Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 518238, 11 pages
http://dx.doi.org/10.1155/2014/518238

Research Article

Existence and Monotone Iteration of Positive Pseudosymmetric Solutions for a Third-Order Four-Point BVP

Xuezhe Lv, Libo Wang, and Minghe Pei

Department of Mathematics, Beihua University, Jilin 132013, China

Correspondence should be addressed to Minghe Pei; peiminghe@163.com

Received 19 January 2014; Revised 6 March 2014; Accepted 13 March 2014; Published 8 April 2014

Academic Editor: Bashir Ahmad

We study the existence and monotone iteration of solutions for a third-order four-point boundary value problem. An existence result of positive, concave, and pseudosymmetric solutions and its monotone iterative scheme are established by using the monotone iterative technique. Meanwhile, as an application of our results, an example is given.

1. Introduction

The third-order equations arise in many areas of applied mathematics and physics, such as the deflection of a curved beam having a constant or varying cross section, three-layer beam, electromagnetic waves, or gravity-driven flows [1], and thus have been studied extensively in the literature; see [1–29] and references therein. Recently, wide attention has been paid to the third-order boundary value problems with nonlocal boundary conditions; see [4, 6–9, 11, 12, 15, 16, 20, 23–30] and references therein.

In 2006, using the monotone iterative technique, Zhou and Ma [30] obtained the existence of positive solutions and established a corresponding iterative scheme for the following third-order p-Laplacian problem:

$$\begin{gathered}\left(\phi_p\left(u''(t)\right)\right)' = q(t) f(t, u(t)), \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \\ u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i).\end{gathered} \tag{1}$$

In 2009, Sun et al. [23] studied the existence of positive solutions for the following third-order p-Laplacian problem:

$$\begin{gathered}\left(\phi_p\left(u''(t)\right)\right)' = q(t) f\left(t, u(t), u'(t), u''(t)\right), \quad t \in (0,1), \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \\ u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i).\end{gathered} \tag{2}$$

By applying a monotone iterative method, the authors obtained the existence of positive solutions for the problem and established iterative schemes for approximating the solutions.

In 2011, Zhang [29] considered the following singular third-order three-point boundary value problem:

$$\begin{gathered}u'''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \qquad u'(1) = \alpha u'(\eta).\end{gathered} \tag{3}$$

The existence and uniqueness of solutions and corresponding iterative scheme to the problem are obtained by applying the cone theory and the Banach contraction mapping principle.

In 2013, Li et al. [7] studied third-order four-point boundary value problem with p-Laplacian of the form

$$\left(\phi_p\left(u''(t)\right)\right)' + q(t) f\left(t, u(t), u'(t)\right) = 0, \quad t \in (0,1),$$
$$u(0) = 0, \quad u(1) = u(\eta), \quad u''\left(\frac{1+\eta}{2}\right) = 0. \tag{4}$$

By using the monotone iterative technique, the existence result of positive pseudosymmetric solutions and its monotone iterative scheme are established for the problem.

Motivated by above works and [31], in this paper, we consider the existence and monotone iteration of positive pseudosymmetric solutions of the following third-order four-point boundary value problem:

$$u'''(t) + q(t) f\left(t, u(t), u'(t), u''(t)\right) = 0, \quad t \in (0,1), \tag{5}$$

subject to boundary conditions

$$\alpha u(0) - \beta u'(0) = 0, \qquad u'(\eta) + u'(1) = 0,$$
$$u''\left(\frac{1+\eta}{2}\right) = 0, \tag{6}$$

where $\eta \in (0,1)$ and $\alpha > 0$, $\beta \geq 0$. Here we say $u^*(t)$ is positive solution of BVP (5), (6), if $u^*(t)$ is the solution of BVP (5), (6) and satisfies $u^*(t) > 0$ for $t \in (0,1]$.

To the best of our knowledge, the pseudosymmetric solutions for the second-order boundary value problem have been studied by some authors, see [31–33]. And [7] is the only one concerned with the third-order boundary value problem. We note that the nonlinearity of f in our problem contains explicitly t and every derivatives of u up to order two.

This work is organized as follows. In Section 2, some notations and preliminaries are introduced. The main results are discussed in Section 3. As an application of our results, an example is given in the last section.

2. Preliminary

In this section, we give a definition and some lemmas which help to simplify the presentation of our main result.

Definition 1 (see [33]). Let $u \in C[0,1]$, $\eta \in (0,1)$. We say u is pseudosymmetric about η on $[0,1]$, if u is symmetric on $[\eta,1]$; that is,

$$u(t) = u(1+\eta-t), \quad \forall t \in [\eta, 1]. \tag{7}$$

Let the Banach space $E = C^2[0,1]$ be endowed with the norm

$$\|u\| = \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty, \left\|u''\right\|_\infty\right\}, \tag{8}$$

where $\|u^{(i)}\|_\infty = \max_{0\leq t\leq 1}|u^{(i)}(t)|$, $i = 0,1,2$. Define a cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \text{ is nonnegative, concave, and}$$
$$\text{pseudosymmetric about } \eta \text{ on } [0,1]\}, \tag{9}$$

and by "$\leq$" denote the induced partial ordering via cone P; that is, for u_1, $u_2 \in E$, $u_1 \leq u_2$ if and only if $u_2 - u_1 \in P$.

For convenience, we denote the following.

(H_0) $q(t)$ is a nonnegative continuous function defined on $(0,1)$, $q(t) \not\equiv 0$ on any subinterval of $(0,1)$. In addition, $\int_0^1 q(t)dt < +\infty$ and

$$q(t) = q(1+\eta-t) \quad \text{on } (\eta, 1). \tag{10}$$

(H_1) $f(t,u,v,w) : [0,1]\times[0,+\infty)\times\mathbb{R}\times(-\infty,0] \to \mathbb{R}$ is continuous,

$$f(t,u,v,w) \leq 0 \quad \text{on } \left[0, \frac{1+\eta}{2}\right] \times [0,+\infty)^2 \times (-\infty, 0], \tag{11}$$

and, for all $(t,u,v,w) \in [\eta,1]\times[0,+\infty)\times\mathbb{R}\times(-\infty,0]$,

$$f(t,u,v,w) = -f(1+\eta-t, u, -v, w). \tag{12}$$

(H_2) $f(t,u,v,w)$ is nonincreasing in u, nonincreasing in v, and nondecreasing in w on $[0,(1+\eta)/2] \times [0,+\infty)^2 \times (-\infty,0]$.

(H_3) $f(t,0,0,0) \not\equiv 0$ on $[0,1]$.

Now, we define an operator $T : P \to C^2\,[0,1]$ as follows: for $u \in P$,

$$(Tu)(t) = \begin{cases} \displaystyle\int_0^t \int_\tau^{(1+\eta)/2} \times \int_{(1+\eta)/2}^r q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ \displaystyle\quad + \frac{\beta}{\alpha} \int_0^{(1+\eta)/2} \times \int_{(1+\eta)/2}^r q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr, \qquad 0 \le t \le \frac{1+\eta}{2}, \\ \displaystyle\int_t^1 \int_{(1+\eta)/2}^\tau \times \int_{(1+\eta)/2}^r q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ \displaystyle\quad + \int_0^\eta \int_\tau^{(1+\eta)/2} \times \int_{(1+\eta)/2}^r q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ \displaystyle\quad + \frac{\beta}{\alpha} \int_0^{(1+\eta)/2} \times \int_{(1+\eta)/2}^r q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr, \qquad \frac{1+\eta}{2} \le t \le 1. \end{cases} \tag{13}$$

Obviously, under assumptions (H_0) and (H_1), the operator T is well defined.

Lemma 2. *Assume that (H_0) and (H_1) hold. Then $u \in P$ is a solution of BVP (5), (6) if and only if $u \in P$ is a fixed point of T.*

Proof. At first we show the necessity. Suppose $u \in P$ is a solution of BVP (5), (6). Then, integrating (5) and using (6) we infer that

$$u''(t) + \int_{(1+\eta)/2}^t q(r) f\left(r, u(r), u'(r), u''(r)\right) dr = 0, \qquad t \in [0,1], \tag{14}$$

$$\alpha u(0) - \beta u'(0) = 0, \qquad u'(\eta) + u'(1) = 0. \tag{15}$$

For $t \in [0, (1+\eta)/2]$, integrating (14) on $[t, (1+\eta)/2]$ and taking into account $u'((1+\eta)/2) = 0$, we get

$$u'(t) = \int_t^{(1+\eta)/2} \int_{(1+\eta)/2}^r q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr, \qquad t \in \left[0, \frac{1+\eta}{2}\right]. \tag{16}$$

Again integrating (16) on $[0,t] \subset [0, (1+\eta)/2]$ one can obtain

$$u(t) = \int_0^t \int_\tau^{(1+\eta)/2} \times \int_{(1+\eta)/2}^r q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau + u(0). \tag{17}$$

But, from (15) and (16), it follows that

$$u(0) = \frac{\beta}{\alpha} \int_0^{(1+\eta)/2} \int_{(1+\eta)/2}^r q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr. \tag{18}$$

Therefore, for $t \in [0, (1+\eta)/2]$, one has

$$u(t) = \int_0^t \int_\tau^{(1+\eta)/2} \times \int_{(1+\eta)/2}^r q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau + \frac{\beta}{\alpha} \int_0^{(1+\eta)/2} \int_{(1+\eta)/2}^r q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr. \tag{19}$$

For $t \in [(1+\eta)/2, 1]$, integrating (14) on $[(1+\eta)/2, t]$, we get

$$u'(t) = -\int_{(1+\eta)/2}^{t} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr, \quad t \in \left[\frac{1+\eta}{2}, 1\right]. \tag{20}$$

Again integrating (20) on $[t, 1] \subset [(1+\eta)/2, 1]$ one obtains

$$u(t) = \int_{t}^{1} \int_{(1+\eta)/2}^{\tau} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau + u(1), \quad t \in \left[\frac{1+\eta}{2}, 1\right]. \tag{21}$$

In particular, we have

$$u\left(\frac{1+\eta}{2}\right) = \int_{(1+\eta)/2}^{1} \int_{(1+\eta)/2}^{\tau} \times \int_{(1+\eta)/2}^{r} q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau + u(1). \tag{22}$$

In (19), we take $t = (1+\eta)/2$, and then

$$u\left(\frac{1+\eta}{2}\right) = \int_{0}^{(1+\eta)/2} \int_{\tau}^{(1+\eta)/2} \times \int_{(1+\eta)/2}^{r} q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau + \frac{\beta}{\alpha} \int_{0}^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr. \tag{23}$$

From (22) and (23) one has

$$u(1) = \int_{0}^{1} \int_{\tau}^{(1+\eta)/2} \times \int_{(1+\eta)/2}^{r} q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau + \frac{\beta}{\alpha} \int_{0}^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr. \tag{24}$$

Hence from (21) it follows that

$$\begin{aligned} u(t) &= \int_{t}^{1} \int_{(1+\eta)/2}^{\tau} \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &\quad + \int_{0}^{1} \int_{\tau}^{(1+\eta)/2} \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &\quad + \frac{\beta}{\alpha} \int_{0}^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr \\ &= \int_{t}^{1} \int_{(1+\eta)/2}^{\tau} \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &\quad + \left(\int_{0}^{\eta} + \int_{\eta}^{(1+\eta)/2} + \int_{(1+\eta)/2}^{1}\right) \times \int_{\tau}^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &\quad + \frac{\beta}{\alpha} \int_{0}^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr, \quad t \in \left[\frac{1+\eta}{2}, 1\right]. \end{aligned} \tag{25}$$

Notice that from (H_0), (H_1), and the fact that $u \in P$, we have

$$\left(\int_{\eta}^{(1+\eta)/2} + \int_{(1+\eta)/2}^{1}\right) \times \int_{\tau}^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau = 0. \tag{26}$$

Therefore

$$\begin{aligned} u(t) &= \int_t^1 \int_{(1+\eta)/2}^{\tau} \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &+ \int_0^{\eta} \int_{\tau}^{(1+\eta)/2} \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &+ \frac{\beta}{\alpha} \int_0^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr, \\ & \qquad t \in \left[\frac{1+\eta}{2}, 1\right]. \end{aligned} \tag{27}$$

This together with (19) implies that $u(t)$ is fixed point of T.

The sufficiency, by direct computation and using the fact that $u \in P$, follows immediately. □

The following lemmas are some properties of the operator T.

Lemma 3. *Assume that (H_0) and (H_1) hold. Then $TP \subset P$.*

Proof. From the definition of T, it is easy to check that Tu is nonnegative on $[0, 1]$ and satisfies (6) for all $u \in P$. Furthermore, since

$$(Tu)''(t) = -\int_{(1+\eta)/2}^{t} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds, \quad t \in [0, 1], \tag{28}$$

it follows that Tu is concave on $[0, 1]$.

Next we prove that Tu is pseudosymmetric about η on $[0, 1]$. In fact, if $t \in [\eta, (1+\eta)/2]$, then $1+\eta-t \in [(1+\eta)/2, 1]$; it follows that

$$\begin{aligned} (Tu)(1+\eta-t) &= \int_{1+\eta-t}^{1} \int_{(1+\eta)/2}^{\tau} \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &+ \int_0^{\eta} \int_{\tau}^{(1+\eta)/2} \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\ &+ \frac{\beta}{\alpha} \int_0^{(1+\eta)/2} \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr. \end{aligned} \tag{29}$$

Note that u is pseudosymmetric about η on $[0, 1]$; that is, $u(t) = u(1+\eta-t)$ for $t \in [\eta, 1]$, and then

$$u'(t) = -u'(1+\eta-t), \qquad u''(t) = u''(1+\eta-t), \quad t \in [\eta, 1]. \tag{30}$$

Thus, for all $r \in [\eta, 1]$, from (H_0) and (H_1), we have

$$\begin{aligned} \int_{(1+\eta)/2}^{r} & q(s) f\left(s, u(s), u'(s), u''(s)\right) ds \\ &= -\int_{(1+\eta)/2}^{r} q(1+\eta-s) \times f\left(1+\eta-s, u(s), -u'(s), u''(s)\right) ds \\ &= -\int_{(1+\eta)/2}^{r} q(1+\eta-s) \times f\left(1+\eta-s, u(1+\eta-s), u'(1+\eta-s), u''(1+\eta-s)\right) ds \\ &= \int_{(1+\eta)/2}^{1+\eta-r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds. \end{aligned} \tag{31}$$

Hence, for $t \in [\eta, (1+\eta)/2]$,

$$\begin{aligned}
&\int_{1+\eta-t}^{1}\int_{(1+\eta)/2}^{\tau} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) \\
&\qquad \times f\left(s, u(s), u'(s), u''(s)\right) ds\,dr\,d\tau \\
&= -\int_{1+\eta-t}^{1}\int_{(1+\eta)/2}^{1+\eta-\tau} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) \\
&\qquad \times f\left(s, u(s), u'(s), u''(s)\right) ds\,dr\,d\tau \\
&= -\int_{t}^{\eta}\int_{(1+\eta)/2}^{\overline{\tau}} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) \\
&\qquad \times f\left(s, u(s), u'(s), u''(s)\right) ds\,dr\,d(-\overline{\tau}) \\
&= \int_{t}^{\eta}\int_{(1+\eta)/2}^{\tau} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) \\
&\qquad \times f\left(s, u(s), u'(s), u''(s)\right) ds\,dr\,d\tau.
\end{aligned} \tag{32}$$

From (29) and (32), it follows that

$$\begin{aligned}
&(Tu)(1+\eta-t) \\
&= \int_{t}^{\eta}\int_{(1+\eta)/2}^{\tau} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\,dr\,d\tau \\
&\quad + \int_{0}^{\eta}\int_{\tau}^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\,dr\,d\tau \\
&\quad + \frac{\beta}{\alpha}\int_{0}^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\,dr \\
&= \int_{0}^{t}\int_{\tau}^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\,dr\,d\tau \\
&\quad + \frac{\beta}{\alpha}\int_{0}^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\,dr \\
&= (Tu)(t), \quad t \in \left[\eta, \frac{1+\eta}{2}\right].
\end{aligned} \tag{33}$$

If $t \in [(1+\eta)/2, 1]$, then $1+\eta-t \in [\eta, (1+\eta)/2]$. From (33), we have

$$(Tu)(1+\eta-t) = (Tu)(1+\eta-(1+\eta-t)) = (Tu)(t), \quad t \in \left[\frac{1+\eta}{2}, 1\right]. \tag{34}$$

This together with (33) implies

$$(Tu)(t) = (Tu)(1+\eta-t), \quad t \in [\eta, 1]. \tag{35}$$

In summary, $Tu \in P$, and then $TP \subset P$. □

The following lemma can be easily verified by a standard argument.

Lemma 4. *Assume that* (H_0) *and* (H_1) *hold. Then* $T : P \to P$ *is completely continuous.*

Lemma 5. *Assume that* (H_0), (H_1), *and* (H_2) *hold. Then* T *is nondecreasing on* P*; that is,* $Tu_1 \leq Tu_2$ *for* $u_1, u_2 \in P$ *with* $u_1 \leq u_2$.

Proof. Let $u_1, u_2 \in P$ with $u_1 \leq u_2$. Then $u_2 - u_1 \in P$. By the definition of P, $u_2(t) - u_1(t)$ is nonnegative, concave, and pseudosymmetric about η on $[0,1]$. Therefore

$$\begin{aligned}
u_2'(t) &\geq u_1'(t), && 0 \leq t \leq \frac{1+\eta}{2}; \\
u_2'(t) &\leq u_1'(t), && \frac{1+\eta}{2} \leq t \leq 1, \\
u_2''(t) &\leq u_1''(t), && t \in [0,1].
\end{aligned} \tag{36}$$

From (H_2) and the definition of T it follows that

$$\begin{aligned}
&(Tu_2)(t) - (Tu_1)(t) \\
&= \int_{0}^{t}\int_{\tau}^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) \\
&\qquad \times \left[f\left(s, u_2(s), u_2'(s), u_2''(s)\right)\right. \\
&\qquad\quad \left. - f\left(s, u_1(s), u_1'(s), u_1''(s)\right)\right] ds\,dr\,d\tau
\end{aligned}$$

$$\begin{aligned}&+\frac{\beta}{\alpha}\int_0^{(1+\eta)/2}\\&\quad\times\int_{(1+\eta)/2}^{r}q(s)\\&\quad\quad\times\left[f\left(s,u_2(s),u_2'(s),u_2''(s)\right)\right.\\&\quad\quad\quad\left.-f\left(s,u_1(s),u_1'(s),u_1''(s)\right)\right]ds\,dr\\&\geq 0,\quad t\in\left[0,\frac{1+\eta}{2}\right],\end{aligned}\tag{37}$$

$$\begin{aligned}&(Tu_2)''(t)-(Tu_1)''(t)\\&=\int_t^{(1+\eta)/2}q(s)\left[f\left(s,u_2(s),u_2'(s),u_2''(s)\right)\right.\\&\quad\quad\left.-f\left(s,u_1(s),u_1'(s),u_1''(s)\right)\right]ds\\&\leq 0,\quad t\in\left[0,\frac{1+\eta}{2}\right].\end{aligned}\tag{38}$$

We now prove that (37) and (38) hold for $t\in[(1+\eta)/2,1]$. In fact, if $t\in[(1+\eta)/2,1]$, then $1+\eta-t\in[\eta,(1+\eta)/2]\subset[0,(1+\eta)/2]$, and hence, from the fact that Tu_1 and Tu_2 are pseudosymmetric about η on $[0,1]$, it follows that, for $t\in[(1+\eta)/2,1]$,

$$\begin{aligned}(Tu_2)(t)-(Tu_1)(t)&=(Tu_2)(1+\eta-t)\\&\quad-(Tu_1)(1+\eta-t)\geq 0,\\(Tu_2)''(t)-(Tu_1)''(t)&=(Tu_2)''(1+\eta-t)\\&\quad-(Tu_1)''(1+\eta-t)\leq 0.\end{aligned}\tag{39}$$

So

$$(Tu_2)(t)-(Tu_1)(t)\geq 0,\quad t\in[0,1],\tag{40}$$

and $(Tu_2)(t)-(Tu_1)(t)$ is concave on $[0,1]$.

Finally, we show that $(Tu_2)(t)-(Tu_1)(t)$ is pseudosymmetric about η on $[0,1]$. To do this we let $S(t)=(Tu_2)(t)-(Tu_1)(t)$. We note that Tu_1, $Tu_2\in P$, $(Tu_1)(t)$ and $(Tu_2)(t)$ are pseudosymmetric about η on $[0,1]$, and thus

$$\begin{aligned}S(1+\eta-t)&=(Tu_2)(1+\eta-t)-(Tu_1)(1+\eta-t)\\&=(Tu_2)(t)-(Tu_1)(t)=S(t),\\&\qquad t\in[\eta,1].\end{aligned}\tag{41}$$

In summary, $Tu_2-Tu_1\in P$; that is, $Tu_1\leq Tu_2$. □

3. Main Result

Now we establish existence result of positive, concave, and pseudosymmetric solutions and its monotone iterative scheme for BVP (5), (6).

Theorem 6. *Assume that* (H_0), (H_1), (H_2), *and* (H_3) *hold. Suppose also that there exist two positive numbers a and b with* $a>b$ *such that*

$$\inf_{t\in(0,(1+\eta)/2]}q(t)f(t,a,a,-a)\geq -b,\tag{42}$$

where a,b *satisfy*

$$\begin{aligned}a\geq\max\Bigg\{&\frac{\beta(1+\eta)^2}{4\alpha}\left(\frac{7}{8}-\frac{1}{2(1+\eta)^2}\right)^{-1},\\&\frac{(1+\eta)^3}{8}+\frac{\beta(1+\eta)^2}{4\alpha},(1+\eta)^2\Bigg\}b.\end{aligned}\tag{43}$$

Then BVP (5), (6) *has positive, concave, and pseudosymmetric solutions* w^*, $v^*\in P$ *with*

$$\begin{aligned}&\|w^*\|_\infty\leq a,\quad \lim_{n\to\infty}T^nw_0=w^*,\\&\text{where } w_0(t)=\frac{at(1+\eta-t)}{2(1+\eta)^2}+\frac{7}{8}a;\\&\|v^*\|_\infty\leq a,\quad \lim_{n\to\infty}T^nv_0=v^*,\quad \text{where } v_0(t)\equiv 0.\end{aligned}\tag{44}$$

Proof. We denote $\overline{P}_a=\{u\in P:\|u\|\leq a\}$. In what follows, we first show that $T\overline{P}_a\subset\overline{P}_a$. To do this, let $u\in\overline{P}_a$; then obviously

$$\begin{aligned}0\leq u(t)\leq\max_{t\in[0,1]}u(t)=\|u\|_\infty\leq a,\quad t\in[0,1],\\ \max_{t\in[0,1]}\left|u''(t)\right|=-\min_{t\in[0,1]}u''(t)\leq a.\end{aligned}\tag{45}$$

Also since $u(t)$ is concave and pseudosymmetric about η on $[0,1]$, then $u'(t)$ is nonincreasing on $[0,1]$, $u'((1+\eta)/2)=0$, and $u'(t)=-u'(1+\eta-t)$ for $t\in[(1+\eta)/2,1]$. Hence $u'(t)\geq 0$ for $t\in[0,(1+\eta)/2]$ and $|u'(t)|$ achieve the maximum at $t=0$. Consequently

$$\max_{t\in[0,1]}\left|u'(t)\right|=u'(0)\leq a.\tag{46}$$

From (H_0), (H_2), and (42), it follows that

$$\begin{aligned}&q(t)f\left(t,u(t),u'(t),u''(t)\right)\\&\quad\geq q(t)f(t,a,a,-a)\\&\quad\geq\inf_{t\in(0,(1+\eta)/2]}q(t)f(t,a,a,-a)\\&\quad\geq -b,\quad t\in\left(0,\frac{1+\eta}{2}\right].\end{aligned}\tag{47}$$

This together with Lemma 3 implies

$$\begin{aligned}
&\|Tu\|_\infty \\
&= (Tu)\left(\frac{1+\eta}{2}\right) \\
&= \int_0^{(1+\eta)/2}\int_\tau^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) \\
&\qquad \times f\left(s, u(s), u'(s), u''(s)\right) ds\, dr\, d\tau \\
&\quad + \frac{\beta}{\alpha}\int_0^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr \\
&\le \frac{(1+\eta)^3}{8} b + \frac{\beta(1+\eta)^2}{4\alpha} b \le a, \\
&\left\|(Tu)'\right\|_\infty \\
&= (Tu)'(0) \\
&= \int_0^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds\, dr \\
&\le \frac{(1+\eta)^2}{8} b < a, \\
&\left\|(Tu)''\right\|_\infty \\
&= \max_{t\in(0,(1+\eta)/2]} \left| -\int_{(1+\eta)/2}^{t} q(s) \right. \\
&\qquad \left. \times f\left(s, u(s), u'(s), u''(s)\right) ds \right| \\
&= -\int_0^{(1+\eta)/2} q(s) f\left(s, u(s), u'(s), u''(s)\right) ds \\
&\le \frac{1+\eta}{2} b < a.
\end{aligned} \tag{48}$$

Hence $\|Tu\| \le a$, and thus $T\overline{P}_a \subset \overline{P}_a$.

Let $w_0(t) = (at(1+\eta-t)/2(1+\eta)^2)+(7/8)a$, $t \in [0,1]$. Then $\|w_0\| = a$, and thus $w_0 \in \overline{P}_a$. Let $w_1 = Tw_0$; then $w_1 \in \overline{P}_a$. Define iterative sequence $\{w_n\}$ as follows:

$$w_{n+1} = Tw_n = T^{n+1} w_0, \quad n = 0, 1, 2, \ldots. \tag{49}$$

Since $T\overline{P}_a \subset \overline{P}_a$, we have $w_n \in \overline{P}_a$, $n = 0, 1, 2, \ldots$. From Lemma 4, $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in \overline{P}_a$ such that

$$w_{n_k}^{(i)}(t) \rightrightarrows w^{*(i)}(t)\,(k \longrightarrow \infty) \quad \text{on } [0,1], \quad i = 0, 1, 2. \tag{50}$$

From the definition of T and (42), for $t \in [0, (1+\eta)/2]$, we have

$$\begin{aligned}
&w_1(t) \\
&= Tw_0(t) \\
&= \int_0^{t}\int_\tau^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) \\
&\qquad \times f\left(s, w_0(s), w_0'(s), w_0''(s)\right) ds\, dr\, d\tau \\
&\quad + \frac{\beta}{\alpha}\int_0^{(1+\eta)/2} \\
&\qquad \times \int_{(1+\eta)/2}^{r} q(s) f\left(s, w_0(s), w_0'(s), w_0''(s)\right) ds\, dr \\
&\le \int_0^{t}\int_\tau^{(1+\eta)/2} \\
&\qquad \times \int_r^{(1+\eta)/2} b\, ds\, dr\, d\tau \\
&\qquad + \frac{\beta}{\alpha}\int_0^{(1+\eta)/2}\int_r^{(1+\eta)/2} b\, ds\, dr \\
&= b\int_0^{t}\int_\tau^{(1+\eta)/2}\left(\frac{1+\eta}{2} - r\right) dr\, d\tau + \frac{\beta}{\alpha} b \\
&\quad \times \int_0^{(1+\eta)/2}\left(\frac{1+\eta}{2} - r\right) dr \\
&\le b\int_0^{t}\frac{1+\eta}{2} d\tau + \frac{\beta}{\alpha} b\left(\frac{1+\eta}{2}\right)^2 \\
&\le \frac{at}{2(1+\eta)} - \frac{at^2}{2(1+\eta)^2} + \frac{7}{8}a \\
&= \frac{at(1+\eta-t)}{2(1+\eta)^2} + \frac{7}{8}a = w_0(t),
\end{aligned} \tag{51}$$

and, for $t \in [(1+\eta)/2, 1]$, we have

$$w_1(t) = w_1(1+\eta-t) \le w_0(1+\eta-t) = w_0(t). \tag{52}$$

Thus $w_0(t) - w_1(t) \ge 0$, $t \in [0,1]$.

On the other hand, since, for $t \in [0,(1+\eta)/2]$, we have

$$\begin{aligned} w_0''(t) - w_1''(t) &= -\frac{a}{(1+\eta)^2} + \int_{(1+\eta)/2}^{t} q(s) f\left(s, w_0(s), w_0'(s), w_0''(s)\right) ds \\ &\le -\frac{a}{(1+\eta)^2} + b \le 0, \end{aligned} \tag{53}$$

and, for $t \in [(1+\eta)/2, 1]$, we have

$$w_0''(t) - w_1''(t) = w_0''(1+\eta-t) - w_1''(1+\eta-t) \le 0, \tag{54}$$

it follows that $w_0''(t) - w_1''(t) \le 0$ on $[0,1]$. Hence $w_0(t) - w_1(t)$ is concave on $[0,1]$.

Also, since, for $t \in [\eta, 1]$,

$$w_0(t) - w_1(t) = w_0(1+\eta-t) - w_1(1+\eta-t), \tag{55}$$

then $w_0(t) - w_1(t)$ is pseudosymmetric about η on $[0,1]$. So $w_1 \le w_0$, and hence from Lemma 5 it follows that $Tw_1 \le Tw_0$; that is, $w_2 \le w_1$. By induction, we can show without any difficulty that

$$w_{n+1} \le w_n, \quad n = 0,1,2,\ldots; \tag{56}$$

that is,

$$w_n - w_{n+1} \in P, \quad n = 0,1,2,\ldots. \tag{57}$$

Thus $w_n(t) - w_{n+1}(t)$ $(n = 0,1,2,\ldots)$ is concave on $[0,1]$ and pseudosymmetric about η on $[0,1]$; consequently

$$w_n'(t) \ge w_{n+1}'(t), \quad t \in \left[0, \frac{1+\eta}{2}\right], \quad n = 0,1,2,\ldots, \tag{58}$$

$$w_n'(t) \le w_{n+1}'(t), \quad t \in \left[\frac{1+\eta}{2}, 1\right], \quad n = 0,1,2,\ldots, \tag{59}$$

$$w_n''(t) \le w_{n+1}''(t), \quad t \in [0,1], \quad n = 0,1,2,\ldots. \tag{60}$$

From (50)–(60), it follows that

$$w_n^{(i)}(t) \rightrightarrows w^{*(i)}(t)\ (n \longrightarrow \infty) \quad \text{on } [0,1], \quad i = 0,1,2; \tag{61}$$

that is, $w_n \to w^*$ $(n \to \infty)$. Let $n \to \infty$ in (49) to obtain

$$w^* = Tw^*. \tag{62}$$

Also, from (H_3), we have $w^*((1+\eta)/2) = \max_{t\in[0,1]} w^*(t) > 0$. This together with the concavity of w^* implies that

$$w^*(t) \ge \frac{w^*((1+\eta)/2) - 0}{((1+\eta)/2) - 0} t = \frac{2}{1+\eta} w^*\left(\frac{1+\eta}{2}\right) t > 0, \quad t \in \left(0, \frac{1+\eta}{2}\right]. \tag{63}$$

Again using the fact that w^* is pseudosymmetric about η on $[0,1]$, we have

$$w^*(t) > 0, \quad t \in \left[\frac{1+\eta}{2}, 1\right]. \tag{64}$$

Hence $w^*(t) > 0$ on $(0,1]$. Therefore, from Lemma 2, w^* is a concave pseudosymmetric positive solution of BVP (5), (6).

Let $v_0(t) \equiv 0$ on $[0,1]$; then $v_0 \in \overline{P}_a$. Set

$$v_{n+1} = Tv_n, \quad n = 0,1,2,\ldots. \tag{65}$$

Then, from Lemma 3, the sequence $\{v_n\}$ is well defined. Since $v_1 \in \overline{P}_a \subset P$, we have $v_1 \ge 0 = v_0$, and thus from Lemma 5 it follows that

$$v_2 = Tv_1 \ge Tv_0 = v_1. \tag{66}$$

By induction we can show that

$$v_{n+1} \ge v_n, \quad n = 0,1,2,\ldots. \tag{67}$$

Similarly to $\{w_n\}$, we can show that there exists $v^* \in \overline{P}_a$ such that $v_n \to v^*$ $(n \to \infty)$. Taking limit in (65), we get $v^* = Tv^*$. Obviously, $v(t) > 0$ on $(0,1]$. Therefore, from Lemma 2, v^* is a concave pseudosymmetric positive solution of BVP (5), (6). This completes the proof of the theorem. □

4. An Example

Consider the following third-order four-point boundary value problem:

$$\begin{aligned} &u'''(t) + 3(3t-2) \times \left(e^{-u/8} + \frac{64}{64+u'^2} + \arctan\frac{u''}{16} + \frac{\pi}{2}\right) = 0, \quad t \in (0,1), \\ &u(0) - \frac{1}{2}u'(0) = 0, \\ &u'\left(\frac{1}{3}\right) + u'(1) = 0, \\ &u''\left(\frac{2}{3}\right) = 0. \end{aligned} \tag{68}$$

Let

$$f(t,u,v,w) = 3(3t-2)\left(e^{-u/8} + \frac{64}{64+v^2} + \arctan\frac{w}{16} + \frac{\pi}{2}\right). \tag{69}$$

Then $f \in C([0,1] \times [0,+\infty) \times \mathbb{R} \times (-\infty,0], \mathbb{R})$. It is easy to see that BVP (68) corresponds to BVP (5), (6) when $q(t) \equiv 1$, $\alpha = 1$, $\beta = 1/2$, and $\eta = 1/3$.

Next we verify that all conditions of Theorem 6 are satisfied. In fact, obviously the conditions (H_0), (H_2), and (H_3) hold. In addition, by the definition of f, we have

$$\begin{aligned}
&f(1+\eta-t,u,-v,w)\\
&\quad = 3(2-3t)\left(e^{-u/8}+\frac{64}{64+(-v)^2}+\arctan\frac{w}{16}+\frac{\pi}{2}\right)\\
&\quad = -3(3t-2)\left(e^{-u/8}+\frac{64}{64+v^2}+\arctan\frac{w}{16}+\frac{\pi}{2}\right)\\
&\quad = -f(t,u,v,w), \quad t\in\left[\frac{1}{3},1\right],\\
&f(t,u,v,w)\le 0 \quad \text{on } \left[0,\frac{2}{3}\right]\times[0,+\infty)^2\times(-\infty,0].
\end{aligned} \tag{70}$$

Hence the condition (H_1) is also satisfied.

Now, we take $a=16$, $b=9$. Then

$$\begin{aligned}
\max&\left\{\frac{\beta(1+\eta)^2}{4\alpha}\left(\frac{7}{8}-\frac{1}{2(1+\eta)^2}\right)^{-1},\right.\\
&\left.\frac{(1+\eta)^3}{8}+\frac{\beta(1+\eta)^2}{4\alpha},(1+\eta)^2\right\}=\frac{16}{9},
\end{aligned} \tag{71}$$

and thus

$$\begin{aligned}
a=\max&\left\{\frac{\beta(1+\eta)^2}{4\alpha}\left(\frac{7}{8}-\frac{1}{2(1+\eta)^2}\right)^{-1},\right.\\
&\left.\frac{(1+\eta)^3}{8}+\frac{\beta(1+\eta)^2}{4\alpha},(1+\eta)^2\right\}b.
\end{aligned} \tag{72}$$

On the other hand, we also have

$$\begin{aligned}
&\inf_{t\in(0,2/3]} f(t,16,16,-16)\\
&= \inf_{t\in(0,2/3]} 3(3t-2)\left(e^{-2}+\frac{64}{64+16^2}+\arctan(-1)+\frac{\pi}{2}\right)\\
&> -9=-b.
\end{aligned} \tag{73}$$

In summary, all conditions of Theorem 6 are satisfied. Hence, from Theorem 6, BVP (68) has concave pseudosymmetric positive solution w^*, $v^*\in P$ with

$$\begin{gathered}
\|w^*\|_\infty\le 16,\quad \lim_{n\to\infty}T^n w_0=w^*,\\
\text{where } w_0(t)=-\frac{9}{2}t^2+6t+14;\\
\|v^*\|_\infty\le 16,\quad \lim_{n\to\infty}T^n v_0=v^*,\quad \text{where } v_0(t)\equiv 0.
\end{gathered} \tag{74}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the anonymous referee for his or her constructive comments and suggestions which led to improvement of the original paper. This work was supported by NSFC (11201008).

References

[1] M. Greguš, *Third Order Linear Differential Equations*, Mathematics and Its Applications, Reidel, Dordrecht, The Netherlands, 1987.

[2] A. R. Aftabizadeh, C. P. Gupta, and J.-M. Xu, "Existence and uniqueness theorems for three-point boundary value problems," *SIAM Journal on Mathematical Analysis*, vol. 20, no. 3, pp. 716–726, 1989.

[3] R. P. Agarwal, "Existence-uniqueness and iterative methods for third-order boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 17, no. 3, pp. 271–289, 1987.

[4] A. Boucherif and N. Al-Malki, "Nonlinear three-point third-order boundary value problems," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1168–1177, 2007.

[5] A. Cabada, M. R. Grossinho, and F. Minhós, "Extremal solutions for third-order nonlinear problems with upper and lower solutions in reversed order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 6, pp. 1109–1121, 2005.

[6] P. W. Eloe and B. Ahmad, "Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 521–527, 2005.

[7] D. Li, L. Wang, and M. Pei, "Existence and monotone iteration of positive pseudosymmetric solutions for a third-order four-point BVP with p-Laplacian," *Abstract and Applied Analysis*, vol. 2013, Article ID 192509, 12 pages, 2013.

[8] J. R. Graef and J. R. L. Webb, "Third order boundary value problems with nonlocal boundary conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1542–1551, 2009.

[9] L.-J. Guo, J.-P. Sun, and Y.-H. Zhao, "Existence of positive solutions for nonlinear third-order three-point boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 3151–3158, 2008.

[10] M. R. Grossinho, F. M. Minhós, and A. I. Santos, "Existence result for a third-order ODE with nonlinear boundary conditions in presence of a sign-type Nagumo control," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 1, pp. 271–283, 2005.

[11] C. P. Gupta and V. Lakshmikantham, "Existence and uniqueness theorems for a third-order three-point boundary value problem," *Nonlinear Analysis*, vol. 16, no. 11, pp. 949–957, 1991.

[12] J. Henderson, "Best interval lengths for boundary value problems for third order Lipschitz equations," *SIAM Journal on Mathematical Analysis*, vol. 18, no. 2, pp. 293–305, 1987.

[13] G. Infante and P. Pietramala, "A third order boundary value problem subject to nonlinear boundary conditions," *Mathematica Bohemica*, vol. 135, no. 2, pp. 113–121, 2010.

[14] D. Jiang and R. P. Agarwal, "A uniqueness and existence theorem for a singular third-order boundary value problem on $[0, \infty)$," *Applied Mathematics Letters*, vol. 15, no. 4, pp. 445–451, 2002.

[15] S. Jin and S. Lu, "Existence of solutions for a third-order multipoint boundary value problem with p-Laplacian," *Journal of the Franklin Institute*, vol. 347, no. 3, pp. 599–606, 2010.

[16] X. Lin, Z. Du, and W. Liu, "Uniqueness and existence results for a third-order nonlinear multi-point boundary value problem," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 187–196, 2008.

[17] R. Ma, "Multiplicity results for a third order boundary value problem at resonance," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 32, no. 4, pp. 493–499, 1998.

[18] F. M. Minhós, "On some third order nonlinear boundary value problems: existence, location and multiplicity results," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 2, pp. 1342–1353, 2008.

[19] J. J. Nieto, "Periodic solutions for third order ordinary differential equations," *Commentationes Mathematicae Universitatis Carolinae*, vol. 32, no. 3, pp. 495–499, 1991.

[20] D. O'Regan, "Topological transversality: applications to third order boundary value problems," *SIAM Journal on Mathematical Analysis*, vol. 18, no. 3, pp. 630–641, 1987.

[21] M. Pei and S. K. Chang, "Existence and uniqueness of solutions for third-order nonlinear boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 1, pp. 23–35, 2007.

[22] I. Rachůnková, "Periodic boundary value problems for third-order differential equations," *Mathematica Slovaca*, vol. 41, no. 3, pp. 241–248, 1991.

[23] B. Sun, J. Zhao, P. Yang, and W. Ge, "Successive iteration and positive solutions for a third-order multipoint generalized right-focal boundary value problem with p-Laplacian," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 1, pp. 220–230, 2009.

[24] Y. Sun, "Positive solutions for third-order three-point nonhomogeneous boundary value problems," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 45–51, 2009.

[25] F. Wang and Y. Cui, "On the existence of solutions for singular boundary value problem of third-order differential equations," *Mathematica Slovaca*, vol. 60, no. 4, pp. 485–494, 2010.

[26] Y. Wang and W. Ge, "Existence of solutions for a third order differential equation with integral boundary conditions," *Computers and Mathematics with Applications*, vol. 53, no. 1, pp. 144–154, 2007.

[27] P. J. Y. Wong, "Constant-sign solutions for a system of third-order generalized right focal problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. e2153–e2163, 2005.

[28] Q. Yao, "Positive solutions of singular third-order three-point boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 207–212, 2009.

[29] P. Zhang, "Iterative solutions of singular boundary value problems of third-order differential equation," *Boundary Value Problems*, vol. 2011, Article ID 483057, 10 pages, 2011.

[30] C. Zhou and D. Ma, "Existence and iteration of positive solutions for a generalized right-focal boundary value problem with p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 409–424, 2006.

[31] H. Pang, M. Feng, and W. Ge, "Existence and monotone iteration of positive solutions for a three-point boundary value problem," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 656–661, 2008.

[32] B. Sun and W. Ge, "Successive iteration and positive pseudo-symmetric solutions for a three-point second-order p-Laplacian boundary value problems," *Applied Mathematics and Computation*, vol. 188, no. 2, pp. 1772–1779, 2007.

[33] R. Avery and J. Henderson, "Existence of three positive pseudo-symmetric solutions for a one dimensional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 395–404, 2003.