

Research Article

Multiple Periodic Solutions of an Impulsive Population System with Delays

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A nonautonomous plant-hare model with impulse is considered. By using the continuation theorem of coincidence degree theory, we present an easily verifiable sufficient condition on the existence of multiple periodic solutions. Though Gao et al. (2014) considered the periodic solutions of plant-hare model, such model with impulses and delay has not been studied in previous paper.

1. Introduction

In recent years, applications of theory differential equations in mathematical ecology have developed rapidly. Various mathematical models have been proposed in the study of population dynamics (see [1–13]). Recently, Gao et al. [5] considered a nonautonomous plant-hare dynamical system with a toxin-determined functional response given by

$$\begin{aligned} \dot{N}(t) &= r(t)N(t) \left[1 - \frac{N(t)}{K} \right] - C(N(t))P(t), \\ \dot{P}(t) &= B(t)C(N(t))P(t) - d(t)P(t), \\ C(N(t)) &= f(N(t)) \left[1 - \frac{f(N(t))}{4G} \right], \\ f(N(t)) &= \frac{e\delta N(t)}{1 + he\delta N(t)}, \end{aligned} \quad (1)$$

where $N(t)$ denotes the density of plant at time t , $P(t)$ denotes the herbivore biomass at time t , $r(t)$ is the plant intrinsic growth rate at time t , $d(t)$ is the per capita rate of herbivore death unrelated to plant toxicity at time t , and $B(t)$ is the conversion rate at time t . e is the encounter rate per unit plant, δ is the fraction of food items encountered that the herbivore ingests, K is the carrying capacity of plant, G measures the toxicity level, and h is the time for handling one unit of

plant. $r(t)$, $d(t)$, and $B(t)$ are continuously positive periodic functions with period ω and e , δ , K , G , and h are five positive real constants.

However, birth of many species is an annual birth pulse; for having more accurate description to the system, we need to consider incorporating the impulsive effect into the differential equations. For more biological view of impulses, one can refer to [14–17]. To describe to a system more accurately, we should consider the following impulsive system with delays:

$$\begin{aligned} \dot{N}(t) &= N(t) \left[r(t) \left(1 - \frac{N(t - \tau_1)}{K} \right) \right. \\ &\quad \left. - \frac{4Ge\delta P(t) + (4Gh - 1)e^2\delta^2 N(t)P(t)}{4G(1 + he\delta N(t))^2} \right], \\ \dot{P}(t) &= P(t) \\ &\quad \times \left[\frac{4Ge\delta B(t)N(t - \tau_2) + (4Gh - 1)e^2\delta^2 B(t)N^2(t)}{4G(1 + he\delta N(t))^2} \right. \\ &\quad \left. - d(t) \right], \quad t \neq t_k; \end{aligned}$$

$$\Delta N(t_k) = N(t_k^+) - N(t_k^-) = c_{1k}N(t_k),$$

$$\Delta P(t_k) = P(t_k^+) - P(t_k^-) = c_{2k}P(t_k), \quad t = t_k, \quad k = 1, 2, \dots, \tag{2}$$

where the assumptions on $r, d, B, e, \delta, K, G,$ and h are the same as before; $c_{jk} \in (-1, \infty)$ ($j = 1, 2, k \in \mathbb{N} = 1, 2, \dots$), $\{t_k\} \in \mathbb{N}$ is a strictly increasing sequence with $t_1 > 0$ and $\lim_{k \rightarrow \infty} t_k = \infty$. We further assume that there exists a $q \in \mathbb{N}$ such that $c_{j(k+q)} = c_{jk}$ ($j = 1, 2$) and $t_{k+q} = t_k + \omega$ for $k \in \mathbb{N}$.

Without loss of generality, we will assume $t_k \neq 0$ for $k = 1, 2, \dots$, and $[0, \omega] \cap \{t_k\} = \{t_1, t_2, \dots, t_m\}$; hence $q = m$. The main purpose of this paper is to derive easily verifiable sufficient conditions for the existence of multiple positive periodic solutions of (2).

2. Preliminaries

In this section, we cite some definitions and lemmas.

Let PC_ω denote the space of ω -periodic functions $\psi : \mathbb{R} \rightarrow \mathbb{R}$ which are continuous for $t \neq t_k$, are continuous from the left for $t \in \mathbb{R}$, and have possible discontinuities of the first kind at points $t = t_k$; that is, the limit from the right of t_k exists but may be different from the value at t_k . We also denote $PC_\omega^1 = \{\psi \in PC_\omega : \dot{\psi} \in PC_\omega\}$.

For the convenience, we list the following definitions and lemmas.

Definition 1 (see [17]). The set $\mathbb{F} \subset PC_\omega$ is said to be quasiequicontinuous in $[0, \omega]$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in \mathbb{F}, k \in \mathbb{Z}, \tau_1, \tau_2 \in (t_{k-1}, t_k) \cap [0, \omega]$, and $|\tau_1 - \tau_2| < \delta$, then

$$|x(\tau_1) - x(\tau_2)| < \epsilon. \tag{3}$$

Lemma 2 (see [17]). *The set $\mathbb{F} \subset PC_\omega$ is relatively compact if and only if*

- (1) \mathbb{F} is bounded, that is, $\|x\| \leq M$, for each $x \in \mathbb{F}$, and some $M > 0$;
- (2) \mathbb{F} is quasiequicontinuous in $[0, \omega]$.

Lemma 3 (see [17]). *Assume that $\psi \in PC_\omega^1$; then the following inequality holds:*

$$\sup_{s \in [0, \omega]} \psi(s) - \inf_{s \in [0, \omega]} \psi(s) \leq \frac{1}{2} \left[\int_0^\omega |\dot{\psi}(s)| ds + \sum_{k=1}^m |\Delta \psi(t_k)| \right]. \tag{4}$$

3. Existence of Multiple Positive Periodic Solutions

In this section, sufficient conditions are obtained for the existence of periodic solutions of (2).

In order to obtain the existence of positive periodic solutions of (2), for convenience, we will summarize in the following a few concepts and results from [8] that will be basic for this section.

Let X, Y be normed vector spaces, let $L : \text{Dom } L \subset X \rightarrow Y$ be a linear mapping, and let $N : X \rightarrow Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$ and $\text{Im } L$ is closed in Y . If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\text{Im } P = \text{Ker } L$ and $\text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$, it follows that $L|_{\text{dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of that map by K_p . If Ω is an open bounded subset of X , the mapping N will be called L -compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

Lemma 4 (see [8]). *Let $\Omega \subset X$ be an open and bounded set. Let L be a Fredholm mapping of index zero and let N be L -compact on $\bar{\Omega}$. Assume,*

- (a) for each $\lambda \in (0, 1)$ and $x \in \partial\Omega \cap \text{Dom } L, Lx \neq \lambda Nx$;
- (b) for each $x \in \partial\Omega \cap \text{Ker } L, QNx \neq 0$;
- (c) $\text{deg}\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom } L$.

Before starting our main result, for the lack of convenience, we denote

$$\begin{aligned} \bar{f} &= \frac{1}{\omega} \int_0^\omega f(t) dt \quad f \in PC_\omega, \\ c_j &= \sum_{k=1}^m \ln(1 + c_{jk}), \quad j = 1, 2, \\ C_1 &= \sum_{k=1}^m |\ln(1 + c_{1k})| + c_1, \\ C_2 &= \sum_{k=1}^m |\ln(1 + c_{2k})| - c_2. \end{aligned} \tag{5}$$

The following assumptions are valid throughout this paper:

- (A₁) $\bar{d} - (c_2/\omega) > 0,$
- (A₂) $1/4h < G < 1/3h,$
- (A₃) $4h(\bar{d} - (c_2/\omega)) \exp[(1/2)(2\bar{r}\omega + C_1)] < \bar{B} < 4Gh^2(\bar{d} - (c_2/\omega))/(4Gh - 1).$

For further convenience, we introduce six positive numbers as follows:

$$\begin{aligned}
 h_{\pm} &= \left(\left[e\delta\bar{B} \exp\left(-\frac{1}{2}(2\bar{r}\omega + C_1)\right) \right. \right. \\
 &\quad \left. \left. - 2he\delta\left(\bar{d} - \frac{c_2}{\omega}\right) \right] \pm \sqrt{\Delta_1} \right) \\
 &\quad \times \left(2\left(\bar{d} - \frac{c_2}{\omega}\right) h^2 e^2 \delta^2 \right)^{-1}, \\
 l_{\pm} &= \left(\left[4Gh^2 e\delta\bar{B} \exp\left(\frac{1}{2}(2\bar{r}\omega + C_1)\right) \right. \right. \\
 &\quad \left. \left. - 2he\delta\left[4Gh^2\left(\bar{d} - \frac{c_2}{\omega}\right) - (4Gh - 1)\bar{B}\right] \right] \pm \sqrt{\Delta_2} \right) \\
 &\quad \times \left(2h^2 e^2 \delta^2 \left[4Gh^2\left(\bar{d} - \frac{c_2}{\omega}\right) - (4Gh - 1)\bar{B}\right] \right)^{-1} \\
 u_{\pm} &= \frac{\left[4Ge\delta\bar{B} - 8Ghe\delta\left(\bar{d} - (c_2/\omega)\right)\right] \pm \sqrt{\Delta_3}}{2\left[4Gh^2 e^2 \delta^2\left(\bar{d} - (c_2/\omega)\right) - (4Gh - 1)e^2 \delta^2 \bar{B}\right]},
 \end{aligned} \tag{6}$$

where

$$\begin{aligned}
 \Delta_1 &= \left[e\delta\bar{B} \exp\left(-\frac{1}{2}(2\bar{r}\omega + C_1)\right) - 2he\delta\left(\bar{d} - \frac{c_2}{\omega}\right) \right]^2 \\
 &\quad - 4h^2 e^2 \delta^2 \left(\bar{d} - \frac{c_2}{\omega}\right)^2, \\
 \Delta_2 &= \left[4Gh^2 e\delta\bar{B} \exp\left(\frac{1}{2}(2\bar{r}\omega + C_1)\right) \right. \\
 &\quad \left. - 2he\delta\left[4Gh^2\left(\bar{d} - \frac{c_2}{\omega}\right) - (4Gh - 1)\bar{B}\right] \right]^2 \\
 &\quad - 4h^2 e^2 \delta^2 \left[4Gh^2\left(\bar{d} - \frac{c_2}{\omega}\right) - (4Gh - 1)\bar{B}\right]^2, \\
 \Delta_3 &= \left[4Ge\delta\bar{B} - 8Ghe\delta\left(\bar{d} - \frac{c_2}{\omega}\right)\right]^2 \\
 &\quad - 16G\left(\bar{d} - \frac{c_2}{\omega}\right)\left[4Gh^2 e^2 \delta^2\left(\bar{d} - \frac{c_2}{\omega}\right) \right. \\
 &\quad \left. - (4Gh - 1)e^2 \delta^2 \bar{B}\right].
 \end{aligned} \tag{7}$$

Under assumptions (A_1) , (A_2) , and (A_3) , it is not difficult to show that

$$l_- < u_- < h_- < h_+ < u_+ < l_+. \tag{8}$$

Theorem 5. *In addition to (A_1) , (A_2) , and (A_3) , suppose that*

$$(A_4) \quad \bar{r} + (c_1/\omega) - (\bar{r}/K) \exp(\ln l_+ + (1/2)(2\bar{r}\omega + C_1)) > 0.$$

Then system (2) has at least two positive ω -periodic solutions.

Proof. Making the change of variables,

$$N(t) = \exp(u_1(t)), \quad P(t) = \exp(u_2(t)). \tag{9}$$

Then, system (2) can be rewritten as

$$\begin{aligned}
 \dot{u}_1(t) &= r(t) - \frac{r(t)}{K} \exp(u_1(t - \tau_1)) \\
 &\quad - \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2 \delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2} \\
 &\equiv f_1(t), \\
 \dot{u}_2(t) &= -d(t) \\
 &\quad + \left((4Ge\delta B(t) \exp(u_1(t - \tau_2))) \right. \\
 &\quad \left. + (4Gh - 1)e^2 \delta^2 B(t) \exp(2u_1(t)) \right) \\
 &\quad \times (4G(1 + he\delta \exp(u_1(t)))^2)^{-1} \\
 &\equiv f_2(t), \quad t \neq t_k, \\
 \Delta u_1(t_k) &= \ln(1 + c_{1k}), \\
 \Delta u_2(t_k) &= \ln(1 + c_{2k}), \quad t = t_k.
 \end{aligned} \tag{10}$$

Take

$$\begin{aligned}
 X &= \{x = (u_1, u_2)^T : u_j \in PC_{\omega}, j = 1, 2, \\
 &\quad x(t + \omega) = x(t)\}, \quad Y = X \times \mathbb{R}^{2q}
 \end{aligned} \tag{11}$$

and define

$$\|x\|_0 = \sum_{j=1}^2 \sup_{t \in [0, \omega]} |u_j(t)|, \quad x = (u_1, u_2) \in X, \tag{12}$$

$$\|y\|_1 = \|x\|_0 + \sum_{j=1}^q \|\xi_j\|, \quad y = [x, \xi_1, \dots, \xi_q] \in Y.$$

Both $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_1)$ are Banach spaces.

Define

$$\begin{aligned}
 \text{Dom } L &= \{x \in X : \dot{x} \in X\}, \quad L : \text{Dom } L \rightarrow Y, \\
 L((u_1, u_2)^T) &= \left(\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}, \begin{pmatrix} \Delta u_1(t_k) \\ \Delta u_2(t_k) \end{pmatrix}_{k=1}^q \right);
 \end{aligned} \tag{13}$$

$N : X \rightarrow Y$,

$$N \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \left(\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \begin{pmatrix} \ln(1 + c_{1k}) \\ \ln(1 + c_{2k}) \end{pmatrix}_{k=1}^q \right); \tag{14}$$

$P : X \rightarrow X, P((u_1, u_2)^T) = (\bar{u}_1, \bar{u}_2)^T; Q : Y \rightarrow Y$,

$$\begin{aligned}
 Q \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \left\{ \begin{pmatrix} m_k \\ n_k \end{pmatrix} \right\}_{k=1}^q \right) \\
 = \left(\begin{pmatrix} \bar{u}_1 + \frac{1}{\omega} \sum_{k=1}^q m_k \\ \bar{u}_2 + \frac{1}{\omega} \sum_{k=1}^q n_k \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{k=1}^q \right).
 \end{aligned} \tag{15}$$

It is not difficult to show that

$$\text{Ker } L = \{x \mid x \in X, (u_1, u_2)^T \in X : (u_1(t), u_2(t))^T = (c_1, c_2)^T \in \mathbb{R}^2, t \in \mathbb{R}\};$$

$$\text{Im } L = \left\{ \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \left\{ \begin{pmatrix} m_k \\ n_k \end{pmatrix} \right\}_{k=1}^q \right] \in Y : \begin{matrix} \omega \bar{u}_1 + \sum_{k=1}^q m_k = 0 \\ \omega \bar{u}_2 + \sum_{k=1}^q n_k = 0 \end{matrix} \right\}. \tag{16}$$

Since $\text{Im } L$ is closed in Y and P and Q are continuous projectors such that

$$\begin{aligned} \text{Im } P &= \text{Ker } L, & \text{Ker } Q &= \text{Im } L = \text{Im } (I - Q), \\ \dim \text{Ker } L &= \text{codim } \text{Im } L = 2, \end{aligned} \tag{17}$$

it follows that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \text{Im } L \rightarrow \text{Dom } L \cap \text{Ker } P$ exists, which is given by

$$\begin{aligned} K_p \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \left\{ \begin{pmatrix} m_k \\ n_k \end{pmatrix} \right\}_{k=1}^q \right] \\ = \left(\begin{array}{l} \int_0^t u_1(s) ds + \sum_{0 < t_k < t} m_k - \frac{1}{\omega} \sum_{k=1}^q m_k - \overline{\int_0^t u_1(s) ds} \\ \int_0^t u_2(s) ds + \sum_{0 < t_k < t} n_k - \frac{1}{\omega} \sum_{k=1}^q n_k - \overline{\int_0^t u_2(s) ds} \end{array} \right). \end{aligned} \tag{18}$$

Then $QN : X \rightarrow Y$ and $K_p(I - Q)N : X \rightarrow X$ are defined by

$$QN \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \left(\begin{array}{l} \left(\frac{\bar{f}_1 + c_1}{\omega} \right), \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{k=1}^q \\ \left(\frac{\bar{f}_2 + c_2}{\omega} \right) \end{array} \right). \tag{19}$$

$$\begin{aligned} K_p(I - Q)N \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \left(\begin{array}{l} \int_0^t f_1(s) ds + \sum_{0 < t_k < t} \ln(1 + c_{1k}) \\ \int_0^t f_2(s) ds + \sum_{0 < t_k < t} \ln(1 + c_{2k}) \end{array} \right) \\ &+ \left(\frac{1}{2} - \frac{t}{\omega} \right) \left(\begin{array}{l} \int_0^\omega f_1(t) dt + c_1 \\ \int_0^\omega f_2(t) dt + c_2 \end{array} \right) \\ &- \frac{1}{\omega} \left(\begin{array}{l} \int_0^\omega \int_0^t f_1(s) ds dt + c_1 \\ \int_0^\omega \int_0^t f_2(s) ds dt + c_2 \end{array} \right). \end{aligned} \tag{20}$$

Clearly, QN and $K_p(I - Q)N$ are continuous. By using the Arzela-Ascoli theorem, it is not difficult to prove that $\overline{K_p(I - Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$.

Moreover, $QN(\bar{\Omega})$ is bounded. Therefore, N is L -compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now, we reach the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem.

Corresponding to the operator equation, $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} \dot{u}_1(t) &= \lambda \left[r(t) - \frac{r(t)}{K} \exp(u_1(t - \tau_1)) \right. \\ &\quad - \left((4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \right. \\ &\quad \quad \left. \left. \times \exp(u_1(t) + u_2(t)) \right) \right. \\ &\quad \left. \times (4G(1 + he\delta \exp(u_1(t)))^2)^{-1} \right], \end{aligned}$$

$$\begin{aligned} \dot{u}_2(t) &= \lambda \left[-d(t) + \left((4Ge\delta B(t) \exp(u_1(t - \tau_2)) \right. \right. \\ &\quad \left. \left. + (4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t)) \right) \right. \\ &\quad \left. \times (4G(1 + he\delta \exp(u_1(t)))^2)^{-1} \right], \\ &\quad \quad \quad t \neq t_k \end{aligned}$$

$$\Delta u_1(t_k) = \lambda \ln(1 + c_{1k}),$$

$$\Delta u_2(t_k) = \lambda \ln(1 + c_{2k}), \quad t = t_k. \tag{21}$$

Suppose that $x = (u_1(t), u_2(t))^T \in X$ is a solution of (21) for a certain $\lambda \in (0, 1)$. Integrating the first equation of (21) over the interval $[0, \omega]$, we obtain

$$\begin{aligned} &\int_0^\omega \frac{r(t)}{K} \exp(u_1(t - \tau_1)) dt \\ &+ \int_0^\omega \frac{4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \exp(u_1(t) + u_2(t))}{4G(1 + he\delta \exp(u_1(t)))^2} dt \\ &= \bar{r}\omega + c_1. \end{aligned} \tag{22}$$

Similarly, integrating the second equation of (21) over the interval $[0, \omega]$, we obtain

$$\begin{aligned} &\int_0^\omega (4Ge\delta B(t) \exp(u_1(t - \tau_2)) \\ &\quad + (4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t))) \\ &\quad \times (4G(1 + he\delta \exp(u_1(t)))^2)^{-1} dt = \bar{d}\omega - c_2. \end{aligned} \tag{23}$$

It follows from the first equation of (21) and (22) and (A_2) that

$$\begin{aligned} &\int_0^\omega |\dot{u}_1(t)| dt \\ &= \lambda \int_0^\omega \left| \left[r(t) - \frac{r(t)}{K} \exp(u_1(t - \tau_1)) \right. \right. \end{aligned}$$

$$\begin{aligned}
 & - \left((4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \right. \\
 & \quad \times \exp(u_1(t) + u_2(t))) \\
 & \quad \left. \times (4G(1 + he\delta \exp(u_1(t)))^2)^{-1} \right) \Big] dt \\
 & < \int_0^\omega r(t) dt + \int_0^\omega \frac{r(t)}{K} \exp(u_1(t - \tau_1)) dt \\
 & + \int_0^\omega (4Ge\delta \exp(u_2(t)) + (4Gh - 1)e^2\delta^2 \\
 & \quad \times \exp(u_1(t) + u_2(t))) \\
 & \quad \times (4G(1 + he\delta \exp(u_1(t)))^2)^{-1} dt \\
 & = \int_0^\omega r(t) dt + \bar{r}\omega + c_1 = 2\bar{r}\omega + c_1;
 \end{aligned} \tag{24}$$

that is,

$$\int_0^\omega |\dot{u}_1(t)| dt < 2\bar{r}\omega + c_1. \tag{25}$$

Similarly, it follows from the second equation of (21) and (23) and (A₂) that

$$\int_0^\omega |\dot{u}_2(t)| dt < 2\bar{d}\omega - c_2. \tag{26}$$

Since (u₁(t), u₂(t))^T ∈ X, there exists ξ_i, η_i ∈ [0, ω], such that

$$u_i(\xi_i) = \inf_{t \in [0, \omega]} u_i(t), \quad u_i(\eta_i) = \sup_{t \in [0, \omega]} u_i(t), \quad i = 1, 2. \tag{27}$$

From (A₂) and (23), we see that

$$\begin{aligned}
 \bar{d}\omega - c_2 & \leq \int_0^\omega \frac{4Ge\delta B(t) \exp(u_1(t - \tau_2))}{4G(1 + he\delta \exp(u_1(t)))^2} dt \\
 & + \int_0^\omega \frac{(4Gh - 1)e^2\delta^2 B(t) \exp(2u_1(t))}{4Gh^2e^2\delta^2 \exp(2u_1(t))} dt,
 \end{aligned} \tag{28}$$

which implies

$$\bar{d} - c_2 \leq \frac{e\delta\bar{B} \exp(u_1(\eta_1))}{(1 + he\delta \exp(u_1(\xi_1)))^2} + \frac{(4Gh - 1)\bar{B}}{4Gh^2}. \tag{29}$$

So

$$\begin{aligned}
 & u_1(\eta_1) \\
 & \geq \ln \frac{\left[4Gh^2 \left(\bar{d} - \frac{c_2}{\omega} \right) - (4Gh - 1)\bar{B} \right] (1 + he\delta \exp(u_1(\xi_1)))^2}{4Gh^2e\delta\bar{B}}.
 \end{aligned} \tag{30}$$

This, combined with Lemma 3, gives

$$\begin{aligned}
 & u_1(\xi_1) + \frac{1}{2}(2\bar{r}\omega + C_1) \\
 & > \ln \left(\left(\left[4Gh^2 \left(\bar{d} - \frac{c_2}{\omega} \right) - (4Gh - 1)\bar{B} \right] \right. \right. \\
 & \quad \left. \left. \times (1 + he\delta \exp(u_1(\xi_1)))^2 \right) \right. \\
 & \quad \left. \times (4Gh^2e\delta\bar{B})^{-1} \right)
 \end{aligned} \tag{31}$$

or equivalently

$$\begin{aligned}
 & \left[4Gh^2 \left(\bar{d} - \frac{c_2}{\omega} \right) - (4Gh - 1)\bar{B} \right] h^2e^2\delta^2 \exp(2u_1(\xi_1)) \\
 & - \left[4Gh^2e\delta\bar{B} \exp\left(\frac{1}{2}(2\bar{r}\omega + C_1)\right) \right. \\
 & \quad \left. - 2he\delta \left(4Gh^2 \left(\bar{d} - \frac{c_2}{\omega} \right) \right. \right. \\
 & \quad \left. \left. - (4Gh - 1)\bar{B} \right) \right] \exp(u_1(\xi_1)) \\
 & + \left[4Gh^2 \left(\bar{d} - \frac{c_2}{\omega} \right) - (4Gh - 1)\bar{B} \right] < 0.
 \end{aligned} \tag{32}$$

In view of (A₁) and (A₃) we have

$$\ln L_- < u_1(\xi_1) < \ln L_+. \tag{33}$$

Similarly, it follows from (A₂) and (23) that

$$\bar{d}\omega - c_2 \geq \int_0^\omega \frac{4Ge\delta B(t) \exp(u_1(t - \tau_2))}{4G(1 + he\delta \exp(u_1(t)))^2} dt, \tag{34}$$

which implies

$$\bar{d} - \frac{c_2}{\omega} \geq \frac{e\delta\bar{B} \exp(u_1(\xi_1))}{(1 + he\delta \exp(u_1(\eta_1)))^2}. \tag{35}$$

So

$$u_1(\xi_1) \leq \ln \frac{\left(\bar{d} - (c_2/\omega) \right) (1 + he\delta \exp(u_1(\eta_1)))^2}{e\delta\bar{B}}. \tag{36}$$

This, combined with Lemma 3, gives

$$\begin{aligned}
 & u_1(\eta_1) - \frac{1}{2}(2\bar{r}\omega + C_1) \\
 & < \ln \frac{\left(\bar{d} - (c_2/\omega) \right) (1 + he\delta \exp(u_1(\eta_1)))^2}{e\delta\bar{B}}
 \end{aligned} \tag{37}$$

or equivalently

$$\begin{aligned}
 & \left(\bar{d} - \frac{c_2}{\omega} \right) h^2e^2\delta^2 \exp(2u_1(\eta_1)) \\
 & - \left[e\delta\bar{B} \exp\left(-\frac{1}{2}(2\bar{r}\omega + C_1)\right) - 2he\delta \left(\bar{d} - \frac{c_2}{\omega} \right) \right] \\
 & \times \exp(u_1(\eta_1)) + \left(\bar{d} - \frac{c_2}{\omega} \right) > 0.
 \end{aligned} \tag{38}$$

It follows from (A_1) and (A_3) that

$$u_1(\eta_1) < \ln h_- \text{ or } u_1(\eta_1) > \ln h_+. \tag{39}$$

It follows from (25), (33), and Lemma 3 that

$$u_1(\eta_1) \leq \ln l_+ + \frac{1}{2}(2\bar{r}\omega + C_1) \triangleq H_{11}. \tag{40}$$

On the other hand, it follows from (A_2) and (22) and (40) that

$$\bar{r}\omega + c_1 \geq \int_0^\omega \frac{4Ge\delta \exp(u_2(\xi_2))}{4G(1 + he\delta \exp(H_{11}))^2} dt, \tag{41}$$

$$\begin{aligned} \bar{r}\omega + c_1 &\leq \int_0^\omega \frac{r(t)}{K} \exp(H_{11}) dt \\ &+ \int_0^\omega e\delta \exp(u_2(\eta_2)) dt \\ &+ \int_0^\omega \frac{e\delta \exp(u_2(\eta_2))}{2} dt. \end{aligned} \tag{42}$$

It follows from (41) that

$$u_2(\xi_2) \leq \ln \frac{(\bar{r} + (c_1/\omega))(1 + he\delta \exp(H_{11}))^2}{e\delta}. \tag{43}$$

This, combined with Lemma 3, gives

$$\begin{aligned} u_2(\eta_2) &\leq \ln \frac{(\bar{r} + (c_1/\omega))(1 + he\delta \exp(H_{11}))^2}{e\delta} \\ &+ \frac{1}{2}(2\bar{d}\omega + C_2) \triangleq H_{21}. \end{aligned} \tag{44}$$

Moreover, because of (A_4) , it follows from (42) that

$$u_2(\eta_2) \geq \ln \frac{2[\bar{r} + (c_1/\omega) - (\bar{r}/K) \exp(H_{11})]}{3e\delta}. \tag{45}$$

This, combined with Lemma 3 again, gives

$$\begin{aligned} u_2(\xi_2) &\geq \ln \frac{2[\bar{r} + (c_1/\omega) - (\bar{r}/K) \exp(H_{11})]}{3e\delta} \\ &- \frac{1}{2}(2\bar{d}\omega + C_2) \triangleq H_{22}. \end{aligned} \tag{46}$$

It follows from (44) and (46) that

$$\sup_{t \in [0, \omega]} |u_2(t)| < \max\{|H_{21}|, |H_{22}|\} \triangleq H_2. \tag{47}$$

Obviously, $\ln l_\pm, \ln h_\pm, H_{11}$, and H_2 are independent of λ .

Now, let us consider QNx with $X = (u_1, u_2)^T \in \mathbb{R}^2$. Note that

$$\begin{aligned} QN(u_1, u_2) &= \left[\bar{r} - \frac{\bar{r}}{K} \exp(u_1) \right. \\ &- \frac{4Ge\delta \exp(u_2) + (4Gh - 1)e^2\delta^2 \exp(u_1 + u_2)}{4G(1 + he\delta \exp(u_1))^2} + \frac{c_1}{\omega}, \\ &- \bar{d} + \frac{4Ge\delta \bar{B} \exp(u_1) + (4Gh - 1)e^2\delta^2 \bar{B} \exp(2u_1)}{4G(1 + he\delta \exp(u_1))^2} \\ &\left. + \frac{c_2}{\omega} \right]. \end{aligned} \tag{48}$$

Because of $(A_1), (A_2), (A_3)$, and (A_4) , we can show that the equation $QN(u_1, u_2) = 0$ has two distinct solutions:

$$\begin{aligned} \tilde{u} &= \left(\ln u_-, \ln \frac{4G(\bar{r} - (\bar{r}/K)u_-)(1 + he\delta u_-)^2}{4Ge\delta + (4Gh - 1)e^2\delta^2 u_-} \right), \\ \hat{u} &= \left(\ln u_+, \ln \frac{4G(\bar{r} - (\bar{r}/K)u_+)(1 + he\delta u_+)^2}{4Ge\delta + (4Gh - 1)e^2\delta^2 u_+} \right). \end{aligned} \tag{49}$$

Choose $C > 0$ such that

$$C > \max \left\{ \left| \ln \frac{4G(\bar{r} - (\bar{r}/K)u_-)(1 + he\delta u_-)^2}{4Ge\delta + (4Gh - 1)e^2\delta^2 u_-} \right|, \left| \ln \frac{4G(\bar{r} - (\bar{r}/K)u_+)(1 + he\delta u_+)^2}{4Ge\delta + (4Gh - 1)e^2\delta^2 u_+} \right| \right\}. \tag{50}$$

Let

$$\begin{aligned} \Omega_1 &= \left\{ x = (u_1, u_2)^T \in X \mid \begin{array}{l} u_1(t) \in (\ln l_-, \ln h_-) \\ \sup_{t \in [0, \omega]} |u_2(t)| < H_2 + C \end{array} \right\}, \\ \Omega_2 &= \left\{ x = (u_1, u_2)^T \in X \mid \begin{array}{l} \inf_{t \in [0, \omega]} u_1(t) \in (\ln l_+, \ln l_+) \\ \sup_{t \in [0, \omega]} u_1(t) \in (\ln h_+, H_{11}) \\ \sup_{t \in [0, \omega]} |u_2(t)| < H_2 + C \end{array} \right\}. \end{aligned} \tag{51}$$

□

Then both Ω_1 and Ω_2 are bounded open subsets of X . It follows from (2) and (50) that $\tilde{u} \in \Omega_1$ and $\hat{u} \in \Omega_2$. With the help of (2), (33), (39), (40), and (47)–(50), it is easy to see that $\Omega_1 \cap \Omega_2 = \emptyset$ and Ω_i satisfies the requirement (a) in Lemma 4 for $i = 1, 2$. Moreover, $QNx \neq 0$ for $x \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap \mathbb{R}^2$. A direct computation gives $\text{deg}\{JQN, \Omega_i \cap \text{Ker } L, 0\} \neq 0$. Here, J is taken as the identity mapping since $\text{Im } Q = \text{Ker } L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 4. Hence, Equation (2) has at least two ω -periodic solutions. This completes the proof of Theorem 5.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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