi), \tag{2}$$

where $' = d/d\xi$, and m is the modulus of Jacobian elliptic functions $(0 \le m \le 1)$.

For a given NLPDE, with independent variables (x, t) and dependent variable u.

Step 1. Use the travelling wave transformation $u(x, t) = u(\xi)$, $\xi = k\,(x - \omega t)$, and reduce the given NLPDE:

$$H(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0, \tag{3}$$

to the following ordinary differential equation

$$G\left(u, u', u'', \ldots\right) = 0, \tag{4}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}\xi$.

Step 2. The solutions of (4) can be supposed as

$$u(\xi) = F\left(\phi^{i}(\xi)\right), \tag{5}$$

where F is a suitable variable transformation, and $\phi(\xi)$ satisfies the following first-order differential equation:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = c_0 + c_2\phi^2(\xi) + c_4\phi^4(\xi) + c_6\phi^6(\xi), \tag{6}$$

which is slightly different from (1) by adding one constant term c_0. The crucial step is to give the solutions of (6). It is difficult to give the general solution of (6); here we only consider twelve solutions expressed by various kinds of Jacobian elliptic functions, which read

$$\phi(\xi) = \frac{1}{2}\left[-\frac{c_4}{c_6}\left(1 \pm f(\xi)\right)\right]^{1/2}, \tag{7}$$

where the function $f(\xi)$ could be expressed through elliptic functions sn, cn, dn, their inverse and different ratios like sn/cn, dn/sn and so on.

Type I. If $c_0 = c_4^3(m^2-1)/(32c_6^2m^2)$, $c_2 = c_4^2(5m^2-1)/(16c_6m^2)$, and $c_6 > 0$, $f(\xi)$ in (7) takes the form

$$f_1(\xi) = \mathrm{sn}(\rho\xi), \quad f_2(\xi) = \frac{1}{m\,\mathrm{sn}(\rho\xi)}, \quad \rho = \frac{c_4\sqrt{c_6}}{2mc_6}. \tag{8}$$

Type II. If $c_0 = c_4^3(1-m^2)/(32c_6^2)$, $c_2 = c_4^2(5-m^2)/(16c_6)$, and $c_6 > 0$, $f(\xi)$ in (7) takes the form

$$f_3(\xi) = m\,\mathrm{sn}(\rho\xi), \quad f_4(\xi) = \frac{1}{\mathrm{sn}(\rho\xi)}, \quad \rho = \frac{c_4\sqrt{c_6}}{2c_6}. \tag{9}$$

Type III. If $c_0 = c_4^3/(32m^2c_6^2)$, $c_2 = c_4^2(4m^2+1)/(16c_6m^2)$, and $c_6 < 0$, $f(\xi)$ in (7) takes the form

$$f_5(\xi) = \mathrm{cn}(\rho\xi), \qquad f_6(\xi) = \frac{\sqrt{1-m^2}\,\mathrm{sn}(\rho\xi)}{\mathrm{dn}(\rho\xi)}, \qquad \rho = \frac{c_4\sqrt{-c_6}}{2mc_6}. \tag{10}$$

Type IV. If $c_0 = c_4^3m^2/(32c_6^2(m^2-1))$, $c_2 = c_4^2(5m^2-4)/(16c_6(m^2-1))$, and $c_6 < 0$, $f(\xi)$ in (7) takes the form

$$f_7(\xi) = \frac{\sqrt{1-m^2}\,\mathrm{dn}(\rho\xi)}{1-m^2}, \qquad f_8(\xi) = \frac{1}{\mathrm{dn}(\rho\xi)}, \qquad \rho = \frac{c_4\sqrt{c_6(m^2-1)}}{2c_6(m^2-1)}. \tag{11}$$

Type V. If $c_0 = c_4^3/(32c_6^2(1-m^2))$, $c_2 = c_4^2(4m^2-5)/(16c_6(m^2-1))$, and $c_6 > 0$, $f(\xi)$ in (7) takes the form

$$f_9(\xi) = \frac{1}{\mathrm{cn}(\rho\xi)}, \qquad f_{10}(\xi) = \frac{\sqrt{1-m^2}\,\mathrm{dn}(\rho\xi)}{(1-m^2)\,\mathrm{sn}(\rho\xi)}, \qquad \rho = \frac{c_4\sqrt{c_6(1-m^2)}}{2c_6(1-m^2)}. \tag{12}$$

Type VI. If $c_0 = m^2c_4^3/(32c_6)$, $c_2 = c_4^2(m^2+4)/(16c_6)$, and $c_6 < 0$, $f(\xi)$ in (7) takes the form

$$f_{11}(\xi) = \mathrm{dn}(\rho\xi), \qquad f_{12} = \frac{\sqrt{1-m^2}}{\mathrm{dn}(\rho\xi)}, \qquad \rho = \frac{c_4\sqrt{-c_6}}{2c_6}. \tag{13}$$

Step 3. Substituting (7) together with (8)–(13) into (5), some new types of Jacobian elliptic function solutions of (3) can be obtained in a unified way.

With the aid of the computer algebraic software *Maple*, the solutions given by (7)–(13) have been verified by putting them back to the original equation (6). To our knowledge, these twelve solutions are firstly reported here. When the modulus m approaches 1 or 0, the Jacobian elliptic functions degenerate to hyperbolic functions and trigonometric functions, respectively.

3. Applications

In this section, three generalized NLS equations with physical interests are chosen to illustrate the effectiveness of the above method.

3.1. The RKL Model. Let us first consider the third-order generalized NLS equation [36], which is proposed by Radhakrishnan, Kundu, and Lakshmanan (RKL). The normalized RKL model can be written as

$$iu_z + u_{tt} + 2|u|^2u + i\alpha u_{ttt} + i\beta\left(|u|^2u\right)_t + i\gamma\left(|u|^4u\right)_t + \delta|u|^4u = 0, \tag{14}$$

which describes the propagation of femtosecond optical pulses. In (14), $u = u(z,t)$ represents a normalized complex slowly varying amplitude of the pulse envelope, and $\alpha, \beta, \gamma, \delta$ are real constants. Some solitary wave solutions and combined Jacobian elliptic function solution were constructed by different methods [37–39].

In order to solve (14), its solutions may be supposed as

$$u(z,t) = \phi(\xi)\,e^{i\eta}, \qquad \xi = kt + \omega z, \qquad \eta = \lambda t + \mu z, \tag{15}$$

where k, ω, λ, and μ are real constants. Substituting (15) into (14) and taking real and imaginary parts separately, we have

$$(3\alpha\lambda - 1) k^2\phi''(\xi) = \left(\alpha\lambda^3 - \lambda^2 - \mu\right)\phi(\xi) + (2-\lambda\beta)\phi^3(\xi) + (\delta - \lambda\gamma)\phi^5(\xi), \tag{16}$$

$$\alpha k^3\phi'''(\xi) + \left(\omega + 2\lambda k - 3\alpha\lambda^2 k\right)\phi'(\xi) + 3\beta k\phi^2(\xi)\phi'(\xi) + 5\gamma k\phi^4(\xi)\phi'(\xi) = 0. \tag{17}$$

There are two cases to discuss.

Case 1. When $3\alpha\lambda - 1 \neq 0$.

Integrating (17) and setting the integration constant to zero, we obtain

$$\alpha k^3\phi''(\xi) = -\left(\omega + 2\lambda k - 3\alpha\lambda^2 k\right)\phi(\xi) - \beta k\phi^3(\xi) - \gamma k\phi^5(\xi). \tag{18}$$

Equations (16) and (18) will be equivalent, provided that

$$\frac{(3\alpha\lambda - 1)k^2}{\alpha k^3} = \frac{\alpha\lambda^3 - \lambda^2 - \mu}{-(\omega + 2\lambda k - 3\alpha\lambda^2 k)} = \frac{2-\lambda\beta}{-\beta k} = \frac{\delta - \lambda\gamma}{-\gamma k}. \tag{19}$$

From which we get

$$\lambda = \frac{\beta - 2\alpha}{2\alpha\beta}, \qquad \gamma = \frac{\beta\delta}{2}, \qquad \omega = \frac{2k\alpha\left(4\beta - 8\alpha + \mu\beta^3\right)}{\beta^2(\beta - 6\alpha)}. \tag{20}$$

Under the parametric constraints (20), integrating (18) and setting the integration constant to zero, we have

$$\phi'^2(\xi) = 2A + c_2\phi^2(\xi) + c_4\phi^4(\xi) + c_6\phi(\xi)^6, \tag{21}$$

where A is an arbitrary integration constant, and c_2, c_4, c_6 are given by

$$c_2 = \frac{8\alpha^3 + 8\alpha^2\mu\beta^3 + \beta^3 - 2\alpha\beta^2 - 4\beta\alpha^2}{4k^2\alpha^2\beta^2(6\alpha - \beta)}, \qquad c_4 = -\frac{\beta}{2\alpha k^2}, \qquad c_6 = -\frac{\beta\delta}{6\alpha k^2}. \tag{22}$$

Up to now, we can obtain twelve Jacobi elliptic function solutions of (14), which can be expressed as the unified form:

$$u_j(z,t) = \frac{1}{2}\left[-\frac{c_4}{c_6}\left(1 \pm f_j(\xi)\right)\right]^{1/2} e^{i(\lambda t + \mu z)}, \qquad \xi = kt + \omega z, \quad j = 1,\ldots,12, \tag{23}$$

where λ, γ, ω are given by (19), c_2, c_4, and c_6 are given by (22), and $f_j(\xi)$ $(j = 1,\ldots,12)$ are given by (8)–(13).

Some solitary wave solutions can be obtained if the modulus m approaches 1. The solution $u_1(z,t)$ given by (23) degenerates to the kink-type solitary wave solution:

$$u_{13}(z,t) = \left\{-\frac{3}{4\delta}\left[1 \pm \tanh\left(\frac{\sqrt{-6\alpha\beta\delta}}{4k\alpha\delta}(kt + \omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t + \mu z)}, \tag{24}$$

where $\alpha\beta\delta < 0$, and the parameter μ is determined by $c_2 = -3\beta/(8\alpha\delta k^2)$.

When $m \to 1$, the solution $u_5(z,t)$ given by (23) degenerates to the bell-type solitary wave solution:

$$u_{14}(z,t) = \left\{-\frac{3}{4\delta}\left[1 \pm \operatorname{sech}\left(\frac{\sqrt{6\alpha\beta\delta}}{4k\alpha\delta}(kt + \omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t + \mu z)}, \tag{25}$$

where $\alpha\beta\delta > 0$, and μ is determined by $c_2 = -15\beta/(32\alpha\delta k^2)$.

When $m \to 1$, the solution $u_2(z,t)$ degenerates to

$$u_{15}(z,t) = \left\{-\frac{3}{4\delta}\left[1 \pm \coth\left(\frac{\sqrt{-6\alpha\beta\delta}}{4k\alpha\delta}(kt + \omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t + \mu z)}, \tag{26}$$

where $\alpha\beta\delta < 0$, and μ is determined by $c_2 = -3\beta/(8\alpha\delta k^2)$.

Some trigonometric function solutions can be obtained if the modulus $m \to 0$; for example, the solution $u_4(z,t)$ becomes

$$u_{16}(z,t) = \left\{-\frac{3}{4\delta}\left[1 \pm \csc\left(\frac{\sqrt{-6\alpha\beta\delta}}{4k\alpha\delta}(kt + \omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t + \mu z)}, \tag{27}$$

where $\alpha\beta\delta < 0$, and μ is determined by $c_2 = -15\beta/(32\alpha\delta k^2)$.

When $m \to 0$, the solution $u_9(z,t)$ becomes

$$u_{17}(z,t) = \left\{-\frac{3}{4\delta}\left[1 \pm \sec\left(\frac{\sqrt{-6\alpha\beta\delta}}{4k\alpha\delta}(kt + \omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t + \mu z)}, \tag{28}$$

where $\alpha\beta\delta < 0$, and μ is determined by $c_2 = -15\beta/(32\alpha\delta k^2)$.

Case 2. When $3\alpha\lambda - 1 = 0$, we have

$$\lambda = \frac{1}{3\alpha}; \tag{29}$$

then setting the coefficients of ϕ, ϕ^3, and ϕ^5 in (16) to zero, respectively, yields

$$\mu = -\frac{2}{27\alpha^2}, \qquad \beta = 6\alpha, \qquad \gamma = 3\alpha\delta. \tag{30}$$

Substituting (29) and (30) into (17) and integrating it with respect to ξ, we have

$$\phi'^2(\xi) = 2A + c_2\,\phi^2(\xi) + c_4\,\phi^4(\xi) + c_6\,\phi^6(\xi), \tag{31}$$

where A is an arbitrary integration constant, and c_2, c_4, c_6 are given by

$$c_2 = -\frac{3\alpha\omega + k}{3\alpha^2 k^3}, \qquad c_4 = -\frac{3}{k^2}, \qquad c_6 = -\frac{\delta}{k^2}. \tag{32}$$

Similar to Case 1, we can also get twelve elliptic function solutions of (14), which read

$$u_j(z,t) = \frac{1}{2}\left[-\frac{c_4}{c_6}\left(1 \pm f_j(\xi)\right)\right]^{1/2} e^{i(\lambda t+\mu z)}, \qquad \xi = kt + \omega z, \quad j = 18,\ldots,29, \tag{33}$$

where the parameters λ, μ, β, γ are given by (29)-(30), c_2, c_4, and c_6 are given by (32), and $f_j(\xi)$ $(j = 18,\ldots,29)$ are given by (8)–(13).

When the modulus $m \to 1$, the solution $u_{18}(z,t)$ given by (33) degenerates to the solitary wave solution:

$$u_{30}(z,t) = \frac{1}{2}\left\{-\frac{3}{\delta}\left[1 \pm \tanh\left(\frac{3\sqrt{-\delta}}{2k\delta}(kt+\omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t+\mu z)}, \tag{34}$$

where $\delta < 0$, and $\omega = k(27\alpha^2 - 4\delta)/(12\alpha\delta)$.

When $m \to 1$, the solution $u_{19}(z,t)$ degenerates to

$$u_{31}(z,t) = \frac{1}{2}\left\{-\frac{3}{\delta}\left[1 \pm \coth\left(\frac{3\sqrt{-\delta}}{2k\delta}(kt+\omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t+\mu z)}, \tag{35}$$

where $\delta < 0$, and $\omega = k(27\alpha^2 - 4\delta)/(12\alpha\delta)$.

When $m \to 1$, the solution $u_{20}(z,t)$ degenerates to the solitary wave solution:

$$u_{32}(z,t) = \frac{1}{2}\left\{-\frac{3}{\delta}\left[1 \pm \operatorname{sech}\left(\frac{3\sqrt{\delta}}{2k\delta}(kt+\omega z)\right)\right]\right\}^{1/2} \times e^{i(\lambda t+\mu z)}, \tag{36}$$

where $\delta > 0$, and $\omega = k(135\alpha^2 - 16\delta)/(48\alpha\delta)$.

All solutions given above have been checked with *Maple* by putting them back into the original equation (14). Among them, only the solutions $u_{13}(z,t)$, $u_{14}(z,t)$, $u_{30}(z,t)$, and $u_{32}(z,t)$ had been found in [38]. To the author's knowledge, the other solutions have not been found before.

3.2. The Generalized Derivative NLS Equation. Next we consider the generalized derivative NLS equation:

$$iu_t + b_1 u_{xx} + \left(b_3|u|^2 + b_5|u|^4\right)u - is_0 u_x - is_2\left(|u|^2 u\right)_x = 0, \tag{37}$$

where b_1, b_3, b_5, s_0, s_2 are real constants. Both pulse and fronts that existed in perturbed system (37) were studied by van Saarloos and Hohenberg [40]. Some solitary wave solutions of (37) were obtained by Huang et al. through a generalized auxiliary expansion method [33]. However, the Jacobian elliptic function solutions of (37) have not been reported in literature.

The solutions of (37) may be supposed as

$$u(x,t) = \phi(\xi)\, e^{i(\psi(\xi)-\omega t)}, \quad \xi = kx - \lambda t, \tag{38}$$

where k and λ are constants to be determined. Substituting (38) into (37) and then separating the real and imaginary parts yield the ordinary differential equations about $\phi(\xi)$ and $\psi(\xi)$:

$$2b_1k^2\phi'(\xi)\psi'(\xi) + b_1k^2\phi(\xi)\psi''(\xi) - 3ks_2\phi^2(\xi)\phi'(\xi) - (ks_0+\lambda)\phi'(\xi) = 0, \tag{39}$$

$$b_1k^2\phi''(\xi) + (\lambda + ks_0)\phi(\xi)\psi'(\xi) - b_1k^2\phi(\xi)\psi'^2(\xi) + ks_2\phi^3(\xi)\psi'(\xi) + \omega\phi(\xi) + b_3\phi^3(\xi) + b_5\phi^5(\xi) = 0. \tag{40}$$

Under the constraint

$$\psi'(\xi) = \frac{\lambda + ks_0}{2b_1k^2} + \frac{3s_2}{4kb_1}\phi^2(\xi), \tag{41}$$

equation (39) is satisfied identically, and (40) becomes

$$b_1k^2\phi''(\xi) + \frac{\lambda^2 + 2\lambda s_0 k + 4\omega k^2 b_1 + s_0^2k^2}{4k^2b_1}\phi(\xi) + \frac{\lambda s_2 + s_2s_0k + 2b_3kb_1}{2kb_1}\phi^3(\xi) + \frac{3s_2^2 + 16b_5b_1}{16b_1}\phi^5(\xi) = 0. \tag{42}$$

Multiplying (42) by $\phi'(\xi)$ and integrating it with respect to ξ, we have

$$\phi'^2(\xi) = 2A + c_2\phi^2(\xi) + c_4\phi^4(\xi) + c_6\phi^6(\xi), \tag{43}$$

where A is an arbitrary integration constant, and c_2, c_4, c_6 are given by

$$c_2 = -\frac{\lambda^2 + 2\lambda s_0 k + 4\omega k^2 b_1 + s_0^2 k^2}{4b_1^2 k^4},$$
$$c_4 = -\frac{\lambda s_2 + s_2 s_0 k + 2b_3 k b_1}{4b_1^2 k^3}, \quad (44)$$
$$c_6 = -\frac{3s_2^2 + 16 b_5 b_1}{48 b_1^2 k^2}.$$

Up to now, we can obtain abundant Jacobi elliptic function solutions of (37), which can be expressed as the unified form:

$$u_j(x,t) = \frac{1}{2}\left[-\frac{c_4}{c_6}\left(1 \pm f_j(\xi)\right)\right]^{1/2} e^{i(\psi(\xi)-\omega t)}, \quad \xi = kx - \lambda t, \quad j = 1, \ldots, 12, \quad (45)$$

where c_2, c_4, and c_6 are given by (44), $\psi(\xi)$ satisfies the constraint given by (41), and $f_j(\xi)$ are given by (8)–(13).

When $m \to 1$, the solution $u_1(x,t)$ given by (45) degenerates to the kink-type solitary wave solution:

$$u_{13}(x,t) = \phi_1(\xi)\, e^{i(\psi(\xi)-\omega t)},$$
$$\phi_1(\xi) = \frac{1}{2}\left\{-\frac{c_4}{c_6}\left[1 \pm \tanh\left(\frac{c_4\sqrt{c_6}}{2mc_6}\xi\right)\right]\right\}^{1/2}, \quad (46)$$
$$\psi'(\xi) = \frac{\lambda + k s_0}{2b_1 k^2} + \frac{3s_2}{4kb_1}\phi_1^2(\xi), \quad \xi = kx - \lambda t,$$

where ω is determined by $c_2 = c_4^2/(4c_6)$.

When $m \to 1$, the solution $u_5(x,t)$ degenerates to the bell-type solitary wave solution:

$$u_{14}(x,t) = \phi_2(\xi)\, e^{i(\psi(\xi)-\omega t)},$$
$$\phi_2(\xi) = \frac{1}{2}\left\{-\frac{c_4}{c_6}\left[1 \pm \operatorname{sech}\left(\frac{c_4\sqrt{-c_6}}{2c_6}\xi\right)\right]\right\}^{1/2}, \quad (47)$$
$$\psi'(\xi) = \frac{\lambda + k s_0}{2b_1 k^2} + \frac{3s_2}{4kb_1}\phi_2^2(\xi), \quad \xi = kx - \lambda t,$$

where ω is determined by $c_2 = 5c_4^2/(16c_6)$. To the authors' knowledge, these fourteen solutions of (37) are firstly reported here.

3.3. The Kundu-Eckhaus Equation. Finally we consider the Kundu-Eckhaus equation with important physical interests:

$$iu_t + u_{xx} + \beta|u|^2 u + 4\delta^2|u|^4 u + i\delta^2\left(|u|^2\right)_x u = 0, \quad (48)$$

which was derived by Kundu [41, 42] and Eckhaus [43–45] independently and then known as Kundu-Eckhaus equation. In (48), the parameters β, δ are real constants. To the author's knowledge, the integrable nonlinear equation (48) has not been investigated for possible exact solutions through elliptic function.

The solutions of (48) may be supposed as

$$u(x,t) = \phi(\xi)\, e^{i(\psi(\xi)-\omega t)}, \quad \xi = kx - \lambda t, \quad (49)$$

where k and λ are constants to be determined. Substituting (49) into (48) and then separating the real and imaginary parts yield the ordinary differential equations about $\phi(\xi)$ and $\psi(\xi)$:

$$-\lambda\phi'(\xi) + 2k^2\phi'(\xi)\psi'(\xi) + k^2\phi(\xi)\psi''(\xi) + 2k\delta^2\phi^2(\xi)\phi'(\xi) = 0, \quad (50)$$

$$\lambda\phi(\xi)\psi'(\xi) + \omega\phi(\xi) + k^2\phi''(\xi) - k^2\phi(\xi)\psi'^2(\xi) + \beta\phi^3(\xi) + 4\delta^2\phi^5(\xi) = 0. \quad (51)$$

Under the constraint

$$\psi'(\xi) = \frac{\lambda}{2k^2} - \frac{\delta^2}{2k}\phi^2(\xi), \quad (52)$$

equation (50) is satisfied identically, and then multiplying (51) by $\phi'(\xi)$ and integrating it with respect to ξ, we have

$$\phi'^2(\xi) = 2A + c_2\phi^2(\xi) + c_4\phi^4(\xi) + c_6\phi^6(\xi), \quad (53)$$

where A is an arbitrary integration constant, and c_2, c_4, c_6 are given by

$$c_2 = -\frac{\lambda^2 + 4\omega k^2}{4k^4}, \qquad c_4 = -\frac{\beta}{2k^2}, \qquad c_6 = \frac{\delta^4 - 16\delta^2}{12k^2}. \quad (54)$$

Up to now, we can obtain twelve Jacobi elliptic function solutions of (48), which can be expressed as the unified form:

$$u_j(x,t) = \frac{1}{2}\left[-\frac{c_4}{c_6}\left(1 \pm f_j(\xi)\right)\right]^{1/2} e^{i(\psi(\xi)-\omega t)}, \quad \xi = kx - \lambda t, \quad j = 1, \ldots, 12, \quad (55)$$

where c_2, c_4, and c_6 are given by (54), $\psi(\xi)$ satisfies the constraint (52), and $f_j(\xi)$ are given by (8)–(13). It is obvious that the solutions of (48) are similar as those of (37) in the form. However, the values of c_2, c_4, c_6 as well as the relation between $\psi(\xi)$ and $\phi(\xi)$ are different.

When the modulus $m \to 1$, we can obtain the solitary wave solutions from (55). If $m \to 0$, the trigonometric function solutions can be also constructed. For the sake of simplicity, these solutions are not listed here.

4. Conclusions

With the aid of symbolic computation software *Maple*, we present abundant families of new periodic wave solutions for the auxiliary equation (6). Based on these new periodic wave solutions, the auxiliary equation method proposed by Sirendaoreji has been improved and applied to three generalized NLS equations with cubic-quintic terms. As a result, a series of new travelling wave solutions have been obtained including not only Jacobian elliptic function solutions but also solitary wave solutions and trigonometric function solutions. These solutions may be important to explain some physical phenomena and find applications in the nonlinear pulse propagation through optical fibers.

The improved auxiliary equation method can be used for solving other nonlinear partial differential equations in mathematical physics, for example, the generalized Pochhammer-Chree equation [46]:

$$u_{tt} - u_{ttxx} - \left(a_1 u + a_3 u^3 + a_5 u^5\right)_{xx} = 0, \tag{56}$$

the (2 + 1)-dimensional cubic-quintic Ginzburg-Landau equation [47]:

$$\begin{aligned} &iu_z + \frac{1}{2}u_{xx} + \frac{1}{2}(\beta - i)\,u_{\tau\tau} + iu + (1 - ir_1)\,|u|^2 u \\ &\quad + ir_2|u|^4 u = 0, \end{aligned} \tag{57}$$

the generalized long-short wave resonance equation [48]:

$$\begin{aligned} iS_t + S_{xx} &= \alpha LS + \gamma|S|^2 S + \delta|S|^4 S, \\ L_t + \beta|S|_x^2 &= 0, \end{aligned} \tag{58}$$

and so on. How to construct other types of exact solutions for nonlinear models with cubic-quintic nonlinear terms? It is an interesting and significant topic, and this problem is still under investigation.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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