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Research Article

Iterative Methods to Solve the Generalized Coupled Sylvester-Conjugate Matrix Equations for Obtaining the Centrally Symmetric (Centrally Antisymmetric) Matrix Solutions

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The iterative method is presented for obtaining the centrally symmetric (centrally antisymmetric) matrix pair (X, Y) solutions of the generalized coupled Sylvester-conjugate matrix equations $A_1X + B_1Y = D_1\overline{X}E_1 + F_1$, $A_2Y + B_2X = D_2\overline{Y}E_2 + F_2$. On the condition that the coupled matrix equations are consistent, we show that the solution pair (X^*, Y^*) can be obtained within finite iterative steps in the absence of round-off error for any initial value given centrally symmetric (centrally antisymmetric) matrix. Moreover, by choosing appropriate initial value, we can get the least Frobenius norm solution for the new generalized coupled Sylvester-conjugate linear matrix equations. Finally, some numerical examples are given to illustrate that the proposed iterative method is quite efficient.

1. Introduction

Many research papers are involved in the system of matrix equation ([1–33]). The following matrix equation

$$AXB = C \tag{1}$$

is a special case of coupled Sylvester linear matrix equations

$$\sum_{j=1}^{n} A_{ij}X_jB_{ij} = C, \quad (i = 1, 2, \ldots, m). \tag{2}$$

In [34], an iterative algorithm was constructed to solve (1) for skew-symmetric matrix X. Navarra et al. studied a representation of the general solution for the matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ [35]. By Moore-Penrose generalized inverse, some necessary and sufficient conditions on the existence of the solution and the expressions for the matrix equation $AX + X^TC = B$ were obtained in ([36]). Deng et al. give the consistent conditions and the general expressions of the Hermitian solutions for (1) [37]. In addition, by extending the well-known Jacobi and Gauss-Seidel iterations for $Ax = b$, Ding et al. gained iterative solutions for matrix equation (1) and the generalized Sylvester matrix equation $AXB + CXD = F$ [38]. The closed form solutions to a family of generalized Sylvester matrix equations were given by utilizing the so-called Kronecker matrix polynomials in ([39]). In recent years, Dehghan and Hajarian considered the solution for the generalized coupled Sylvester matrix equations [40] $AXB+CYD = M$, $EXF+GYH = N$ and presented a modified conjugate gradient method to solve the matrix equations over generalized bisymmetric matrix pair (X, Y). Liang and Liu proposed a modified conjugate gradient method to solve the following problem [41]:

$$\begin{aligned} A_1XB_1 + C_1X^TD_1 &= F_1, \\ A_2XB_2 + C_2X^TD_2 &= F_2. \end{aligned} \tag{3}$$

In the present paper, we conceive efficient algorithm to solve the following generalized coupled Sylvester-conjugate linear matrix equations for centrally symmetric (centrally antisymmetric) matrix pair (X, Y):

$$A_1X + B_1Y = D_1\overline{X}E_1 + F_1,$$
$$A_2Y + B_2X = D_2\overline{Y}E_2 + F_2, \quad (4)$$

where $A_i, B_i, D_i \in C^{p\times m}$, $E_i \in C^{m\times m}$, $F_i \in C^{p\times m}$ $(i = 1, 2)$ are given constant matrices, and $X, Y \in C^{m\times m}$ are unknown matrices to be solved. When $A_2 = B_2 = D_2 = 0$ and $F_2 = 0$, the problem (4) becomes the problem studied in [42]. When $A_2 = B_2 = D_2 = 0$, $F_2 = 0$, and $A_1 = I$, this system becomes the Yakubovich-conjugate matrix equation investigated in [43]. When $B_1 = A_2 = B_2 = D_2 = 0$, $F_2 = 0$, and $A_1 = I$, the problem (4) becomes the equation considered in [44]. When $A_2 = B_2 = D_2 = 0$ and $F_1 = F_2 = 0$, the problem (4) becomes the equation in [45]. When $B_1 = A_2 = B_2 = D_2 = 0$, $F_2 = 0$, and $D_1 = I$, (4) becomes the equation in [46].

It is known that modified conjugate gradient (MCG) method is the most popular iterative method for solving the system of linear equation

$$Ax = b, \quad (5)$$

where $x \in R^n$ is an unknown vector, $A \in R^{m\times n}$ is a given matrix, and $b \in R^m$ is constant vector. By the definition of the Kronecker product, matrix equations can be transformed into the system (5). Then the MCG can be applied to various linear matrix equations [44, 45]. Based on this idea, in this paper, we propose a modified conjugate gradient method to solve the system (4) and show that a solution pair (X^*, Y^*) can be obtained within finite iterative steps in the absence of round-off error for any initial value given centrally symmetric (centrally antisymmetric) matrix. Furthermore, by choosing appropriate initial value matrix pair, we can obtain the least Frobenius norm solution for (4).

As a matter of convenience, some terminology used throughout the paper follows.

$C^{m\times n}$ is the set of $m \times n$ complex matrices and $R^{m\times n}$ is the set of $m \times n$ all real matrices. For $A \in C^{m\times n}$, we write $\mathrm{Re}(A)$, $\mathrm{Im}(A)$, $\overline{A}$, A^T, A^H, A^{-1}, $\|A\|_F$, and $\mathscr{R}(A)$ to denote the real part, the imaginary part, the conjugation, transpose, conjugate transpose, the inverse, the Frobenius norm, and the column space of the matrix A, respectively. $\mathrm{Diag}\{A_1, A_2, \ldots, A_n\}$ denotes the block diagonal matrix, where $A_i \in R^{m\times m}$ $(i = 1, 2, \ldots, n)$. For any $A = (a_{ij})$, $B = (b_{ij})$, $A\otimes B$ denotes the Kronecker product defined as $A\otimes B = (a_{ij}B)$. For the matrix $X = (x_1, x_2, \ldots, x_n) \in C^{m\times n}$, $\mathrm{vec}(X)$ denotes the vec operator defined as $\mathrm{vec}(X) = (x_1^T, x_2^T, \ldots, x_n^T)^T$. We use I to denote the identity matrix of size implied by context.

Definition 1. Let $S \in R^{m\times m}$ and $S = (e_m, e_{m-1}, \ldots, e_1)$, where e_j $(j = 1, 2, \ldots, m)$ denotes the column vector whose jth element is 1 and the other elements are zeros. An $m \times m$ complex matrix X is said to be a centrally symmetric (centrally antisymmetric) matrix if $SXS = X$ $(SXS = -X)$, denote the set of all centrally symmetric (centrally antisymmetric) matrices by $CSC^{m\times m}(CASC^{m\times m})$.

The rest of this paper is organized as follows. In Section 2, we construct modified conjugate gradient (MCG) method for solving the system (4) and show that a solution pair (X^*, Y^*) for (4) can be obtained by the MCG method within finite iterative steps in the absence of round-off error for any initial value given centrally symmetric (centrally antisymmetric) matrix. Furthermore, we demonstrate that the least Frobenius norm solution can be obtained by choosing a special kind of initial matrix. Also we give some numerical examples which illustrate that the introduced iterative algorithm is efficient in Section 3. Conclusions are arranged in Section 4.

2. The Iterative Method for Solving the Matrix Equations (4)

In this section, we present the modified conjugate gradient method (MCG) for solving the system (4). Firstly, we recall that the definition of inner product came from [42].

The inner product in space $C^{m\times n}$ is defined as

$$\langle A, B\rangle = \mathrm{Re}\left[\mathrm{tr}\left(A^HB\right)\right]. \quad (6)$$

By Theorem 1 in [42], we know that the inner product defined by (6) satisfies the following three axioms:

(1) symmetry: $\langle A, B\rangle = \langle B, A\rangle$;

(2) linearity in the first argument:

$$\langle \delta_1A_1 + \delta_2A_2, B\rangle = \delta_1\langle A_1, B\rangle + \delta_2\langle A_2, B\rangle, \quad (7)$$

where δ_1 and δ_2 are real constants;

(3) positive definiteness: $\langle A, A\rangle > 0$, for all $A \neq 0$.

For all real constants δ_1, δ_2, by (1) and (2), we get

$$\begin{aligned}\left\langle A, \delta_1B_1 + \delta_2B_2\right\rangle &= \langle \delta_1B_1 + \delta_2B_2, A\rangle \\ &= \delta_1\langle B_1, A\rangle + \delta_2\langle B_2, A\rangle \\ &= \delta_1\left\langle A, B_1\right\rangle + \delta_2\left\langle A, B_2\right\rangle;\end{aligned} \quad (8)$$

namely, the inner product defined by (6) is linear in the second argument.

By the relation between the matrix trace and the conjugate operation, we get

$$\langle A, B\rangle = \mathrm{Re}\left[\mathrm{tr}\left(A^HB\right)\right] = \mathrm{Re}\left[\overline{\mathrm{tr}\left(A^HB\right)}\right] = \mathrm{Re}\left[\mathrm{tr}\left(\overline{A^HB}\right)\right]. \quad (9)$$

The norm of a matrix generated by this inner product space is denoted by $\|\cdot\|$. Then for $A \in C^{m\times n}$, we obtain

$$\|A\|^2 = \langle A, A\rangle = \mathrm{Re}\left[\mathrm{tr}\left(A^HA\right)\right]. \quad (10)$$

What is the relationship between this norm and the Frobenius norm? It is well known that $\|A\|_F^2 = \mathrm{tr}(A^HA)$ and A^HA is a Hermite matrix. Then by the knowledge of algebra, we know that $\mathrm{tr}(A^HA)$ is real; hence, $\mathrm{tr}(A^HA) = \mathrm{Re}[\mathrm{tr}(A^HA)]$.

This shows that $\|A\|_F = \|A\|$. Another interesting relationship is that

$$\begin{aligned}
\|A\|^2 &= \|\mathrm{Re}(A) + i\,\mathrm{Im}(A)\|^2 \\
&= \mathrm{Re}\left\{\mathrm{tr}\left[(\mathrm{Re}\,(A) + i\,\mathrm{Im}\,(A))^H (\mathrm{Re}\,(A) + i\,\mathrm{Im}\,(A))\right]\right\} \\
&= \mathrm{Re}\left\{\mathrm{tr}\left[\left(\mathrm{Re}\,(A)^T - i\,\mathrm{Im}\,(A)^T\right)(\mathrm{Re}\,(A) + i\,\mathrm{Im}\,(A))\right]\right\} \\
&= \mathrm{Re}\left[\mathrm{tr}\left(\mathrm{Re}\,(A)^T\,\mathrm{Re}\,(A) + i\,\mathrm{Re}\,(A)^T\,\mathrm{Im}\,(A)\right.\right. \\
&\qquad \left.\left. - i\,\mathrm{Im}\,(A)^T\,\mathrm{Re}\,(A) + \mathrm{Im}\,(A)^T\,\mathrm{Im}\,(A)\right)\right] \\
&= \mathrm{Re}\left[\mathrm{tr}\left(\mathrm{Re}\,(A)^T\,\mathrm{Re}\,(A)\right) + \mathrm{tr}\left(\mathrm{Im}\,(A)^T\,\mathrm{Im}\,(A)\right)\right. \\
&\qquad \left. + i\,\mathrm{tr}\left(\mathrm{Re}\,(A)^T\,\mathrm{Im}\,(A)\right) - i\,\mathrm{tr}\left(\mathrm{Im}\,(A)^T\,\mathrm{Re}\,(A)\right)\right] \\
&= \mathrm{tr}\left(\mathrm{Re}\,(A)^T\,\mathrm{Re}\,(A)\right) + \mathrm{tr}\left(\mathrm{Im}\,(A)^T\,\mathrm{Im}\,(A)\right) \\
&= \|\mathrm{Re}(A)\|_F^2 + \|\mathrm{Im}(A)\|_F^2.
\end{aligned} \tag{11}$$

That is, $\|A\|^2 = \|\,\mathrm{Re}(A)\|_F^2 + \|\,\mathrm{Im}(A)\|_F^2$.

In the following we present some algorithms. The ordinary conjugate gradient (CG) method to solve (5) is as follows [47].

Algorithm 2 (CG method). Consider the following steps.

Step 1. Input A, b. Choose the initial vectors x_0 and set $k := 0$; calculate $r_0 = b - Ax_0$, $p_0 = r_0$.

Step 2. If $r_k = 0$ or $r_k \neq 0$ and $p_k = 0$, stop; otherwise, calculate

$$x_{k+1} = x_k + \frac{\langle r_k, r_k\rangle}{\langle p_k, p_k\rangle} p_k. \tag{12}$$

Step 3. Update the sequences

$$\begin{aligned}
r_{k+1} &= b - Ax_{k+1}, \\
p_{k+1} &= r_{k+1} + \frac{\langle r_{k+1}, r_{k+1}\rangle}{\langle r_k, r_k\rangle} p_k.
\end{aligned} \tag{13}$$

Step 4. Set $k := k + 1$; return to Step 2.

It is known that the size of the linear equation (5) will be large, when (4) is transformed to a linear equation (5) by the Kronecker product. Therefore, the iterative Algorithm 2 will consume much more computer time and memory space once increasing the dimensionality of coefficient matrix.

In view of these considerations, we construct the following so-called modified conjugate gradient (MCG) method to solve (4).

Algorithm 3 (MCG method for centrally symmetric matrix version). Consider the following steps.

Step 1. Input appropriate dimension matrices A_i, B_i, D_i, E_i, and F_i, $(i = 1, 2)$. Choose the initial matrices $X_1 \in CSC^{m\times m}$ and $Y_1 \in CSC^{m\times m}$, $S = (e_m, e_{m-1}, \ldots, e_1)$ in Definition 1. Compute

$$\begin{aligned}
R_1 &:= \begin{pmatrix} R_1^{(1)} & 0 \\ 0 & R_1^{(2)} \end{pmatrix}, \\
R_1^{(1)} &= D_1\overline{X_1}E_1 + F_1 - A_1X_1 - B_1Y_1, \\
R_1^{(2)} &= D_2\overline{Y_1}E_2 + F_2 - A_2Y_1 - B_2X_1, \\
\widetilde{R}_1 &:= \begin{pmatrix} \widetilde{R}_1^{(1)} & 0 \\ 0 & \widetilde{R}_1^{(2)} \end{pmatrix}, \\
\widetilde{R}_1^{(1)} &= A_1^H R_1^{(1)} + B_2^H R_1^{(2)} - D_1^T \overline{R_1^{(1)}} E_1^T, \\
\widetilde{R}_1^{(2)} &= A_2^H R_1^{(2)} + B_1^H R_1^{(1)} - D_2^T \overline{R_1^{(2)}} E_2^T, \\
M_1 &= \frac{1}{2}\left(\widetilde{R}_1^{(1)} + S\widetilde{R}_1^{(1)}S\right), \qquad N_1 = \frac{1}{2}\left(\widetilde{R}_1^{(2)} + S\widetilde{R}_1^{(2)}S\right);
\end{aligned} \tag{14}$$

set $k := 1$.

Step 2. If $R_k = 0$ or $R_k \neq 0$, $M_k = N_k = 0$, stop; otherwise, go to Step 3.

Step 3. Update the sequences

$$\begin{aligned}
X_{k+1} &= X_k + \alpha_k M_k, \qquad Y_{k+1} = Y_k + \alpha_k N_k, \\
R_{k+1} &:= \begin{pmatrix} R_{k+1}^{(1)} & 0 \\ 0 & R_{k+1}^{(2)} \end{pmatrix}, \\
R_{k+1}^{(1)} &= D_1\overline{X_{k+1}}E_1 + F_1 - A_1X_{k+1} - B_1Y_{k+1}, \\
R_{k+1}^{(2)} &= D_2\overline{Y_{k+1}}E_2 + F_2 - A_2Y_{k+1} - B_2X_{k+1}, \\
\widetilde{R}_{k+1} &:= \begin{pmatrix} \widetilde{R}_{k+1}^{(1)} & 0 \\ 0 & \widetilde{R}_{k+1}^{(2)} \end{pmatrix}, \\
\widetilde{R}_{k+1}^{(1)} &= A_1^H R_{k+1}^{(1)} + B_2^H R_{k+1}^{(2)} - D_1^T \overline{R_{k+1}^{(1)}} E_1^T, \\
\widetilde{R}_{k+1}^{(2)} &= A_2^H R_{k+1}^{(2)} + B_1^H R_{k+1}^{(1)} - D_2^T \overline{R_{k+1}^{(2)}} E_2^T, \\
M_{k+1} &= \frac{1}{2}\left(\widetilde{R}_{k+1}^{(1)} + S\widetilde{R}_{k+1}^{(1)}S\right) + \beta_k M_k, \\
N_{k+1} &= \frac{1}{2}\left(\widetilde{R}_{k+1}^{(2)} + S\widetilde{R}_{k+1}^{(2)}S\right) + \beta_k N_k,
\end{aligned} \tag{15}$$

where

$$\alpha_k := \frac{\|R_k\|^2}{\|M_k\|^2 + \|N_k\|^2}, \qquad \beta_k := \frac{\|R_{k+1}\|^2}{\|R_k\|^2}. \tag{16}$$

Step 4. Set $k := k + 1$; return to Step 2.

Algorithm 4 (MCG method for centrally antisymmetric matrix version). Consider the following steps.

Step 1. Input matrices A_i, B_i, D_i, E_i, and F_i, $(i = 1, 2)$. Choose the initial matrix $X_1 \in CASC^{m\times m}$ and $Y_1 \in CASC^{m\times m}$, $S = (e_m, e_{m-1}, \ldots, e_1)$ in Definition 1. Compute

$$\begin{gathered} R_1 := \begin{pmatrix} R_1^{(1)} & 0 \\ 0 & R_1^{(2)} \end{pmatrix}, \\ R_1^{(1)} = D_1\overline{X_1}E_1 + F_1 - A_1X_1 - B_1Y_1, \\ R_1^{(2)} = D_2\overline{Y_1}E_2 + F_2 - A_2Y - B_2X_1, \\ \widetilde{R}_1 := \begin{pmatrix} \widetilde{R}_1^{(1)} & 0 \\ 0 & \widetilde{R}_1^{(2)} \end{pmatrix}, \\ \widetilde{R}_1^{(1)} = A_1^HR_1^{(1)} + B_2^HR_1^{(2)} - D_1^T\overline{R_1^{(1)}}E_1^T, \\ \widetilde{R}_1^{(2)} = A_2^HR_1^{(2)} + B_1^HR_1^{(1)} - D_2^T\overline{R_1^{(2)}}E_2^T, \\ M_1 = \frac{1}{2}\left(\widetilde{R}_1^{(1)} - S\widetilde{R}_1^{(1)}S\right), \qquad N_1 = \frac{1}{2}\left(\widetilde{R}_1^{(2)} - S\widetilde{R}_1^{(2)}S\right); \end{gathered} \tag{17}$$

set $k := 1$.

Step 2. If $R_k = 0$ or $R_k \neq 0$, $M_k = N_k = 0$, stop; otherwise, go to Step 3.

Step 3. Update the sequences

$$\begin{gathered} X_{k+1} = X_k + \alpha_kM_k, \qquad Y_{k+1} = Y_k + \alpha_kN_k, \\ R_{k+1} := \begin{pmatrix} R_{k+1}^{(1)} & 0 \\ 0 & R_{k+1}^{(2)} \end{pmatrix}, \\ R_{k+1}^{(1)} = D_1\overline{X_{k+1}}E_1 + F_1 - A_1X_{k+1} - B_1Y_{k+1}, \\ R_{k+1}^{(2)} = D_2\overline{Y_{k+1}}E_2 + F_2 - A_2Y_{k+1} - B_2X_{k+1}, \\ \widetilde{R}_{k+1} := \begin{pmatrix} \widetilde{R}_{k+1}^{(1)} & 0 \\ 0 & \widetilde{R}_{k+1}^{(2)} \end{pmatrix}, \end{gathered} \tag{18}$$

$$\begin{gathered} \widetilde{R}_{k+1}^{(1)} = A_1^HR_{k+1}^{(1)} + B_2^HR_{k+1}^{(2)} - D_1^T\overline{R_{k+1}^{(1)}}E_1^T, \\ \widetilde{R}_{k+1}^{(2)} = A_2^HR_{k+1}^{(2)} + B_1^HR_{k+1}^{(1)} - D_2^T\overline{R_{k+1}^{(2)}}E_2^T, \\ M_{k+1} = \frac{1}{2}\left(\widetilde{R}_{k+1}^{(1)} - S\widetilde{R}_{k+1}^{(1)}S\right) + \beta_kM_k, \\ N_{k+1} = \frac{1}{2}\left(\widetilde{R}_{k+1}^{(2)} - S\widetilde{R}_{k+1}^{(2)}S\right) + \beta_kN_k, \end{gathered} \tag{19}$$

where

$$\alpha_k := \frac{\|R_k\|^2}{\|M_k\|^2 + \|N_k\|^2}, \qquad \beta_k := \frac{\|R_{k+1}\|^2}{\|R_k\|^2}. \tag{20}$$

Step 4. Set $k := k + 1$; return to Step 2.

Now, we will show that the sequence matrix pair $\{X_k, Y_k\}$ generated by Algorithm 3 converges to the solution (X^*, Y^*) for (4) within finite iterative steps in the absence of round-off error for any initial value over centrally symmetric (centrally antisymmetric) matrix.

Lemma 5. *Let the sequences $\{R_k\}$, $\{M_k\}$, $\{N_k\}$, $\{\widetilde{R}_j^{(1)}\}$, $\{\widetilde{R}_j^{(2)}\}$, and $\{\alpha_k\}$ generated by Algorithm 3; then have*

$$\left\langle R_{k+1}, R_j\right\rangle = \left\langle R_k, R_j\right\rangle - \alpha_k\left(\left\langle M_k, \widetilde{R}_j^{(1)}\right\rangle + \left\langle N_k, \widetilde{R}_j^{(2)}\right\rangle\right), \quad k, j = 1, 2, \ldots. \tag{21}$$

Proof. By Algorithm 3 and (20), we get

$$\begin{aligned} R_{k+1}^{(1)} &= D_1\overline{X_{k+1}}E_1 + F_1 - A_1X_{k+1} - B_1Y_{k+1} \\ &= D_1\left(\overline{X_k} + \alpha_k\overline{M_k}\right)E_1 + F_1 - A_1\left(X_k + \alpha_kM_k\right) \\ &\quad - B_1\left(Y_k + \alpha_kN_k\right) \\ &= D_1\overline{X_k}E_1 + F_1 - A_1X_k - B_1Y_k \\ &\quad + \alpha_k\left(D_1\overline{M_k}E_1 - A_1M_k - B_1N_k\right) \\ &= R_k^{(1)} + \alpha_k\left(D_1\overline{M_k}E_1 - A_1M_k - B_1N_k\right). \end{aligned} \tag{22}$$

In a similar way, we can get

$$R_{k+1}^{(2)} = R_k^{(2)} + \alpha_k\left(D_2\overline{N_k}E_2 - A_2N_k - B_2M_k\right). \tag{23}$$

This together with the definition of inner product yields that

$$\begin{aligned} &\left\langle R_{k+1}^{(1)}, R_j^{(1)}\right\rangle \\ &\quad = \left\langle R_k^{(1)}, R_j^{(1)}\right\rangle \\ &\qquad + \alpha_k\left(\left\langle D_1\overline{M_k}E_1, R_j^{(1)}\right\rangle - \left\langle A_1M_k, R_j^{(1)}\right\rangle\right. \\ &\qquad\qquad \left. - \left\langle B_1N_k, R_j^{(1)}\right\rangle\right) \\ &\quad = \left\langle R_k^{(1)}, R_j^{(1)}\right\rangle + \alpha_k \\ &\qquad \times\left\{\mathrm{Re}\left[\mathrm{tr}\left(E_1^HM_k^TD_1^HR_j^{(1)}\right)\right] - \mathrm{Re}\left[\mathrm{tr}\left(M_k^HA_1^HR_j^{(1)}\right)\right]\right. \\ &\qquad\qquad \left. - \mathrm{Re}\left[\mathrm{tr}\left(N_k^HB_1^HR_j^{(1)}\right)\right]\right\} \\ &\quad = \left\langle R_k^{(1)}, R_j^{(1)}\right\rangle + \alpha_k \\ &\qquad \times\left\{\mathrm{Re}\left[\mathrm{tr}\left(\overline{E_1^HM_k^TD_1^HR_j^{(1)}}\right)\right] - \left\langle M_k, A_1^HR_j^{(1)}\right\rangle\right. \\ &\qquad\qquad \left. - \left\langle N_k, B_1^HR_j^{(1)}\right\rangle\right\} \\ &\quad = \left\langle R_k^{(1)}, R_j^{(1)}\right\rangle + \alpha_k \\ &\qquad \times\left\{\mathrm{Re}\left[\mathrm{tr}\left(E_1^TM_k^HD_1^T\overline{R_j^{(1)}}\right)\right] - \left\langle M_k, A_1^HR_j^{(1)}\right\rangle\right. \\ &\qquad\qquad \left. - \left\langle N_k, B_1^HR_j^{(1)}\right\rangle\right\} \end{aligned}$$

$$
\begin{aligned}
&= \left\langle R_k^{(1)}, R_j^{(1)} \right\rangle + \alpha_k \\
&\quad \times \left\{ \operatorname{Re}\left[\operatorname{tr}\left(M_k^H D_1^T \overline{R_j^{(1)}} E_1^T \right) \right] - \left\langle M_k, A_1^H R_j^{(1)} \right\rangle \right. \\
&\qquad \left. - \left\langle N_k, B_1^H R_j^{(1)} \right\rangle \right\} \\
&= \left\langle R_k^{(1)}, R_j^{(1)} \right\rangle + \alpha_k \\
&\quad \times \left(\left\langle M_k, D_1^T \overline{R_j^{(1)}} E_1^T \right\rangle - \left\langle M_k, A_1^H R_j^{(1)} \right\rangle - \left\langle N_k, B_1^H R_j^{(1)} \right\rangle \right) \\
&= \left\langle R_k^{(1)}, R_j^{(1)} \right\rangle + \alpha_k \\
&\quad \times \left(\left\langle M_k, D_1^T \overline{R_j^{(1)}} E_1^T - A_1^H R_j^{(1)} \right\rangle - \left\langle N_k, B_1^H R_j^{(1)} \right\rangle \right), \\
&\left\langle R_{k+1}^{(2)}, R_j^{(2)} \right\rangle \\
&= \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left(\left\langle D_2 \overline{N_k} E_2, R_j^{(2)} \right\rangle - \left\langle A_2 N_k, R_j^{(2)} \right\rangle - \left\langle B_2 M_k, R_j^{(2)} \right\rangle \right) \\
&= \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left\{ \operatorname{Re}\left[\operatorname{tr}\left(E_2^H N_k^T D_2^H R_j^{(2)} \right) \right] - \operatorname{Re}\left[\operatorname{tr}\left(N_k^H A_2^H R_j^{(2)} \right) \right] \right. \\
&\qquad \left. - \operatorname{Re}\left[\operatorname{tr}\left(M_k^H B_2^H R_j^{(2)} \right) \right] \right\} \\
&= \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left\{ \operatorname{Re}\left[\operatorname{tr}\left(\overline{E_2^H N_k^T D_2^H R_j^{(2)}} \right) \right] - \left\langle N_k, A_2^H R_j^{(2)} \right\rangle \right. \\
&\qquad \left. - \left\langle M_k, B_2^H R_j^{(2)} \right\rangle \right\} \\
&= \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left\{ \operatorname{Re}\left[\operatorname{tr}\left(E_2^T N_k^H D_2^T \overline{R_j^{(2)}} \right) \right] - \left\langle N_k, A_2^H R_j^{(2)} \right\rangle \right. \\
&\qquad \left. - \left\langle M_k, B_2^H R_j^{(2)} \right\rangle \right\} \\
&= \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left\{ \operatorname{Re}\left[\operatorname{tr}\left(N_k^H D_2^T \overline{R_j^{(2)}} E_2^T \right) \right] - \left\langle N_k, A_2^H R_j^{(2)} \right\rangle \right. \\
&\qquad \left. - \left\langle M_k, B_2^H R_j^{(2)} \right\rangle \right\} \\
&= \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left(\left\langle N_k, D_2^T \overline{R_j^{(2)}} E_2^T \right\rangle - \left\langle N_k, A_2^H R_j^{(2)} \right\rangle - \left\langle M_k, B_2^H R_j^{(2)} \right\rangle \right) \\
&= \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left(\left\langle N_k, D_2^T \overline{R_j^{(2)}} E_2^T - A_2^H R_j^{(2)} \right\rangle - \left\langle M_k, B_2^H R_j^{(2)} \right\rangle \right).
\end{aligned} \tag{24}
$$

Then by the updated formulas of R_k, $\widetilde{R}_j^{(1)}$, and $\widetilde{R}_j^{(2)}$, we obtain

$$
\begin{aligned}
&\left\langle R_{k+1}, R_j \right\rangle \\
&= \left\langle R_{k+1}^{(1)}, R_j^{(1)} \right\rangle + \left\langle R_{k+1}^{(2)}, R_j^{(2)} \right\rangle \\
&= \left\langle R_k^{(1)}, R_j^{(1)} \right\rangle + \alpha_k \\
&\quad \times \left(\left\langle M_k, D_1^T \overline{R_j^{(1)}} E_1^T - A_1^H R_j^{(1)} \right\rangle - \left\langle N_k, B_1^H R_j^{(1)} \right\rangle \right) \\
&\quad + \left\langle R_k^{(2)}, R_j^{(2)} \right\rangle + \alpha_k \\
&\quad \times \left(\left\langle N_k, D_2^T \overline{R_j^{(2)}} E_2^T - A_2^H R_j^{(2)} \right\rangle - \left\langle M_k, B_2^H R_j^{(2)} \right\rangle \right) \\
&= \langle R_k, R_j \rangle + \alpha_k \\
&\quad \times \left(\left\langle M_k, D_1^T \overline{R_j^{(1)}} E_1^T - A_1^H R_j^{(1)} - B_2^H R_j^{(2)} \right\rangle \right. \\
&\qquad \left. + \left\langle N_k, D_2^T \overline{R_j^{(2)}} E_2^T - A_2^H R_j^{(2)} - B_1^H R_j^{(1)} \right\rangle \right) \\
&= \left\langle R_k, R_j \right\rangle - \alpha_k \left(\left\langle M_k, \widetilde{R}_j^{(1)} \right\rangle + \left\langle N_k, \widetilde{R}_j^{(2)} \right\rangle \right),
\end{aligned} \tag{25}
$$

which completes the proof. □

Lemma 6. *Let the sequences, $\{R_k\}$, $\{M_k\}$, and $\{N_k\}$, be generated by Algorithm 3; one has*

$$
\left\langle R_i, R_j \right\rangle = 0, \qquad \left\langle M_i, M_j \right\rangle + \langle N_i, N_j \rangle = 0, \qquad i, j = 1, 2, \ldots, k, i \neq j. \tag{26}
$$

Proof. Firstly, we prove

$$
\left\langle R_i, R_j \right\rangle = 0, \quad \left\langle M_i, M_j \right\rangle + \left\langle N_i, N_j \right\rangle = 0, \quad 1 \le j < i \le k. \tag{27}
$$

By mathematical induction, for $k = 2$, by Lemma 5, and noticing $M_j \in CSC^{m\times m}$, $N_j \in CSC^{m\times m}$ $(j = 1, 2, \ldots, k)$ generated by Algorithm 3, we get

$$
\begin{aligned}
&\langle R_2, R_1 \rangle \\
&= \langle R_1, R_1 \rangle - \alpha_1 \left(\left\langle M_1, \widetilde{R}_1^{(1)} \right\rangle + \left\langle N_1, \widetilde{R}_1^{(2)} \right\rangle \right) \\
&= \left\| R_1 \right\|^2 - \alpha_1 \left(\left\langle M_1, \frac{\widetilde{R}_1^{(1)} + S\widetilde{R}_1^{(1)} S}{2} \right\rangle \right. \\
&\qquad \left. + \left\langle N_1, \frac{\widetilde{R}_1^{(2)} + S\widetilde{R}_1^{(2)} S}{2} \right\rangle \right) \\
&= \left\| R_1 \right\|^2 - \frac{\left\| R_1 \right\|^2}{\left\| M_1 \right\|^2 + \left\| N_1 \right\|^2} \left(\langle M_1, M_1 \rangle + \langle N_1, N_1 \rangle \right) \\
&= \left\| R_1 \right\|^2 - \left\| R_1 \right\|^2 = 0,
\end{aligned} \tag{28}
$$

where the second equality is from the fact

$$\left\langle M_1, S\widetilde{R}_1^{(1)}S\right\rangle = \left\langle M_1, \widetilde{R}_1^{(1)}\right\rangle,$$
$$\left\langle N_1, S\widetilde{R}_1^{(2)}S\right\rangle = \left\langle N_1, \widetilde{R}_1^{(2)}\right\rangle. \tag{29}$$

In addition, by (19), (20), and Lemma 5, we have

$$\begin{aligned}
&\langle M_2, M_1\rangle + \langle N_2, N_1\rangle \\
&= \left\langle \frac{\widetilde{R}_2^{(1)} + S\widetilde{R}_2^{(1)}S}{2} + \beta_1 M_1, M_1\right\rangle \\
&\quad + \left\langle \frac{\widetilde{R}_2^{(2)} + S\widetilde{R}_2^{(2)}S}{2} + \beta_1 N_1, N_1\right\rangle \\
&= \left\langle \widetilde{R}_2^{(1)}, M_1\right\rangle + \left\langle \widetilde{R}_2^{(2)}, N_1\right\rangle \\
&\quad + \beta_1\left(\|M_1\|^2 + \|N_1\|^2\right) \\
&= \frac{1}{\alpha_1}\left(\langle R_1, R_2\rangle - \langle R_2, R_2\rangle\right) \\
&\quad + \beta_1\left(\|M_1\|^2 + \|N_1\|^2\right) = 0,
\end{aligned} \tag{30}$$

where the second equality is from (6) and the fact

$$\left\langle M_1, S\widetilde{R}_2^{(1)}S\right\rangle = \left\langle M_1, \widetilde{R}_2^{(1)}\right\rangle,$$
$$\left\langle N_1, S\widetilde{R}_2^{(2)}S\right\rangle = \left\langle N_1, \widetilde{R}_2^{(2)}\right\rangle. \tag{31}$$

Therefore, (27) holds for $k = 2$.

Suppose that (27) holds, for $k = l$ $(l \geq 2)$. For $k = l + 1$, it follows from Lemma 5, (9) that

$$\begin{aligned}
&\langle R_{l+1}, R_l\rangle \\
&= \langle R_l, R_l\rangle - \alpha_l\left(\left\langle M_l, \widetilde{R}_l^{(1)}\right\rangle + \left\langle N_l, \widetilde{R}_l^{(2)}\right\rangle\right) \\
&= \|R_l\|^2 - \alpha_l \\
&\quad \times \left(\left\langle M_l, \frac{\widetilde{R}_l^{(1)} + S\widetilde{R}_l^{(1)}S}{2}\right\rangle + \left\langle N_l, \frac{\widetilde{R}_l^{(2)} + S\widetilde{R}_l^{(2)}S}{2}\right\rangle\right) \\
&= \|R_l\|^2 - \alpha_l \\
&\quad \times \left(\langle M_l, M_l - \beta_{l-1}M_{l-1}\rangle + \langle N_l, N_l - \beta_{l-1}N_{l-1}\rangle\right) \\
&= \|R_l\|^2 - \alpha_l\left(\|M_l\|^2 + \|N_l\|^2\right) = 0,
\end{aligned} \tag{32}$$

where the fourth equality holds by the induction assumption. Combining (19) and (20) and by induction with the above result, we obtain

$$\begin{aligned}
&\langle M_{l+1}, M_l\rangle + \langle N_{l+1}, N_l\rangle \\
&= \left\langle \frac{\widetilde{R}_{l+1}^{(1)} + S\widetilde{R}_{l+1}^{(1)}S}{2} + \beta_l M_l, M_l\right\rangle \\
&\quad + \left\langle \frac{\widetilde{R}_{l+1}^{(2)} + S\widetilde{R}_{l+1}^{(2)}S}{2} + \beta_l N_l, N_l\right\rangle \\
&= \left\langle \widetilde{R}_{l+1}^{(1)}, M_l\right\rangle + \left\langle \widetilde{R}_{l+1}^{(2)}, N_l\right\rangle + \beta_l\left(\|M_l\|^2 + \|N_l\|^2\right) \\
&= \frac{1}{\alpha_l}\left(\langle R_l, R_{l+1}\rangle - \langle R_{l+1}, R_{l+1}\rangle\right) \\
&\quad + \beta_l\left(\|M_l\|^2 + \|N_l\|^2\right) = 0,
\end{aligned} \tag{33}$$

where the third equality is from Lemma 5.

For $j = 1$, by Lemma 5 and the induction, we have

$$\begin{aligned}
&\langle R_{l+1}, R_1\rangle \\
&= \langle R_l, R_1\rangle - \alpha_l\left(\left\langle M_l, \widetilde{R}_1^{(1)}\right\rangle + \left\langle N_l, \widetilde{R}_1^{(2)}\right\rangle\right) \\
&= -\alpha_l\left(\left\langle M_l, \frac{\widetilde{R}_1^{(1)} + S\widetilde{R}_1^{(1)}S}{2}\right\rangle + \left\langle N_l, \frac{\widetilde{R}_1^{(2)} + S\widetilde{R}_1^{(2)}S}{2}\right\rangle\right) \\
&= -\alpha_l\left(\langle M_l, M_1\rangle + \langle N_l, N_1\rangle\right) = 0.
\end{aligned} \tag{34}$$

Analogously, for $j = 2, 3, \ldots, l - 1$, then we obtain

$$\begin{aligned}
&\langle R_{l+1}, R_j\rangle \\
&= \langle R_l, R_j\rangle - \alpha_l\left(\left\langle M_l, \widetilde{R}_j^{(1)}\right\rangle + \left\langle N_l, \widetilde{R}_j^{(2)}\right\rangle\right) \\
&= -\alpha_l\left(\left\langle M_l, \frac{\widetilde{R}_j^{(1)} + S\widetilde{R}_j^{(1)}S}{2}\right\rangle\right. \\
&\qquad \left. + \left\langle N_l, \frac{\widetilde{R}_j^{(2)} + S\widetilde{R}_j^{(2)}S}{2}\right\rangle\right) \\
&= -\alpha_l\left(\left\langle M_l, M_j - \beta_{j-1}M_{j-1}\right\rangle\right. \\
&\qquad \left. + \left\langle N_l, N_j - \beta_{j-1}N_{j-1}\right\rangle\right) \\
&= -\alpha_l\left(\left\langle M_l, M_j\right\rangle + \left\langle N_l, N_j\right\rangle - \beta_{j-1}\right. \\
&\qquad \left. \times\left(\left\langle M_l, M_{j-1}\right\rangle + \left\langle N_l, N_{j-1}\right\rangle\right)\right) = 0.
\end{aligned} \tag{35}$$

In addition, from Lemma 5 and the induction, for $j = 1, 2, \ldots, l-1$, we get

$$
\begin{aligned}
&\langle M_{l+1}, M_j \rangle + \langle N_{l+1}, N_j \rangle \\
&\quad = \left\langle \frac{\widetilde{R}_{l+1}^{(1)} + S\widetilde{R}_{l+1}^{(1)} S}{2}, M_j \right\rangle \\
&\qquad + \left\langle \frac{\widetilde{R}_{l+1}^{(2)} + S\widetilde{R}_{l+1}^{(2)} S}{2}, N_j \right\rangle \\
&\qquad + \beta_l \left(\left\langle M_l, M_j \right\rangle + \left\langle N_l, N_j \right\rangle \right) \\
&\quad = \left\langle \widetilde{R}_{l+1}^{(1)}, M_j \right\rangle + \left\langle \widetilde{R}_{l+1}^{(2)}, N_j \right\rangle \\
&\quad = \frac{1}{\alpha_l} \left(\left\langle R_j, R_{l+1} \right\rangle - \left\langle R_{j+1}, R_{l+1} \right\rangle \right) = 0.
\end{aligned}
\tag{36}
$$

So (27) holds, for $k = l+1$. By induction principle, (27) holds, for all $1 \leq j < i \leq k$. For $j > i$, we obtain

$$
\begin{aligned}
\langle R_i, R_j \rangle &= \langle R_j, R_i \rangle = 0, \\
\langle M_i, M_j \rangle + \langle N_i, N_j \rangle &= \langle M_j, M_i \rangle + \langle N_j, N_i \rangle = 0,
\end{aligned}
\tag{37}
$$

which completes the proof. □

Lemma 7. *Suppose that the system of matrix equations* (4) *is consistent; let* (X^*, Y^*) *be an arbitrary solution pair of* (4). *Then for any initial matrices* $X_1 \in CMC^{m\times m}$, $Y_1 \in CMC^{m\times m}$, *the sequences* $\{X_k\}$, $\{Y_k\}$, $\{R_k\}$, $\{M_k\}$, *and* $\{N_k\}$ *generated by Algorithm 3 satisfy*

$$
\left\langle X^* - X_k, M_k \right\rangle + \left\langle Y^* - Y_k, N_k \right\rangle = \left\| R_k \right\|^2, \quad k = 1, 2, \ldots, n. \tag{38}
$$

Proof. The conclusion is accomplished by mathematical induction.

Firstly, we notice that the sequences pair (X_k, Y_k), $(k = 1, 2, \ldots)$ generated by Algorithm 3 are all central symmetric matrices since initial matrix pair (X_1, Y_1) is centrally symmetric matrix. Then for $k = 1$, it follows from Algorithm 3 that

$$
\begin{aligned}
&\left\langle X^* - X_1, M_1 \right\rangle \\
&\quad = \left\langle X^* - X_1, \frac{\widetilde{R}_1^{(1)} + S\widetilde{R}_1^{(1)} S}{2} \right\rangle = \left\langle X^* - X_1, \widetilde{R}_1^{(1)} \right\rangle \\
&\quad = \left\langle X^* - X_1, A_1^H R_1^{(1)} + B_2^H R_1^{(2)} - D_1^T \overline{R_1^{(1)}} E_1^T \right\rangle \\
&\quad = \left\langle X^* - X_1, A_1^H R_1^{(1)} \right\rangle + \left\langle X^* - X_1, B_2^H R_1^{(2)} \right\rangle \\
&\qquad - \left\langle X^* - X_1, D_1^T \overline{R_1^{(1)}} E_1^T \right\rangle \\
&\quad = \operatorname{Re}\left[\operatorname{tr}\left(\left(X^* - X_1 \right)^H A_1^H R_1^{(1)} \right) \right] \\
&\qquad + \operatorname{Re}\left[\operatorname{tr}\left(\left(X^* - X_1 \right)^H B_2^H R_1^{(2)} \right) \right] \\
&\qquad - \operatorname{Re}\left[\operatorname{tr}\left(\left(X^* - X_1 \right)^H D_1^T \overline{R_1^{(1)}} E_1^T \right) \right] \\
&\quad = \left\langle A_1 \left(X^* - X_1 \right), R_1^{(1)} \right\rangle + \left\langle B_2 \left(X^* - X_1 \right), R_1^{(2)} \right\rangle \\
&\qquad - \operatorname{Re}\left[\operatorname{tr}\left(\overline{\left(X^* - X_1 \right)^H D_1^T \overline{R_1^{(1)}} E_1^T} \right) \right] \\
&\quad = \left\langle A_1 \left(X^* - X_1 \right), R_1^{(1)} \right\rangle + \left\langle B_2 \left(X^* - X_1 \right), R_1^{(2)} \right\rangle \\
&\qquad - \operatorname{Re}\left[\operatorname{tr}\left(\left(X^* - X_1 \right)^T D_1^H R_1^{(1)} E_1^H \right) \right] \\
&\quad = \left\langle A_1 \left(X^* - X_1 \right), R_1^{(1)} \right\rangle + \left\langle B_2 \left(X^* - X_1 \right), R_1^{(2)} \right\rangle \\
&\qquad - \operatorname{Re}\left[\operatorname{tr}\left(E_1^H \left(X^* - X_1 \right)^T D_1^H R_1^{(1)} \right) \right] \\
&\quad = \left\langle A_1 \left(X^* - X_1 \right), R_1^{(1)} \right\rangle + \left\langle B_2 \left(X^* - X_1 \right), R_1^{(2)} \right\rangle \\
&\qquad - \left\langle D_1 \left(\overline{X^* - X_1} \right) E_1, R_1^{(1)} \right\rangle \\
&\quad = \left\langle A_1 \left(X^* - X_1 \right) - D_1 \left(\overline{X^* - X_1} \right) E_1, R_1^{(1)} \right\rangle \\
&\qquad + \left\langle B_2 \left(X^* - X_1 \right), R_1^{(2)} \right\rangle .
\end{aligned}
\tag{39}
$$

In the same way, we can get

$$
\begin{aligned}
&\left\langle Y^* - Y_1, N_1 \right\rangle \\
&\quad = \left\langle A_2 \left(Y^* - Y_1 \right) - D_2 \left(\overline{Y^* - Y_1} \right) E_2, R_1^{(2)} \right\rangle \\
&\qquad + \left\langle B_1 \left(Y^* - Y_1 \right), R_1^{(1)} \right\rangle .
\end{aligned}
\tag{40}
$$

This shows that

$$
\begin{aligned}
&\left\langle X^* - X_1, M_1 \right\rangle + \left\langle Y^* - Y_1, N_1 \right\rangle \\
&\quad = \left\langle A_1 \left(X^* - X_1 \right) - D_1 \left(\overline{X^* - X_1} \right) E_1, R_1^{(1)} \right\rangle \\
&\qquad + \left\langle B_2 \left(X^* - X_1 \right), R_1^{(2)} \right\rangle \\
&\qquad + \left\langle A_2 \left(Y^* - Y_1 \right) - D_2 \left(\overline{Y^* - Y_1} \right) E_2, R_1^{(2)} \right\rangle \\
&\qquad + \left\langle B_1 \left(Y^* - Y_1 \right), R_1^{(1)} \right\rangle \\
&\quad = \Big\langle A_1 \left(X^* - X_1 \right) + B_1 \left(Y^* - Y_1 \right) \\
&\qquad\quad - D_1 \left(\overline{X^* - X_1} \right) E_1, R_1^{(1)} \Big\rangle \\
&\qquad + \Big\langle A_2 \left(Y^* - Y_1 \right) + B_2 \left(X^* - X_1 \right) \\
&\qquad\quad - D_2 \left(\overline{Y^* - Y_1} \right) E_2, R_1^{(2)} \Big\rangle
\end{aligned}
$$

$$= \left\langle F_1 - A_1 X_1 - B_1 Y_1 + D_1 \overline{X_1} E_1, R_1^{(1)} \right\rangle + \left\langle F_2 - A_2 Y_1 - B_2 X_1 + D_2 \overline{Y_1} E_2, R_1^{(2)} \right\rangle = \left\| R_1^{(1)} \right\|^2 + \left\| R_1^{(2)} \right\|^2 = \| R_1 \|^2. \tag{41}$$

That is, (38) holds, for $k = 1$.

Assume (38) holds, for $k = l$. For $k = l + 1$, it follows from the updated formulas of X_{l+1}, Y_{l+1} that

$$\begin{aligned} \langle X^* - X_{l+1}, M_l \rangle &= \langle X^* - X_l - \alpha_l M_l, M_l \rangle \\ &= \langle X^* - X_l, M_l \rangle - \alpha_l \| M_l \|^2, \\ \langle Y^* - Y_{l+1}, N_l \rangle &= \langle Y^* - Y_l - \alpha_l N_l, N_l \rangle \\ &= \langle Y^* - Y_l, N_l \rangle - \alpha_l \| N_l \|^2. \end{aligned} \tag{42}$$

Then

$$\begin{aligned} &\langle X^* - X_{l+1}, M_{l+1} \rangle \\ &= \left\langle X^* - X_{l+1}, \frac{\widetilde{R}_{l+1}^{(1)} + S \widetilde{R}_{l+1}^{(1)} S}{2} + \beta_l M_l \right\rangle \\ &= \left\langle X^* - X_{l+1}, \widetilde{R}_{l+1}^{(1)} \right\rangle + \beta_l \langle X^* - X_{l+1}, M_l \rangle \\ &= \left\langle X^* - X_{l+1}, \widetilde{R}_{l+1}^{(1)} \right\rangle + \beta_l \left(\langle X^* - X_l, M_l \rangle - \alpha_l \| M_l \|^2 \right), \\ &\langle Y^* - Y_{l+1}, N_{l+1} \rangle \\ &= \left\langle Y^* - Y_{l+1}, \frac{\widetilde{R}_{l+1}^{(2)} + S \widetilde{R}_{l+1}^{(2)} S}{2} + \beta_l N_l \right\rangle \\ &= \left\langle Y^* - Y_{l+1}, \widetilde{R}_{l+1}^{(2)} \right\rangle + \beta_l \langle Y^* - Y_{l+1}, N_l \rangle \\ &= \left\langle Y^* - Y_{l+1}, \widetilde{R}_{l+1}^{(2)} \right\rangle + \beta_l \left(\langle Y^* - Y_l, N_l \rangle - \alpha_l \| N_l \|^2 \right). \end{aligned} \tag{43}$$

On the other hand, we have

$$\begin{aligned} &\left\langle X^* - X_{l+1}, \widetilde{R}_{l+1}^{(1)} \right\rangle + \left\langle Y^* - Y_{l+1}, \widetilde{R}_{l+1}^{(2)} \right\rangle \\ &= \left\langle X^* - X_{l+1}, A_1^H R_{l+1}^{(1)} + B_2^H R_{l+1}^{(2)} - D_1^T \overline{R_{l+1}^{(1)}} E_1^T \right\rangle \\ &\quad + \left\langle Y^* - Y_{l+1}, A_2^H R_{l+1}^{(2)} + B_1^H R_{l+1}^{(1)} - D_2^T \overline{R_{l+1}^{(2)}} E_2^T \right\rangle \\ &= \left\langle A_1 (X^* - X_{l+1}), R_{l+1}^{(1)} \right\rangle + \left\langle B_2 (X^* - X_{l+1}), R_{l+1}^{(2)} \right\rangle \\ &\quad - \operatorname{Re} \left[\operatorname{tr} \left((X^* - X_{l+1})^H D_1^T \overline{R_{l+1}^{(1)}} E_1^T \right) \right] \\ &\quad + \left\langle A_2 (Y^* - Y_{l+1}), R_{l+1}^{(2)} \right\rangle \\ &\quad + \left\langle B_1 (Y^* - Y_{l+1}), R_{l+1}^{(1)} \right\rangle \\ &\quad - \operatorname{Re} \left[\operatorname{tr} \left((Y^* - Y_{l+1})^H D_2^T \overline{R_{l+1}^{(2)}} E_2^T \right) \right] \\ &= \left\langle A_1 (X^* - X_{l+1}) + B_1 (Y^* - Y_{l+1}), R_{l+1}^{(1)} \right\rangle \\ &\quad - \operatorname{Re} \left[\operatorname{tr} \left((X^* - X_{l+1})^T D_1^H R_{l+1}^{(1)} E_1^H \right) \right] \\ &\quad + \left\langle A_2 (Y^* - Y_{l+1}) + B_2 (X^* - X_{l+1}), R_{l+1}^{(2)} \right\rangle \\ &\quad - \operatorname{Re} \left[\operatorname{tr} \left((Y^* - Y_{l+1})^T D_2^H R_{l+1}^{(2)} E_2^H \right) \right] \\ &= \left\langle A_1 (X^* - X_{l+1}) + B_1 (Y^* - Y_{l+1}), R_{l+1}^{(1)} \right\rangle \\ &\quad - \operatorname{Re} \left[\operatorname{tr} \left(E_1^H (X^* - X_{l+1})^T D_1^H R_{l+1}^{(1)} \right) \right] \\ &\quad + \left\langle A_2 (Y^* - Y_{l+1}) + B_2 (X^* - X_{l+1}), R_{l+1}^{(2)} \right\rangle \\ &\quad - \operatorname{Re} \left[\operatorname{tr} \left(E_2^H (Y^* - Y_{l+1})^T D_2^H R_{l+1}^{(2)} \right) \right] \\ &= \Big\langle A_1 (X^* - X_{l+1}) + B_1 (Y^* - Y_{l+1}) \\ &\qquad - D_1 \left(\overline{X^* - X_{l+1}} \right) E_1, R_{l+1}^{(1)} \Big\rangle \\ &\quad + \Big\langle A_2 (Y^* - Y_{l+1}) + B_2 (X^* - X_{l+1}) \\ &\qquad - D_2 \left(\overline{Y^* - Y_{l+1}} \right) E_2, R_{l+1}^{(2)} \Big\rangle \\ &= \left\langle F_1 - A_1 X_{l+1} - B_1 Y_{l+1} + D_1 \overline{X_{l+1}} E_1, R_{l+1}^{(1)} \right\rangle \\ &\quad + \left\langle F_2 - A_2 Y_{l+1} - B_2 X_{l+1} + D_2 \overline{Y_{l+1}} E_2, R_{l+1}^{(2)} \right\rangle \\ &= \left\| R_{l+1}^{(1)} \right\|^2 + \left\| R_{l+1}^{(2)} \right\|^2 = \| R_{l+1} \|^2. \end{aligned} \tag{44}$$

Therefore, by (20) we get

$$\begin{aligned} &\langle X^* - X_{l+1}, M_{l+1} \rangle + \langle Y^* - Y_{l+1}, N_{l+1} \rangle \\ &= \left\langle X^* - X_{l+1}, \frac{\widetilde{R}_{l+1}^{(1)} + S \widetilde{R}_{l+1}^{(1)} S}{2} + \beta_l M_l \right\rangle \\ &\quad + \left\langle Y^* - Y_{l+1}, \frac{\widetilde{R}_{l+1}^{(2)} + S \widetilde{R}_{l+1}^{(2)} S}{2} + \beta_l N_l \right\rangle \\ &= \left\langle X^* - X_{l+1}, \widetilde{R}_{l+1}^{(1)} \right\rangle + \beta_l \left(\langle X^* - X_l, M_l \rangle - \alpha_l \| M_l \|^2 \right) \\ &\quad + \left\langle Y^* - Y_{l+1}, \widetilde{R}_{l+1}^{(2)} \right\rangle + \beta_l \left(\langle Y^* - Y_l, N_l \rangle - \alpha_l \| N_l \|^2 \right) \\ &= \left\langle X^* - X_{l+1}, \widetilde{R}_{l+1}^{(1)} \right\rangle + \left\langle Y^* - Y_{l+1}, \widetilde{R}_{l+1}^{(2)} \right\rangle \\ &\quad + \beta_l \big[\langle X^* - X_l, M_l \rangle + \langle Y^* - Y_l, N_l \rangle \\ &\qquad - \alpha_l \left(\| M_l \|^2 + \| N_l \|^2 \right) \big] \\ &= \left\langle X^* - X_{l+1}, \widetilde{R}_{l+1}^{(1)} \right\rangle + \left\langle Y^* - Y_{l+1}, \widetilde{R}_{l+1}^{(2)} \right\rangle \\ &\quad + \beta_l \left[\| R_l \|^2 - \frac{\| R_l \|^2}{\| M_l \|^2 + \| N_l \|^2} \left(\| M_l \|^2 + \| N_l \|^2 \right) \right] \\ &= \left\langle X^* - X_{l+1}, \widetilde{R}_{l+1}^{(1)} \right\rangle + \left\langle Y^* - Y_{l+1}, \widetilde{R}_{l+1}^{(2)} \right\rangle = \| R_{l+1} \|^2. \end{aligned} \tag{45}$$

Hence, the proof is completed. □

Remark 8. Lemma 7 implies that if (4) is consistent, then $\|M_k\|^2 + \|N_k\|^2 \neq 0$ when $R_k \neq 0$. Conversely, if there exists a positive integer k_0 such that $R_{k_0} \neq 0$ and $\|M_{k_0}\|^2 + \|N_{k_0}\|^2 = 0$ in the iteration process of Algorithm 3, then (4) is inconsistent; we will study this condition with other papers in the future.

Remark 9. The above lemmas are achieved under the assumption that initial value is centrally symmetric matrix. Similarly, if the initial matrix is centrally antisymmetric matrix, we can get the same conclusions easily (see Definition 1). Hence, we need not show these results in detail; in the following content, we only discuss the version when $X, Y \in CMC^{m\times m}$.

Theorem 10. *Suppose the system* (4) *is consistent; then, for any initial matrix $X_1 \in CMC^{m\times m}$, $Y_1 \in CMC^{m\times m}$, an exact solution of* (4) *can be derived at most $2pm + 1$ iteration steps by Algorithm 3.*

Proof. Assume $R_k \neq 0$, for $k = 1, 2, \ldots, 2pm$. It follows from Lemma 7 that $\|M_k\|^2 + \|N_k\|^2 \neq 0$ for $k = 1, 2, \ldots, 2pm$. Then R_{2pm+1} can be derived by Algorithm 3. According to Lemma 6, we know $\langle R_i, R_j\rangle = 0$, for $i, j = 1, 2, \ldots, 2pm + 1$, $i \neq j$. Then the matrix sequence of $R_1, R_2, \ldots, R_{2pm}$ is an orthogonal basis of the linear space

$$\mathscr{H} = \left\{ H \mid H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \right\}, \tag{46}$$

where $H_1, H_2 \in R^{p\times m}$. Since $R_{2pm+1} \in \mathscr{H}$ and $\langle R_{2pm+1}, R_k\rangle = 0$, for $k = 1, 2, \ldots, 2pm$, hence $R_{2pm+1} = 0$, which completes the proof. □

Although the proof is trivial, the consequences of this result are of major importance.

When (4) is consistent, the solution of (4) is not unique. Then we need to find the unique least Frobenius norm solution of (4). Next, we introduce the following lemma.

Lemma 11. *Suppose $A \in R^{m\times n}$, $b \in R^m$, and the linear matrix equation $Ax = b$ has a solution $x^* \in \mathscr{R}(A^T)$; then x^* is the unique least Frobenius norm solution of $Ax = b$.*

For a rigorous proof of this lemma above the reader is referred to [46, 48].

Lemma 12. *Suppose $A \in C^{m\times n}$, $b \in C^m$, and the linear matrix equation $Ax = b$ has a solution $x^* \in R^n$. If $x^* \in \mathscr{R}[(\mathrm{Re}(A)^T, \mathrm{Im}(A)^T)]$, then x^* is the unique least Frobenius norm solution of $Ax = b$.*

Proof. Let $A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$ and $b = \mathrm{Re}(b) + i\,\mathrm{Im}(b)$. Then $Ax = b$ can be written as

$$\begin{aligned}(\mathrm{Re}(A) + i\,\mathrm{Im}(A))\,x &= \mathrm{Re}(A)\,x + i\,\mathrm{Im}(A)\,x \\ &= \mathrm{Re}(b) + i\,\mathrm{Im}(b).\end{aligned} \tag{47}$$

It shows that

$$\mathrm{Re}(A)\,x = \mathrm{Re}(b), \qquad \mathrm{Im}(A)\,x = \mathrm{Im}(b), \tag{48}$$

or

$$\begin{pmatrix}\mathrm{Re}(A) \\ \mathrm{Im}(A)\end{pmatrix} x = \begin{pmatrix}\mathrm{Re}(b) \\ \mathrm{Im}(b)\end{pmatrix}. \tag{49}$$

Since

$$\begin{aligned} x^* &\in \mathscr{R}\left[\left(\mathrm{Re}(A)^T, \mathrm{Im}(A)^T\right)\right] \\ &= \mathscr{R}\left[\begin{pmatrix}\mathrm{Re}(A) \\ \mathrm{Im}(A)\end{pmatrix}^T\right],\end{aligned} \tag{50}$$

this together with Lemma 11 completes the result. □

In order to get the least Frobenius norm solution of (4), we need to transform the problem (4).

Let $X = \mathrm{Re}(X) + i\,\mathrm{Im}(X)$, $Y = \mathrm{Re}(Y) + i\,\mathrm{Im}(Y)$. Then the problem (4) can be equivalently written as

$$\begin{aligned} &A_1(\mathrm{Re}(X) + i\,\mathrm{Im}(X)) + B_1(\mathrm{Re}(Y) + i\,\mathrm{Im}(Y)) \\ &\quad = D_1(\mathrm{Re}(X) - i\,\mathrm{Im}(X))\,E_1 + F_1, \\ &A_2(\mathrm{Re}(Y) + i\,\mathrm{Im}(Y)) + B_2(\mathrm{Re}(X) + i\,\mathrm{Im}(X)) \\ &\quad = D_2(\mathrm{Re}(Y) - i\,\mathrm{Im}(Y))\,E_2 + F_2,\end{aligned} \tag{51}$$

or

$$\begin{aligned} &A_1\mathrm{Re}(X) - D_1\mathrm{Re}(X)\,E_1 + iA_1\mathrm{Im}(X) \\ &\quad + iD_1\mathrm{Im}(X)\,E_1 + B_1\mathrm{Re}(Y) + iB_1\mathrm{Im}(Y) = F_1, \\ &A_2\mathrm{Re}(Y) - D_2\mathrm{Re}(Y)\,E_2 + iA_2\mathrm{Im}(Y) \\ &\quad + iD_2\mathrm{Im}(Y)\,E_2 + B_2\mathrm{Re}(X) + iB_2\mathrm{Im}(X) = F_2.\end{aligned} \tag{52}$$

This together with the definition of the Kronecker product yields

$$W\begin{pmatrix}\mathrm{vec}[\mathrm{Re}(X)] \\ \mathrm{vec}[\mathrm{Im}(X)] \\ \mathrm{vec}[\mathrm{Re}(Y)] \\ \mathrm{vec}[\mathrm{Im}(Y)]\end{pmatrix} = \begin{pmatrix}\mathrm{vec}(F_1) \\ \mathrm{vec}(F_2)\end{pmatrix}, \tag{53}$$

where

$$W = \begin{pmatrix} I\otimes A_1 - E_1^T\otimes D_1 & i\left(I\otimes A_1 + E_1^T\otimes D_1\right) & I\otimes B_1 & i\left(I\otimes B_1\right) \\ I\otimes B_2 & i\left(I\otimes B_2\right) & I\otimes A_2 - E_2^T\otimes D_2 & i\left(I\otimes A_2 + E_2^T\otimes D_2\right)\end{pmatrix}. \tag{54}$$

By some simple calculating, we have

$$W = \operatorname{Re}(W) + i \operatorname{Im}(W), \tag{55}$$

where

$$\operatorname{Re}(W) = \begin{pmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \end{pmatrix}, \tag{56}$$

$$\operatorname{Im}(W) = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \end{pmatrix}, \tag{57}$$

$$\begin{aligned}
N_{11} &= I \otimes \operatorname{Re}(A_1) - \operatorname{Re}(E_1)^T \otimes \operatorname{Re}(D_1) \\
&\quad + \operatorname{Im}(E_1)^T \otimes \operatorname{Im}(D_1), \\
N_{12} &= -I \otimes \operatorname{Im}(A_1) - \operatorname{Im}(E_1)^T \otimes \operatorname{Re}(D_1) \\
&\quad - \operatorname{Re}(E_1)^T \otimes \operatorname{Im}(D_1), \\
N_{13} &= I \otimes \operatorname{Re}(B_1), \qquad N_{14} = -I \otimes \operatorname{Im}(B_1), \\
N_{21} &= I \otimes \operatorname{Re}(B_2), \qquad N_{22} = -I \otimes \operatorname{Im}(B_2), \\
N_{23} &= I \otimes \operatorname{Re}(A_2) - \operatorname{Re}(E_2)^T \otimes \operatorname{Re}(D_2) \\
&\quad + \operatorname{Im}(E_2)^T \otimes \operatorname{Im}(D_2), \\
N_{24} &= -I \otimes \operatorname{Im}(A_2) - \operatorname{Im}(E_2)^T \otimes \operatorname{Re}(D_2) \\
&\quad - \operatorname{Re}(E_2)^T \otimes \operatorname{Im}(D_2), \\
L_{11} &= I \otimes \operatorname{Im}(A_1) - \operatorname{Im}(E_1)^T \otimes \operatorname{Re}(D_1) \\
&\quad - \operatorname{Re}(E_1)^T \otimes \operatorname{Im}(D_1), \\
L_{12} &= I \otimes \operatorname{Re}(A_1) + \operatorname{Re}(E_1)^T \otimes \operatorname{Re}(D_1) \\
&\quad - \operatorname{Im}(E_1)^T \otimes \operatorname{Im}(D_1), \\
L_{13} &= I \otimes \operatorname{Im}(B_1), \qquad L_{14} = I \otimes \operatorname{Re}(B_1), \\
L_{21} &= I \otimes \operatorname{Im}(B_2), \qquad L_{22} = I \otimes \operatorname{Re}(B_2), \\
L_{23} &= I \otimes \operatorname{Im}(A_2) - \operatorname{Im}(E_2)^T \otimes \operatorname{Re}(D_2) \\
&\quad - \operatorname{Re}(E_2)^T \otimes \operatorname{Im}(D_2), \\
L_{24} &= I \otimes \operatorname{Re}(A_2) + \operatorname{Re}(E_2)^T \otimes \operatorname{Re}(D_2) \\
&\quad - \operatorname{Im}(E_2)^T \otimes \operatorname{Im}(D_2).
\end{aligned} \tag{58}$$

Particularly, when $X \in CSC^{m\times m}$, X can be substituted with $(X + SXS)/2$ in (51); then we obtain

$$\overline{W} \cdot \begin{pmatrix} \operatorname{vec}[\operatorname{Re}(X)] \\ \operatorname{vec}[\operatorname{Im}(X)] \\ \operatorname{vec}[\operatorname{Re}(Y)] \\ \operatorname{vec}[\operatorname{Im}(Y)] \end{pmatrix} = \begin{pmatrix} \operatorname{vec}(F_1) \\ \operatorname{vec}(F_2) \end{pmatrix}, \tag{59}$$

where $\overline{W} := (1/2)[(I+K)(\operatorname{Re}(W)^T, (I+K)(\operatorname{Im}(W)^T)]^T$, $K := \operatorname{Diag}\{\overline{S}, \overline{S}, \overline{S}, \overline{S}\}$, $\overline{S} := S^T \otimes S = S \otimes S \in R^{m^2\times m^2}$ (see Definition 1).

Obviously, if $x \in \mathscr{R}(Z)$, then $x \in \mathscr{R}((1/2)Z)$, where Z is a matrix. So, from the above analysis, we can get the result.

Lemma 13. *Let $\overline{W} \in C^{4m^2\times 4m^2}$, $b \in C^{4m^2}$, and the linear matrix equation $\overline{W}x = b$ has a solution $x^* \in R^{4m^2}$, where $x = (\operatorname{vec}[\operatorname{Re}(X)]^T, \operatorname{vec}[\operatorname{Re}(Y)]^T, \operatorname{vec}[\operatorname{Im}(X)]^T, \operatorname{vec}[\operatorname{Im}(Y)]^T)^T$. If $x^* \in \mathscr{R}[(I+K)(\operatorname{Re}(W)^T, (I+K)\operatorname{Im}(W)^T)]$, then x^* is the unique least Frobenius norm solution of $\overline{W}x = b$ in* (59).

Theorem 14. *Suppose the system* (4) *is consistent; if one chooses the initial matrix pair*

$$\begin{aligned}
X_1 &= A_1^H U_1 + B_2^H V_1 - D_1^T \overline{U_1} E_1^T + S A_1^H U_1 S \\
&\quad + S B_2^H V_1 S - S D_1^T \overline{U_1} E_1^T S,
\end{aligned} \tag{60}$$

$$\begin{aligned}
Y_1 &= A_2^H V_1 + B_1^H U_1 - D_2^T \overline{V_1} E_2^T + S A_2^H V_1 S \\
&\quad + S B_1^H U_1 S - S D_2^T \overline{V_1} E_2^T S,
\end{aligned} \tag{61}$$

where $U_1, V_1 \in C^{p\times m}$ are two arbitrary matrices, especially, taking $X_1 = Y_1 = 0 \in R^{m\times m}$, the solution (X^*, Y^*) given by Algorithm 3 is the unique least Frobenius norm solution of (4).

Proof. If X_1, Y_1 has the form of (60), (61), respectively, by step 2 of Algorithm 3, we have

$$\begin{aligned}
X_2 &= X_1 + \alpha_1 M_1 = X_1 + \alpha_1 \left(\frac{1}{2} \left(\widetilde{R}_1^{(1)} + S \widetilde{R}_1^{(1)} S \right) \right) \\
&= A_1^H U_1 + B_2^H V_1 - D_1^T \overline{U_1} E_1^T + S A_1^H U_1 S \\
&\quad + S B_2^H V_1 S - S D_1^T \overline{U_1} E_1^T S \\
&\quad + \alpha_1 \left[\frac{1}{2} \left(A_1^H R_1^{(1)} + B_2^H R_1^{(2)} - D_1^T \overline{R_1^{(1)}} E_1^T + S A_1^H R_1^{(1)} S \right.\right. \\
&\qquad \left.\left. + S B_2^H R_1^{(2)} S - S D_1^T \overline{R_1^{(1)}} E_1^T S \right) \right] \\
&= A_1^H \left(U_1 + \frac{1}{2} \alpha_1 R_1^{(1)} \right) + B_2^H \left(V_1 + \frac{1}{2} \alpha_1 R_1^{(2)} \right) \\
&\quad - D_1^T \overline{\left(U_1 + \frac{1}{2} \alpha_1 R_1^{(1)} \right)} E_1^T + S A_1^H \left(U_1 + \frac{1}{2} \alpha_1 R_1^{(1)} \right) S \\
&\quad + S B_2^H \left(V_1 + \frac{1}{2} \alpha_1 R_1^{(2)} \right) S - S D_1^T \overline{\left(U_1 + \frac{1}{2} \alpha_1 R_1^{(1)} \right)} E_1^T S \\
&:= A_1^H U_2 + B_2^H V_2 - D_1^T \overline{U_2} E_1^T + S A_1^H U_2 S \\
&\quad + S B_2^H V_2 S - S D_1^T \overline{U_2} E_1^T S,
\end{aligned}$$

$$\begin{aligned} Y_2 &= Y_1 + \alpha_1 N_1 = Y_1 + \alpha_1 \left(\frac{1}{2} \left(\widetilde{R}_1^{(2)} + S\widetilde{R}_1^{(2)} S \right) \right) \\ &= A_2^H V_1 + B_1^H U_1 - D_2^T \overline{V_1} E_2^T + S A_2^H V_1 S \\ &\quad + S B_1^H U_1 S - S D_2^T \overline{V_1} E_2^T S \\ &\quad + \alpha_1 \left[\frac{1}{2} \left(A_2^H R_1^{(2)} + B_1^H R_1^{(1)} - D_2^T \overline{R_1^{(2)}} E_2^T + S A_2^H R_1^{(2)} S \right.\right. \\ &\qquad \left.\left. + S B_1^H R_1^{(1)} S - S D_2^T \overline{R_1^{(2)}} E_2^T S \right) \right] \\ &= A_2^H \left(V_1 + \frac{1}{2}\alpha_1 R_1^{(2)} \right) + B_1^H \left(U_1 + \frac{1}{2}\alpha_1 R_1^{(1)} \right) \\ &\quad - D_2^T \left(\overline{V_1 + \frac{1}{2}\alpha_1 R_1^{(2)}} \right) E_2^T + S A_2^H \left(V_1 + \frac{1}{2}\alpha_1 R_1^{(2)} \right) S \\ &\quad + S B_1^H \left(U_1 + \frac{1}{2}\alpha_1 R_1^{(1)} \right) S - S D_2^T \left(\overline{V_1 + \frac{1}{2}\alpha_1 R_1^{(2)}} \right) E_2^T S \\ &:= A_2^H V_2 + B_1^H U_2 - D_2^T \overline{V_2} E_2^T + S A_2^H V_2 S \\ &\quad + S B_1^H U_2 S - S D_2^T \overline{V_2} E_2^T S, \end{aligned} \tag{62}$$

where $U_2 := U_1 + (1/2)\alpha_1 R_1^{(1)}$ and $V_2 := V_1 + (1/2)\alpha_1 R_1^{(2)}$. Since

$$\begin{aligned} M_2 &= \frac{1}{2} \left(\widetilde{R}_2^{(1)} + S\widetilde{R}_2^{(1)} S \right) + \beta_1 M_1 \\ &= \frac{1}{2} \left(A_1^H R_2^{(1)} + B_2^H R_2^{(2)} - D_1^T \overline{R_2^{(1)}} E_1^T + S A_1^H R_2^{(1)} S \right. \\ &\qquad \left. + S B_2^H R_2^{(2)} S - S D_1^T \overline{R_2^{(1)}} E_1^T S \right) \\ &\quad + \frac{\beta_1}{2} \left(A_1^H R_1^{(1)} + B_2^H R_1^{(2)} - D_1^T \overline{R_1^{(1)}} E_1^T + S A_1^H R_1^{(1)} S \right. \\ &\qquad \left. + S B_2^H R_1^{(2)} S - S D_1^T \overline{R_1^{(1)}} E_1^T S \right), \\ N_2 &= \frac{1}{2} \left(\widetilde{R}_2^{(2)} + S\widetilde{R}_2^{(2)} S \right) + \beta_1 N_1 \\ &= \frac{1}{2} \left(A_1^H R_2^{(2)} + B_2^H R_2^{(1)} - D_1^T \overline{R_2^{(2)}} E_1^T + S A_1^H R_2^{(2)} S \right. \\ &\qquad \left. + S B_2^H R_2^{(1)} S - S D_1^T \overline{R_2^{(2)}} E_1^T S \right) \\ &\quad + \frac{\beta_1}{2} \left(A_1^H R_1^{(2)} + B_2^H R_1^{(1)} - D_1^T \overline{R_1^{(2)}} E_1^T + S A_1^H R_1^{(2)} S \right. \\ &\qquad \left. + S B_2^H R_1^{(1)} S - S D_1^T \overline{R_1^{(2)}} E_1^T S \right), \end{aligned} \tag{63}$$

we have

$$\begin{aligned} X_3 &= X_2 + \alpha_2 M_2 \\ &= A_1^H U_3 + B_2^H V_3 - D_1^T \overline{U_3} E_1^T + S A_1^H U_3 S + S B_2^H V_3 S \\ &\quad - S D_1^T \overline{U_3} E_1^T S, \\ Y_3 &= Y_2 + \alpha_2 N_2 \\ &= A_2^H V_3 + B_1^H U_3 - D_2^T \overline{V_3} E_2^T + S A_2^H V_3 S + S B_1^H U_3 S \\ &\quad - S D_2^T \overline{V_3} E_2^T S, \end{aligned} \tag{64}$$

where $U_3 := U_2 + (\alpha_2/2)R_2^{(1)} + (\beta_1\alpha_2/2)R_1^{(1)}$ and $V_3 := V_2 + (\alpha_2/2)R_2^{(2)} + (\beta_1\alpha_2/2)R_1^{(2)}$. By parity of reasoning, we can prove that

$$\begin{aligned} X_{k+1} &= A_1^H U_{k+1} + B_2^H V_{k+1} - D_1^T \overline{U_{k+1}} E_1^T \\ &\quad + S A_1^H U_{k+1} S + S B_2^H V_{k+1} S - S D_1^T \overline{U_{k+1}} E_1^T S, \\ Y_{k+1} &= A_2^H V_{k+1} + B_1^H U_{k+1} - D_2^T \overline{V_{k+1}} E_2^T \\ &\quad + S A_2^H V_{k+1} S + S B_1^H U_{k+1} S - S D_2^T \overline{V_{k+1}} E_2^T S, \end{aligned} \tag{65}$$

where $U_{k+1}, V_{k+1} \in C^{p\times m}$. This together with Theorem 10 yields that

$$\begin{aligned} X_{k+1} \longrightarrow X^* &= A_1^H \widehat{U} + B_2^H \widehat{V} - D_1^T \overline{\widehat{U}} E_1^T + S A_1^H \widehat{U} S \\ &\quad + S B_2^H \widehat{V} S - S D_1^T \overline{\widehat{U}} E_1^T S \quad (k \longrightarrow \infty), \\ Y_{k+1} \longrightarrow Y^* &= A_2^H \widehat{V} + B_1^H \widehat{U} - D_2^T \overline{\widehat{V}} E_2^T + S A_2^H \widehat{V} S \\ &\quad + S B_1^H \widehat{U} S - S D_2^T \overline{\widehat{V}} E_2^T S \quad (k \longrightarrow \infty), \end{aligned} \tag{66}$$

where $\widehat{U}, \widehat{V} \in R^{p\times m}$. Since

$$\begin{aligned} X^* &= A_1^H \widehat{U} + B_2^H \widehat{V} - D_1^T \overline{\widehat{U}} E_1^T + S A_1^H \widehat{U} S \\ &\quad + S B_2^H \widehat{V} S - S D_1^T \overline{\widehat{U}} E_1^T S \\ &= \left[\operatorname{Re}(A_1) + i \operatorname{Im}(A_1) \right]^H \left[\operatorname{Re}(\widehat{U}) + i \operatorname{Im}(\widehat{U}) \right] \\ &\quad + \left[\operatorname{Re}(B_2) + i \operatorname{Im}(B_2) \right]^H \left[\operatorname{Re}(\widehat{V}) + i \operatorname{Im}(\widehat{V}) \right] \\ &\quad - \left[\operatorname{Re}(D_1) + i \operatorname{Im}(D_1) \right]^T \left[\overline{\operatorname{Re}(\widehat{U}) + i \operatorname{Im}(\widehat{U})} \right] \\ &\quad \times \left[\operatorname{Re}(E_1) + i \operatorname{Im}(E_1) \right]^T + S \left[\operatorname{Re}(A_1) + i \operatorname{Im}(A_1) \right]^H \\ &\quad \times \left[\operatorname{Re}(\widehat{U}) + i \operatorname{Im}(\widehat{U}) \right] S + S \left[\operatorname{Re}(B_2) + i \operatorname{Im}(B_2) \right]^H \\ &\quad \times \left[\operatorname{Re}(\widehat{V}) + i \operatorname{Im}(\widehat{V}) \right] S - S \left[\operatorname{Re}(D_1) + i \operatorname{Im}(D_1) \right]^T \\ &\quad \times \left[\overline{\operatorname{Re}(\widehat{U}) + i \operatorname{Im}(\widehat{U})} \right] \left[\operatorname{Re}(E_1) + i \operatorname{Im}(E_1) \right]^T S \end{aligned}$$

$$
\begin{aligned}
&= \left[\operatorname{Re}(A_1)^T - i\operatorname{Im}(A_1)^T\right]\left[\operatorname{Re}(\widehat{U}) + i\operatorname{Im}(\widehat{U})\right] \\
&\quad + \left[\operatorname{Re}(B_2)^T - i\operatorname{Im}(B_2)^T\right]\left[\operatorname{Re}(\widehat{V}) + i\operatorname{Im}(\widehat{V})\right] \\
&\quad - \left[\operatorname{Re}(D_1)^T + i\operatorname{Im}(D_1)^T\right]\left[\operatorname{Re}(\widehat{U}) - i\operatorname{Im}(\widehat{U})\right] \\
&\quad \times \left[\operatorname{Re}(E_1)^T + i\operatorname{Im}(E_1)^T\right] + S\left[\operatorname{Re}(A_1)^T - i\operatorname{Im}(A_1)^T\right] \\
&\quad \times \left[\operatorname{Re}(\widehat{U}) + i\operatorname{Im}(\widehat{U})\right] S + S\left[\operatorname{Re}(B_2)^T - i\operatorname{Im}(B_2)^T\right] \\
&\quad \times \left[\operatorname{Re}(\widehat{V}) + i\operatorname{Im}(\widehat{V})\right] S - S\left[\operatorname{Re}(D_1)^T + i\operatorname{Im}(D_1)^T\right] \\
&\quad \times \left[\operatorname{Re}(\widehat{U}) - i\operatorname{Im}(\widehat{U})\right]\left[\operatorname{Re}(E_1)^T + i\operatorname{Im}(E_1)^T\right] S \\
&= \operatorname{Re}(A_1)^T \operatorname{Re}(\widehat{U}) + i\operatorname{Re}(A_1)^T \operatorname{Im}(\widehat{U}) \\
&\quad - i\operatorname{Im}(A_1)^T \operatorname{Re}(\widehat{U}) + \operatorname{Im}(A_1)^T \operatorname{Im}(\widehat{U}) \\
&\quad + \operatorname{Re}(B_2)^T \operatorname{Re}(\widehat{V}) + i\operatorname{Re}(B_2)^T \operatorname{Im}(\widehat{V}) \\
&\quad - i\operatorname{Im}(B_2)^T \operatorname{Re}(\widehat{V}) + \operatorname{Im}(B_2)^T \operatorname{Im}(\widehat{V}) \\
&\quad - \operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad - i\operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad + i\operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad - \operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad - i\operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad + \operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad - \operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad - i\operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad + S\operatorname{Re}(A_1)^T \operatorname{Re}(\widehat{U}) S + iS\operatorname{Re}(A_1)^T \operatorname{Im}(\widehat{U}) S \\
&\quad - iS\operatorname{Im}(A_1)^T \operatorname{Re}(\widehat{U}) S + S\operatorname{Im}(A_1)^T \operatorname{Im}(\widehat{U}) S \\
&\quad + S\operatorname{Re}(B_2)^T \operatorname{Re}(\widehat{V}) S + iS\operatorname{Re}(B_2)^T \operatorname{Im}(\widehat{V}) S \\
&\quad - iS\operatorname{Im}(B_2)^T \operatorname{Re}(\widehat{V}) S + S\operatorname{Im}(B_2)^T \operatorname{Im}(\widehat{V}) S \\
&\quad - S\operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T S \\
&\quad - iS\operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T S \\
&\quad + iS\operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T S \\
&\quad - S\operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T S \\
&\quad - iS\operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T S \\
&\quad + S\operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T S \\
&\quad - S\operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T S \\
&\quad - iS\operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T S,
\end{aligned}
\tag{67}
$$

we have

$$
\begin{aligned}
\operatorname{Re}(X^*) &= \operatorname{Re}(A_1)^T \operatorname{Re}(\widehat{U}) + \operatorname{Im}(A_1)^T \operatorname{Im}(\widehat{U}) \\
&\quad + \operatorname{Re}(B_2)^T \operatorname{Re}(\widehat{V}) + \operatorname{Im}(B_2)^T \operatorname{Im}(\widehat{V}) \\
&\quad - \operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad - \operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad + \operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad - \operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad + S\operatorname{Re}(A_1)^T \operatorname{Re}(\widehat{U}) S + S\operatorname{Im}(A_1)^T \operatorname{Im}(\widehat{U}) S \\
&\quad + S\operatorname{Re}(B_2)^T \operatorname{Re}(\widehat{V}) S + S\operatorname{Im}(B_2)^T \operatorname{Im}(\widehat{V}) S \\
&\quad - S\operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T S \\
&\quad - S\operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T S \\
&\quad + S\operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T S \\
&\quad - S\operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T S, \\
\operatorname{Im}(X^*) &= \operatorname{Re}(A_1)^T \operatorname{Im}(\widehat{U}) - \operatorname{Im}(A_1)^T \operatorname{Re}(\widehat{U}) \\
&\quad + \operatorname{Re}(B_2)^T \operatorname{Im}(\widehat{V}) - \operatorname{Im}(B_2)^T \operatorname{Re}(\widehat{V}) \\
&\quad - \operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad + \operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad - \operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T \\
&\quad - \operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T \\
&\quad + S\operatorname{Re}(A_1)^T \operatorname{Im}(\widehat{U}) S - S\operatorname{Im}(A_1)^T \operatorname{Re}(\widehat{U}) S \\
&\quad + S\operatorname{Re}(B_2)^T \operatorname{Im}(\widehat{V}) S - S\operatorname{Im}(B_2)^T \operatorname{Re}(\widehat{V}) S \\
&\quad - S\operatorname{Re}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Im}(E_1)^T S \\
&\quad + S\operatorname{Re}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Re}(E_1)^T S \\
&\quad - S\operatorname{Im}(D_1)^T \operatorname{Re}(\widehat{U}) \operatorname{Re}(E_1)^T S \\
&\quad - S\operatorname{Im}(D_1)^T \operatorname{Im}(\widehat{U}) \operatorname{Im}(E_1)^T S.
\end{aligned}
\tag{68}
$$

Let $\gamma = (\text{vec}[\text{Re}(\widehat{U})]^T, \text{vec}[\text{Re}(\widehat{V})]^T, \text{vec}[\text{Im}(\widehat{U})]^T, \text{vec}[\text{Im}(\widehat{V})]^T)^T$ and K in (59); using Kronecker product, we get

$$\begin{aligned}
&\text{vec}[\text{Re}(X^*)] \\
&\quad = (I+K) \\
&\quad\times \begin{pmatrix} I\otimes\text{Re}(A_1) - \text{Re}(E_1)^T\otimes\text{Re}(D_1) + \text{Im}(E_1)^T\otimes\text{Im}(D_1) \\ I\otimes\text{Re}(B_2) \\ I\otimes\text{Im}(A_1) - \text{Im}(E_1)^T\otimes\text{Re}(D_1) - \text{Re}(E_1)^T\otimes\text{Im}(D_1) \\ I\otimes\text{Im}(B_2) \end{pmatrix}^T \\
&\quad\cdot \begin{pmatrix} \text{vec}[\text{Re}(\widehat{U})] \\ \text{vec}[\text{Re}(\widehat{V})] \\ \text{vec}[\text{Im}(\widehat{U})] \\ \text{vec}[\text{Im}(\widehat{V})] \end{pmatrix} = (I+K)\cdot\left(N_{11}^T, N_{21}^T, L_{11}^T, L_{21}^T\right)\gamma, \\
&\text{vec}[\text{Im}(X^*)] \\
&\quad = (I+K) \\
&\quad\times \begin{pmatrix} -I\otimes\text{Im}(A_1) - \text{Im}(E_1)^T\otimes\text{Re}(D_1) - \text{Re}(E_1)^T\otimes\text{Im}(D_1) \\ -I\otimes\text{Im}(B_2) \\ I\otimes\text{Re}(A_1) + \text{Re}(E_1)^T\otimes\text{Re}(D_1) - \text{Im}(E_1)^T\otimes\text{Im}(D_1) \\ I\otimes\text{Re}(B_2) \end{pmatrix}^T \\
&\quad\cdot \begin{pmatrix} \text{vec}[\text{Re}(\widehat{U})] \\ \text{vec}[\text{Re}(\widehat{V})] \\ \text{vec}[\text{Im}(\widehat{U})] \\ \text{vec}[\text{Im}(\widehat{V})] \end{pmatrix} = (I+K)\cdot\left(N_{12}^T, N_{22}^T, L_{12}^T, L_{22}^T\right)\gamma.
\end{aligned} \tag{69}$$

In the same way, we can prove that

$$\begin{aligned}
&\text{vec}[\text{Re}(Y^*)] \\
&\quad = (I+K) \\
&\quad\times \begin{pmatrix} I\otimes\text{Re}(B_1) \\ I\otimes\text{Re}(A_2) - \text{Re}(E_2)^T\otimes\text{Re}(D_2) + \text{Im}(E_2)^T\otimes\text{Im}(D_2) \\ I\otimes\text{Im}(B_1) \\ I\otimes\text{Im}(A_2) - \text{Im}(E_2)^T\otimes\text{Re}(D_2) - \text{Re}(E_2)^T\otimes\text{Im}(D_2) \end{pmatrix}^T \\
&\quad\cdot \begin{pmatrix} \text{vec}[\text{Re}(\widehat{U})] \\ \text{vec}[\text{Re}(\widehat{V})] \\ \text{vec}[\text{Im}(\widehat{U})] \\ \text{vec}[\text{Im}(\widehat{V})] \end{pmatrix} = (I+K)\cdot\left(N_{13}^T, N_{23}^T, L_{13}^T, L_{23}^T\right)\gamma, \\
&\text{vec}[\text{Im}(Y^*)] \\
&\quad = (I+K) \\
&\quad\times \begin{pmatrix} -I\otimes\text{Im}(B_1) \\ -I\otimes\text{Im}(A_2) - \text{Im}(E_2)^T\otimes\text{Re}(D_2) - \text{Re}(E_2)^T\otimes\text{Im}(D_2) \\ I\otimes\text{Re}(B_1) \\ I\otimes\text{Re}(A_2) + \text{Re}(E_2)^T\otimes\text{Re}(D_2) - \text{Im}(E_2)^T\otimes\text{Im}(D_2) \end{pmatrix}^T \\
&\quad\cdot \begin{pmatrix} \text{vec}[\text{Re}(\widehat{U})] \\ \text{vec}[\text{Re}(\widehat{V})] \\ \text{vec}[\text{Im}(\widehat{U})] \\ \text{vec}[\text{Im}(\widehat{V})] \end{pmatrix} = (I+K)\cdot\left(N_{14}^T, N_{24}^T, L_{14}^T, L_{24}^T\right)\gamma.
\end{aligned} \tag{70}$$

This shows that

$$\begin{aligned}
&\begin{pmatrix} \text{vec}[\text{Re}(X^*)] \\ \text{vec}[\text{Im}(X^*)] \\ \text{vec}[\text{Re}(Y^*)] \\ \text{vec}[\text{Im}(Y^*)] \end{pmatrix} \\
&\quad = (I+K)\begin{pmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \end{pmatrix}^T \gamma \\
&\quad = (I+K)\begin{pmatrix} \text{Re}(W) \\ \text{Im}(W) \end{pmatrix}^T \gamma = \begin{pmatrix} \text{Re}(W)(I+K) \\ \text{Im}(W)(I+K) \end{pmatrix}^T \gamma.
\end{aligned} \tag{71}$$

Notice that $X^*, Y^* \in CMC^{m\times m}$ and $K^T = K$ in (59). By (56) and (57), we have

$$\begin{pmatrix} \text{vec}[\text{Re}(X^*)] \\ \text{vec}[\text{Im}(X^*)] \\ \text{vec}[\text{Re}(Y^*)] \\ \text{vec}[\text{Im}(Y^*)] \end{pmatrix} \in \mathscr{R}\left[(I+K)\,\text{Re}(W)^T,\right. \\ \left.(I+K)\,\text{Im}(W)^T\right]. \tag{72}$$

It follows from Lemma 11, Lemma 12, and (59) that

$$\begin{aligned}
&\|\text{vec}[\text{Re}(X^*)]\|^2 + \|\text{vec}[\text{Im}(X^*)]\|^2 \\
&\quad + \|\text{vec}[\text{Re}(Y^*)]\|^2 + \|\text{vec}[\text{Im}(Y^*)]\|^2 \le \|\bar{\xi}\|^2,
\end{aligned} \tag{73}$$

where

$$\bar{\xi} = \begin{pmatrix} \text{vec}[\text{Re}(\overline{X})] \\ \text{vec}[\text{Im}(\overline{X})] \\ \text{vec}[\text{Re}(\overline{Y})] \\ \text{vec}[\text{Im}(\overline{Y})] \end{pmatrix} \in R^{4m^2} \tag{74}$$

for all solutions $(\overline{X}, \overline{Y})$ of (4). Since

$$\begin{aligned}
&\|\text{vec}[\text{Re}(X^*)]\|^2 + \|\text{vec}[\text{Im}(X^*)]\|^2 \\
&\quad = \|\text{Re}(X^*)\|_F^2 + \|\text{Im}(X^*)\|_F^2 = \|X^*\|^2, \\
&\|\text{vec}[\text{Re}(Y^*)]\|^2 + \|\text{vec}[\text{Im}(Y^*)]\|^2 \\
&\quad = \|\text{Re}(Y^*)\|_F^2 + \|\text{Im}(Y^*)\|_F^2 = \|Y^*\|^2,
\end{aligned} \tag{75}$$

we obtain

$$\begin{aligned}
&\|\text{vec}[\text{Re}(X^*)]\|^2 + \|\text{vec}[\text{Im}(X^*)]\|^2 \\
&\quad + \|\text{vec}[\text{Re}(Y^*)]\|^2 + \|\text{vec}[\text{Im}(Y^*)]\|^2 \\
&\quad = \|X^*\|^2 + \|Y^*\|^2 \le \|\overline{X}\|^2 + \|\overline{Y}\|^2,
\end{aligned} \tag{76}$$

for all solutions $(\overline{X}, \overline{Y})$ of (4). Hence (X^*, Y^*) is the unique least Frobenius norm solution of (4). □

3. Numerical Experiments

In this section, we report some numerical results to support our Algorithm 3. The iterations have been carried out by MATLAB R2011b (7.13), Intel(R) Core(TM) i7-2670QM, CPU 2.20 GHZ, RAM 8.GB PC Environment.

Example 1. We consider (4) with the following matrices:

$$A1=\begin{pmatrix}1-i & 1+i & 2-2i & -3-i & 4+i\\ 3+2i & 4-3i & 2-i & 2+i & 1-2i\\ 0+i & 4+i & 7-i & 2-i & 4+2i\\ -1+i & -1-i & -1+i & 2-i & 4+i\\ 4+i & 4-i & 3+i & 2-i & 1+2i\end{pmatrix},\quad A2=\begin{pmatrix}2-i & 2+i & 3-i & 1+i & 1+i\\ 0-2i & 5-i & 4+i & -2+i & -2+i\\ 2+i & 3-i & 4+i & 1+i & 1+i\\ 2-i & 0+i & 2-2i & 0-i & 1+i\\ -3-i & -3+i & 1+i & 2-i & 2+i\end{pmatrix},$$

$$B1=\begin{pmatrix}4+i & 3+i & 4-i & 4-i & 1+i\\ -2-i & -2+i & 3+i & 4-3i & 4-i\\ 5+i & 6-i & 5+i & 0+i & 1-i\\ 5-i & 4-i & 5+i & 3-i & 3-3i\\ 1-2i & 2-2i & 0+i & 0-i & 1-i\end{pmatrix},\quad B2=\begin{pmatrix}-2-i & -i & -2+i & 3+i & 2-i\\ 6-i & 5-i & 4+i & 4-i & 3-i\\ 2-i & 3+i & 2-i & 1+i & 1-i\\ 1+3i & 1-2i & 2-i & 4+i & 1-3i\\ 0-i & 0+i & 2-i & 3-i & 2+i\end{pmatrix},$$

$$D1=\begin{pmatrix}1-i & 2+i & 3-i & 1-i & 2-i\\ 0+i & 1+i & 2-i & 3+i & 1-3i\\ 4-i & 4-i & 2-i & 1+i & 3+i\\ 1-i & 0-i & 0-i & -1-i & i\\ 2-i & 4+i & 5+i & 3+i & 2-i\end{pmatrix},\quad D2=\begin{pmatrix}1+i & 2-i & 3+i & -1+i & -1+i\\ 3-i & 0-i & 0+i & 3-i & 3-i\\ 1-i & 2+i & 3-3i & 0+2i & -2-3i\\ -1+i & -1-i & 2+2i & 2-2i & 3+i\\ 5-i & 4-i & 5+i & 4+i & 4-4i\end{pmatrix},$$

$$E1=\begin{pmatrix}1-i & 2-i & 1-i & 2+i & 1-i\\ 3+i & 3-i & 3+i & 1-i & 2+i\\ 1-i & 2-i & 3-i & -4-i & -4+i\\ 5-i & 5-i & 5+i & 4+i & 4-i\\ 2-i & -2+i & -2+i & -2+i & 1-i\end{pmatrix},\quad E2=\begin{pmatrix}2+i & 4+i & 3+i & 2-i & 1-i\\ 0+i & -3+i & -3+i & -3-i & 2-i\\ 1+i & 1+i & 1-i & 0-i & 0-i\\ 1-i & 1+i & 2+i & 3-i & 4-i\\ 2-i & 3+i & 4+i & 5-i & 2+i\end{pmatrix},\tag{77}$$

$$F1=10^2\cdot\begin{pmatrix}-1.2100+1.1500i & -1.1100+0.8000i & -1.5500+0.2300i & -0.2800+0.2000i & -0.4300+0.9500i\\ -0.9400+1.5300i & -1.1200+0.6000i & -1.5900-0.1400i & -0.8200+0.4400i & -0.3700+1.0300i\\ -2.5200+1.5700i & -2.1900+1.4200i & -2.6600-0.1500i & -0.4200+0.2800i & -0.5900+1.3200i\\ 0.5100+0.1500i & 0.6800+0.2500i & 0.3100+0.3300i & 0.6700-0.1500i & 0.4300-0.0500i\\ -2.5700+1.7800i & -2.2700+1.1300i & -2.6400-0.5200i & -1.0900+0.5600i & -0.8500+1.6700i\end{pmatrix},$$

$$F2=10^2\cdot\begin{pmatrix}-0.0900-0.0500i & -0.4400-0.2700i & -0.6500-0.2700i & -1.1700+0.1100i & -0.8100-0.3600i\\ -0.6700-0.0200i & -0.0300-0.1700i & -0.1200-0.2400i & 0.2800+0.7800i & -0.7800+0.9300i\\ -0.1700+0.7100i & -0.4400+0.2500i & -0.6800+0.6800i & 0.0100+1.3100i & -0.7000+1.0900i\\ -0.3000-0.2400i & 0.3100-0.1300i & 0.2300-0.5800i & -0.0600-0.2400i & -0.0700-0.3600i\\ -1.5800+0.2400i & -1.4700-0.8400i & -2.0700-0.2300i & -2.2000+2.6500i & -3.6200+0.9700i\end{pmatrix}.$$

We can show that the matrix equation (4) is consistent. Choose the initial matrix pair $X_1=0, Y_1=0$; by Algorithm 3, we obtain its exact solution:

$$X_{116}=\begin{pmatrix}2.0000+1.0000i & 3.0000-1.0000i & 1.0000+1.0000i & 3.0000+1.0000i & 1.0000-1.0000i\\ -1.0000+2.0000i & 0.0000+1.0000i & 2.0000-1.0000i & 3.0000+2.0000i & 1.0000+2.0000i\\ 1.0000+1.0000i & 1.0000-1.0000i & -0.0000-1.0000i & 1.0000-1.0000i & 1.0000+1.0000i\\ 1.0000+2.0000i & 3.0000+2.0000i & 2.0000-1.0000i & 0.0000+1.0000i & -1.0000+2.0000i\\ 1.0000-1.0000i & 3.0000+1.0000i & 1.0000+1.0000i & 3.0000-1.0000i & 2.0000+1.0000i\end{pmatrix},$$

$$Y_{116}=\begin{pmatrix}2.0000-1.0000i & 1.0000-1.0000i & 1.0000-1.0000i & 3.0000-1.0000i & 1.0000+1.0000i\\ -1.0000-2.0000i & 0.0000+1.0000i & 2.0000+1.0000i & 3.0000-2.0000i & 2.0000+2.0000i\\ 2.0000+1.0000i & 1.0000-1.0000i & 0.0000-1.0000i & 1.0000-1.0000i & 2.0000+1.0000i\\ 2.0000+2.0000i & 3.0000-2.0000i & 2.0000+1.0000i & 0.0000+1.0000i & -1.0000-2.0000i\\ 1.0000+1.0000i & 3.0000-1.0000i & 1.0000-1.0000i & 1.0000-1.0000i & 2.0000-1.0000i\end{pmatrix}.\tag{78}$$

The corresponding residual and least Frobenius norm are

$$\|R_{116}\|_F = 4.6203 \times 10^{-11}, \qquad \|X_{116}\|_F = 9.1726, \qquad \|Y_{116}\|_F = 7.5890. \tag{79}$$

The results above are presented in Figure 1, where $r_k = \|R_k\|$, $\delta_k = (\|X_k - X^*\| + \|Y_k - Y^*\|)/(\|X^*\| + \|Y^*\|)$.

Example 2. Consider (4) with the following matrices:

$$A1 = \begin{pmatrix} 5+6i & 3+i & 3-5i \\ 3-6i & 2-6i & 7+3i \\ 0+6i & 1+7i & 3+6i \end{pmatrix}, \qquad A2 = \begin{pmatrix} 5-6*i & 2-3i & 3+2i \\ 6+2*i & 3-i & 3+5i \\ 3-6*i & 2+3i & 5+4i \\ 0+3*i & 3-4i & 3+2i \end{pmatrix},$$

$$B1 = \begin{pmatrix} 3+2i & 2i & 4+2i \\ 5-3i & 3+i & 2+5i \\ 3+6i & 2-6i & 6+3i \\ 0+6i & 1+7i & 3+6i \end{pmatrix}, \qquad B2 = \begin{pmatrix} 3+2i & 2+3i & 4+2i \\ 3-3i & 3-1i & 4+5i \\ 3+2i & 2-6i & 7+3i \\ 0-6i & 1+7i & 3+6i \end{pmatrix},$$

$$D1 = \begin{pmatrix} 5+6i & 2+3i & 6+2i \\ 2+3i & 3+1i & 3+5i \\ 3-2i & 2-6i & 5-3i \\ 0+6i & 1+i & 3+6i \end{pmatrix}, \qquad D2 = \begin{pmatrix} 5+3i & 2+3i & 6+2i \\ 5+3i & 3-1i & 3-5i \\ 3+4i & 2-3i & 4+3i \\ 0-6i & 1+6i & 3+6i \end{pmatrix},$$

$$E1 = \begin{pmatrix} 5+6i & 2+3i & 6+1i \\ 5-3i & 3-1i & 2-5i \\ 2+6i & 2-5i & 8-3i \end{pmatrix}, \qquad E2 = \begin{pmatrix} 8-6i & 1+3i & 4+1i \\ 5-3i & 4-1i & 2+2i \\ 2+8i & 1+3i & 9-2i \end{pmatrix}, \tag{80}$$

$$F1 = 10^2 \cdot \begin{pmatrix} 0.4500 - 5.6000i & -1.0300 - 1.6100i & -4.4300 - 2.9100i \\ 0.9100 - 3.3300i & -0.3400 - 0.3800i & -2.2400 - 1.8000i \\ -3.6300 + 0.4600i & 0.5000 + 0.8100i & -0.6500 + 4.2400i \\ 2.8300 - 2.5900i & 0.4400 - 0.6200i & -1.2100 - 2.8100i \end{pmatrix},$$

$$F2 = 10^2 \cdot \begin{pmatrix} -4.8800 - 3.5100i & -0.3100 - 4.0900i & -4.4000 - 3.8400i \\ -2.5500 + 2.3800i & -1.6500 - 1.8100i & -4.2900 - 0.3900i \\ -3.0200 - 2.8300i & 0.5300 - 2.9700i & -3.5600 - 3.7000i \\ -3.4400 - 3.8200i & 0.0100 - 1.3000i & -0.2200 + 0.2600i \end{pmatrix}.$$

We can show that the matrix equation (4) is also consistent. Choose the initial matrix pair $X_1 = 0$, $Y_1 = 0$; from Algorithm 3, we obtain its exact solution only the 29th iterative step:

$$X_{29} = \begin{pmatrix} 1.0000 + 1.0000i & -0.0000 - 2.0000i & 2.0000 - 1.0000i \\ 3.0000 + 1.0000i & 2.0000 + 2.0000i & 3.0000 + 1.0000i \\ 2.0000 - 1.0000i & -0.0000 - 2.0000i & 1.0000 + 1.0000i \end{pmatrix},$$
$$Y_{29} = \begin{pmatrix} 2.0000 + 1.0000i & 2.0000 - 1.0000i & 5.0000 - 1.0000i \\ 1.0000 - 1.0000i & 3.0000 - 1.0000i & 1.0000 - 1.0000i \\ 5.0000 - 1.0000i & 2.0000 - 1.0000i & 2.0000 + 1.0000i \end{pmatrix}. \tag{81}$$

The corresponding residual and least Frobenius norm are

$$\|R_{29}\|_F = 9.9651 \times 10^{-13}, \qquad \|X_{29}\|_F = 5.9876, \qquad \|Y_{29}\|_F = 8.2249. \tag{82}$$

The above results are presented in Figure 2, where $r_k = \|R_k\|$, $\delta_k = (\|X_k - X^*\| + \|Y_k - Y^*\|)/(\|X^*\| + \|Y^*\|)$.

4. Conclusion

Iterative method is proposed to solve the generalized coupled Sylvester-conjugate linear matrix equations $A_1X + B_1Y = D_1\overline{X}E_1 + F_1$, $A_2Y + B_2X = D_2\overline{Y}E_2 + F_2$ for center-symmetry (center-antisymmetry) matrix pair (X, Y). When (4) is consistent, utilizing the Kronecker product, it has been revealed that an exact solution can be obtained by the proposed algorithm within finite iterative steps in the absence of round-off error for any initial value chosen center-symmetry (center-antisymmetry) matrix. Furthermore, we show that the least Frobenius norm solution is obtained by choosing a special kind of initial matrix. Finally, some numerical examples were given to show the efficiency for the proposed method.

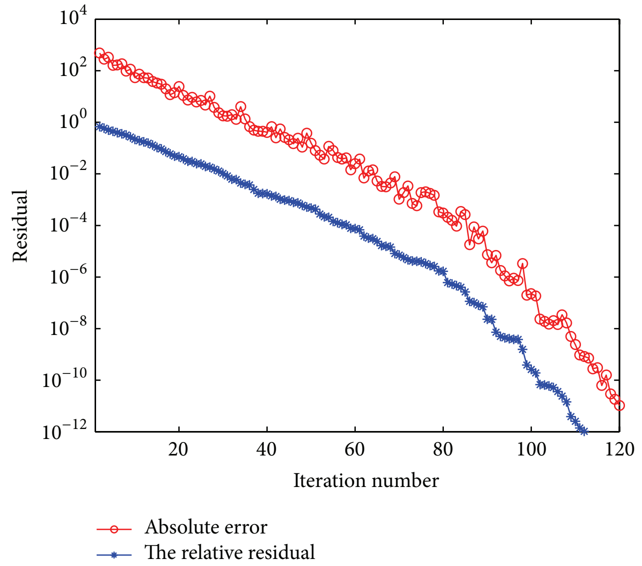

Figure 1: The relative error of solution and the residual for Example 1.

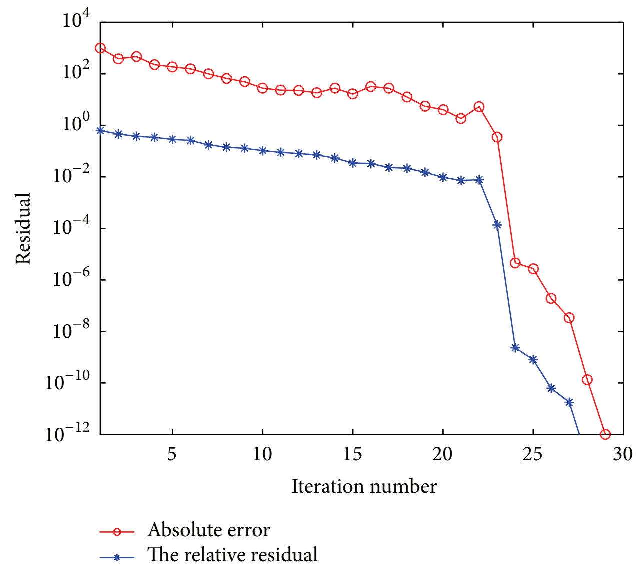

Figure 2: The relative error of solution and the residual for Example 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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