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Research Article

Existence and Multiple Positive Solutions for Boundary Value Problem of Fractional Differential Equation with p-Laplacian Operator

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This paper investigates the existence, multiplicity, nonexistence, and uniqueness of positive solutions to a kind of two-point boundary value problem for nonlinear fractional differential equations with p-Laplacian operator. By using fixed point techniques combining with partially ordered structure of Banach space, we establish some criteria for existence and uniqueness of positive solution of fractional differential equations with p-Laplacian operator in terms of different value of parameter. In particular, the dependence of positive solution on the parameter was obtained. Finally, several illustrative examples are given to support the obtained new results. The study of illustrative examples shows that the obtained results are applicable.

1. Introduction

In this paper, we consider boundary value problem for a fractional differential equation with p-Laplacian operator

$$\begin{aligned} &\left(\phi_p\left(D_{0+}^{\alpha}x(t)\right)\right)' + \lambda\omega(t) f(t, x(t)) = 0, \quad 0 < t < 1, \\ &x'(0) = x''(0) = 0, \\ &x(1) = \eta\int_0^1 g(s)\,x(s)\,ds, \end{aligned} \tag{1}$$

where $2 < \alpha \le 3, 0 < \eta < 1$ is a constant and $\lambda > 0$ is a parameter. $p > 1$, $\phi_p(s)$ is the p-Laplacian operator; that is, $\phi_p(s) = |s|^{p-2}s$, $\phi_p^{-1}(s) = \phi_q(s)$, $1/p + 1/q = 1$. D denotes the Caputo fractional derivative.

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various fields of sciences and engineering such as physics, chemistry, aerodynamics, electrodynamics of complex medium, electrical circuits, and biology (see [1–7] and their references). Differential equations with p-Laplacian arise naturally in non-Newtonian mechanics, nonlinear elasticity, glaciology, population biology, combustion theory, and nonlinear flow laws. Since the p-Laplacian operator and fractional calculus arise from so many applied fields, the fractional p-Laplacian differential equations are worth studying. Recently, there have appeared a very large number of papers which are devoted to the existence of solutions of boundary value problems and initial value problems for the fractional differential equations (see [8–12]), and the existence of solutions of boundary value problems for the fractional p-Laplacian differential equations has just begun in recent years (see [13–22]). On the other hand, there are few papers that consider the eigenvalue intervals of fractional boundary value problems (see [23, 24]). In [23], the author discussed the following system:

$$\begin{aligned} &D_{0+}^{\alpha}x(t) + \lambda h(t) f(x(t)) = 0, \quad 0 < t < 1, \\ &\qquad 2 < \alpha \le 3, \\ &x(0) = x'(0) = x''(0) = x''(1) = 0. \end{aligned} \tag{2}$$

By the use of appropriate conditions with respect to $\lim_{t\to 0}(f(t)/t) = f_0$ and $\lim_{t\to\infty}(f(t)/t) = f_\infty$, the author proved that the above problem has at least one or two positive solutions for some λ, where $f_0 = 0, l, \infty$, $f_\infty = 0, l, \infty$, $0 < l < \infty$. But almost all the results which the author obtained depend on both f_0 and f_∞; the case depends on one of f_0 and f_∞; the author only discussed $f_0 = 0, \infty$ and $f_\infty = 0, \infty$; the case $f_0 = l$ or $f_\infty = l$ has not been discussed. On the other hand, there exist several results on the existence of one solution to fractional p-Laplacian boundary value problems (BVPs); there are, to the best of our knowledge, relatively few results on the nonexistence and the uniqueness of positive solutions to fractional p-Laplacian differential equation with parameter.

Motivated by the above questions, in this paper, we will establish several sufficient conditions for the existence of positive solutions of (1) by using fixed point theorem and fixed point index theory.

The work is organized in the following fashion. In Section 2, we provide some necessary background. In particular, we will introduce some lemmas and definitions associated with fixed point index theory. The main results will be stated and proved in Section 3. Two examples are given in Section 4.

2. Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Definition 1 (see [1]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds, \tag{3}$$

provided that the right side is point-wise defined on $(0, +\infty)$ and Γ is the Gamma function.

Definition 2 (see [1]). The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds, \tag{4}$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0, +\infty)$.

Lemma 3 (see [1]). *Let $\alpha > 0$, and assume that $x \in C[0,1]$, and then the fractional differential equation $D_0^{\alpha} x(t) = 0$ has unique solutions:*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \quad i = 0, 1, 2, \ldots, n-1, \quad n-1 < \alpha \le n. \tag{5}$$

Lemma 4 (see [1]). *Let $\alpha > 0$. Then,*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{6}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$, $n-1 < \alpha \le n$.

Lemma 5 (see [25]). *Let P be a cone in a real Banach space E, and let Ω be a bounded open set of E. Assume that operator $A : P \cap \overline{\Omega} \to P$ is completely continuous. If there exists a $x_0 > 0$ such that*

$$x - Ax \neq t x_0, \quad \forall x \in P \cap \partial\Omega, \ t \ge 0, \tag{7}$$

then $i(A, P \cap \Omega, P) = 0$.

Lemma 6 (see [26]). *Let E be a Banach space and $P \subseteq E$ a cone, and Ω_1 and Ω_2 are open set with $0 \in \Omega_1, \overline{\Omega}_1 \subseteq \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_2$

or

(ii) $\|Tu\| \ge \|u\|, u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|, u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, we present some new results on the existence, multiplicity, nonexistence, and the uniqueness of positive solution of problem (1) and dependence of the positive solution $x_\lambda(t)$ on the parameter λ.

Let $E = C[0,1]$; then E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{0 \le t \le 1} |x(t)|$.

Lemma 7. *Let $y(t) \in C(0,1) \cap L^1(0,1)$ and $y(t) \ge 0$; then the solution of the problem*

$$\begin{aligned} &\left(\phi_p\left(D_{0+}^{\alpha} x(t)\right)\right)' + \lambda y(t) = 0, \quad 0 < t < 1, \\ &x'(0) = x''(0) = 0, \\ &x(1) = \eta \int_0^1 g(s)\, x(s)\, ds \end{aligned} \tag{8}$$

is given by

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \left\{ \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\lambda \int_0^s y(\tau)\, d\tau\right) ds \right. \\ & \left. - \int_0^t (t-s)^{\alpha-1} \phi_q\left(\lambda \int_0^s y(\tau)\, d\tau\right) ds \right\} \\ & + \frac{\eta}{\Gamma(\alpha)\left(1 - \eta \int_0^1 g(s)\, ds\right)} \\ & \times \left\{ \int_0^1 g(s) \right. \\ & \times \left(\int_0^1 (1-\tau)^{\alpha-1} \phi_q\left(\lambda \int_0^{\tau} y(\mu)\, d\mu\right) d\tau \right. \\ & - \int_0^s (s-\tau)^{\alpha-1} \\ & \left. \left. \times \phi_q\left(\lambda \int_0^{\tau} y(\mu)\, d\mu\right) d\tau \right) ds \right\}, \end{aligned} \tag{9}$$

$$x(t) \geq 0, \quad \min_{0\leq t\leq 1} x(t) \geq \sigma \|x\|, \tag{10}$$

where $2 < \alpha \leq 3, 0 < \eta < 1, 1 - \eta \int_0^1 g(s)ds > 0$ *and* $\sigma = \eta \int_0^1 (1-s)g(s)ds / (1 - \eta \int_0^1 sg(s)ds)$.

Proof. It is easy to see by integration of (8) that

$$\phi_p(D_{0+}^{\alpha} x(t)) - \phi_p(D_{0+}^{\alpha} x(0)) = -\lambda \int_0^t y(s)\,ds. \tag{11}$$

By the boundary condition $x'(0) = x''(0) = 0$, we can easily get that $D_{0+}^{\alpha} x(0) = 0$, and we obtain that

$$\phi_p(D_{0+}^{\alpha} x(t)) = -\lambda \int_0^t y(s)\,ds; \tag{12}$$

that is

$$D_{0+}^{\alpha} x(t) = -\phi_q\left(\lambda \int_0^t y(s)\,ds\right). \tag{13}$$

By Lemmas 3 and 4, we get that

$$x(t) = -I^{\alpha}\phi_q\left(\lambda \int_0^t y(s)\,ds\right) + c_0 + c_1 t + c_2 t^2. \tag{14}$$

Using the boundary condition $x'(0) = x''(0) = 0$, $x(1) = \eta \int_0^1 g(s)x(s)ds$, we get $c_1 = c_2 = 0$ and

$$\begin{aligned} c_0 = {} & \frac{1}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\ & \times \int_0^1 (1-s)^{\alpha-1}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds \\ & - \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\ & \times \int_0^1 g(t)\left(\int_0^t (t-s)^{\alpha-1}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds\right)dt. \end{aligned} \tag{15}$$

Thus,

$$\begin{aligned} x(t) = {} & -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds \\ & + \frac{1}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\ & \times \int_0^1 (1-s)^{\alpha-1}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds \\ & - \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\ & \times \int_0^1 g(t)\left(\int_0^t (t-s)^{\alpha-1}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds\right)dt \\ = {} & \frac{1}{\Gamma(\alpha)}\left\{\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds\right. \\ & \left. - \int_0^t (t-s)^{\alpha-1}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds\right\} \\ & + \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\ & \times \left\{\int_0^1 g(s)\left(\int_0^1 (1-\tau)^{\alpha-1}\phi_q\left(\lambda\int_0^\tau y(\mu)\,d\mu\right)d\tau\right.\right. \\ & - \int_0^s (s-\tau)^{\alpha-1} \\ & \left.\left. \times \phi_q\left(\lambda\int_0^\tau y(\mu)\,d\mu\right)d\tau\right)ds\right\}. \end{aligned} \tag{16}$$

Direct differentiation of (9) implies

$$\begin{aligned} & x'(t) \\ & = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds \leq 0. \end{aligned} \tag{17}$$

By differentiation of (17), we get

$$\begin{aligned} & x''(t) \\ & = -\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q\left(\lambda\int_0^s y(\tau)\,d\tau\right)ds \leq 0. \end{aligned} \tag{18}$$

Thus the solution of problem (8) is nonincreasing and concave on $[0,1]$, and

$$\|x\| = x(0), \quad \min_{0\leq t\leq 1} x(t) = x(1). \tag{19}$$

On the other hand, as we assume that $y(t) \geq 0$, we see that

$$\begin{aligned} x(1) = {} & \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\ & \times \left\{\int_0^1 g(s)\left(\int_0^1 (1-\tau)^{\alpha-1}\phi_q\left(\lambda\int_0^\tau y(\mu)\,d\mu\right)d\tau\right.\right. \\ & - \int_0^s (s-\tau)^{\alpha-1} \\ & \left.\left. \times \phi_q\left(\lambda\int_0^\tau y(\mu)\,d\mu\right)d\tau\right)ds\right\} \\ = {} & \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\ & \times \left\{\int_0^1 g(s)\left(\int_0^s \left[(1-\tau)^{\alpha-1} - (s-\tau)^{\alpha-1}\right]\right.\right. \end{aligned}$$

$$\times \phi_q\left(\lambda \int_0^\tau y(\mu)\,d\mu\right) d\tau + \int_s^1 (1-\tau)^{\alpha-1}\phi_q\left(\lambda \int_0^\tau y(\mu)\,d\mu\right) d\tau\right) ds\Big\} \geq 0. \tag{20}$$

Therefore, $x(t) \geq 0$ and is concave for $t \in [0,1]$. So for every $t \in [0,1]$,

$$x(t) \geq t x(1) + (1-t)x(0). \tag{21}$$

Therefore,

$$\eta \int_0^1 x(t) g(t)\,dt \geq \eta x(1) \int_0^1 t g(t)\,dt + \eta x(0) \int_0^1 (1-t) g(t)\,dt. \tag{22}$$

Since $x(1) = \eta \int_0^1 x(t)g(t)dt$, $1 - \eta \int_0^1 g(t)dt > 0$, we have

$$x(1) \geq \frac{\eta \int_0^1 (1-t) g(t)\,dt}{1 - \eta \int_0^1 t g(t)\,dt} x(0); \tag{23}$$

that is,

$$\min_{0\leq t\leq 1} x(t) = x(1) \geq \frac{\eta \int_0^1 (1-t) g(t)\,dt}{1 - \eta \int_0^1 t g(t)\,dt} x(0) = \sigma \|x\|. \tag{24}$$

The Lemma is proved. □

We construct a cone P in E by

$$P = \left\{x \in E : x \geq 0, \min_{0\leq t\leq 1} x(t) \geq \sigma \|x\|\right\}, \tag{25}$$

where σ is defined in Lemma 7. It is easy to see P is a closed convex cone of E.

Define $T : P \to E$ by

$$\begin{aligned}(Tx)(t) &= \frac{1}{\Gamma(\alpha)}\left\{\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right) ds - \int_0^t (t-s)^{\alpha-1}\phi_q\left(\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right) ds\right\} \\ &+ \frac{\eta}{\Gamma(\alpha)\left(1 - \eta \int_0^1 g(s)\,ds\right)} \times \left\{\int_0^1 g(s) \times \left(\int_0^1 (1-\tau)^{\alpha-1} \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right) d\tau - \int_0^s (s-\tau)^{\alpha-1} \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right) d\tau\right) ds\right\}.\end{aligned} \tag{26}$$

It is clear that the fixed points of the operator $\phi_q(\lambda)T$ are the solutions of the boundary value problems (1).

We make the following hypotheses:

(H1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous;

(H2) $\omega : (0,1) \to [0,\infty)$ is continuous and not identical zero on any closed subinterval of $(0,1)$ with $0 < \int_0^1 \omega(t)dt < +\infty$;

(H3) $1 - \eta \int_0^1 g(s)ds > 0$;

(H4) $f_\phi^0 \in [0,\infty]$, where $f_\phi^0 := \lim_{x\to 0}(f(t,x)/\phi_p(x))$ uniformly for $t \in [0,1]$;

(H5) $f_\phi^\infty \in [0,\infty]$, where $f_\phi^\infty := \lim_{x\to \infty}(f(t,x)/\phi_p(x))$ uniformly for $t \in [0,1]$;

(H6) $f(t,x) > 0$ for any $t \in [0,1]$ and $x > 0$.

Lemma 8. *Assume that (H1)–(H3) hold. Then $T : P \to E$ is completely continuous.*

Proof. First, we show that T is continuous. It is easy to see $T(P) \subset P$. For $x_n(t) \in E$ and $x_n(t) \to x(t)$, as $n \to \infty$, by the continuous of $f(t,x(t))$, we get $f(t,x_n(t)) \to f(t,x(t))$, as $n \to \infty$. This implies that

$$\phi_q\left(\int_0^t w(s) f(s, x_n(s))\,ds\right) \longrightarrow \phi_q\left(\int_0^t w(s) f(s, x(s))\,ds\right). \tag{27}$$

So we have

$$\sup_{t\in[0,1]}\left|\phi_q\left(\int_0^t w(s) f(s, x_n(s))\,ds\right) - \phi_q\left(\int_0^t w(s) f(s, x(s))\,ds\right)\right| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{28}$$

Denote $\Pi = |(Tx_n)(t) - (Tx)(t)|$; then

$$\Pi \leq \frac{1}{\Gamma(\alpha)}\left\{\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(\int_0^s \omega(\tau) f(\tau, x_n(\tau))\,d\tau\right) - \phi_q\left(\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right)\right| ds\right.$$

$$
\begin{aligned}
&+\int_0^t (t-s)^{\alpha-1}\left|\phi_q\left(\int_0^s \omega(\tau) f(\tau, x_n(\tau))\,d\tau\right)-\phi_q\right.\\
&\qquad\left.\times\left(\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right)\right|ds\Bigg\}\\
&+\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\\
&\times\Bigg\{\int_0^1 g(s)\\
&\quad\times\Bigg(\int_0^1 (1-\tau)^{\alpha-1}\\
&\qquad\times\left|\phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_n(\mu))\,d\mu\right)\right.\\
&\qquad\left.-\phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right)\right|d\tau\\
&\quad+\int_0^s (s-\tau)^{\alpha-1}\\
&\qquad\times\left|\phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_n(\mu))\,d\mu\right)-\phi_q\right.\\
&\qquad\left.\times\left(\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right)\right|d\tau\Bigg)ds\Bigg\}\\
&\le \sup_{s\in[0,1]}\left|\phi_q\left(\int_0^s w(\tau) f(\tau, x_n(\tau))\,d\tau\right)\right.\\
&\qquad\left.-\phi_q\left(\int_0^s w(\tau) f(\tau, x(\tau))\,d\tau\right)\right|\\
&\quad\times\frac{2}{\Gamma(\alpha+1)}\left(1+\frac{\eta\int_0^1 g(s)\,ds}{1-\eta\int_0^1 g(s)\,ds}\right)\longrightarrow 0,\\
&\qquad n\longrightarrow\infty.
\end{aligned}
\tag{29}
$$

Hence, T is continuous.

Second, we show that T is compact.

For $x(t)\in B_r=\{x\in P:\|x\|\le r\}$, by condition (H1), $f(t,x(t))<\infty$. Denote $M=\max_{0\le t\le 1,\|x\|\le r} f(t,x(t))$.

Then

$$
\begin{aligned}
&|Tx(t)|\\
&\le\frac{1}{\Gamma(\alpha)}\Bigg\{\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right)\right|ds\\
&\qquad+\int_0^t (t-s)^{\alpha-1}\left|\phi_q\left(\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right)\right|ds\Bigg\}\\
&\quad+\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\\
&\quad\times\Bigg\{\int_0^1 g(s)\\
&\qquad\times\Bigg(\int_0^1 (1-\tau)^{\alpha-1}\\
&\qquad\quad\times\left|\phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right)\right|d\tau\\
&\qquad+\int_0^s (s-\tau)^{\alpha-1}\\
&\qquad\quad\times\left|\phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right)\right|d\tau\Bigg)ds\Bigg\}\\
&\le\frac{1}{\Gamma(\alpha)}\\
&\quad\times\Bigg\{\int_0^1 (1-s)^{\alpha-1}\left|\phi_q\left(M\int_0^s \omega(\tau)\,d\tau\right)\right|ds\\
&\qquad+\int_0^t (t-s)^{\alpha-1}\left|\phi_q\left(M\int_0^s \omega(\tau)\,d\tau\right)\right|ds\Bigg\}\\
&\quad+\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\\
&\quad\times\Bigg\{\int_0^1 g(s)\Bigg(\int_0^1 (1-\tau)^{\alpha-1}\\
&\qquad\quad\times\left|\phi_q\left(M\int_0^\tau \omega(\mu)\,d\mu\right)d\tau\right|\\
&\qquad+\int_0^s (s-\tau)^{\alpha-1}\\
&\qquad\quad\times\left|\phi_q\left(M\int_0^\tau \omega(\mu)\,d\mu\right)\right|d\tau\Bigg)ds\Bigg\}\\
&\le\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\phi_q\left(M\int_0^1 \omega(\tau)\,d\tau\right)\\
&\quad\times\int_0^1 g(s)\left(\int_0^1 (1-\tau)^{\alpha-1}d\tau+\int_0^s (s-\tau)^{\alpha-1}d\tau\right)ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\phi_q\left(M\int_0^1 \omega(\tau)\,d\tau\right)\\
&\quad\times\left\{\int_0^1 (1-s)^{\alpha-1}ds+\int_0^t (t-s)^{\alpha-1}ds\right\}\\
&\le\frac{2\phi_q\left(M\int_0^1 \omega(\tau)\,d\tau\right)}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(\tau)\,d\tau\right)}<\infty;
\end{aligned}
\tag{30}
$$

that is, T maps bounded sets into bounded sets in P.

For $x(t)\in B_r$, $0<t_1<t_2<1$,

$$
\begin{aligned}
&|Tx(t_2)-Tx(t_1)|\\
&\quad=\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}\phi_q\left(\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right)ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_0^{t_1} (t_1 - s)^{\alpha-1} \\
&\qquad \times \phi_q \left(\int_0^s \omega(\tau) f(\tau, x(\tau))\, d\tau \right) ds \Bigg| \\
&\le \frac{1}{\Gamma(\alpha)} \phi_q \left(M \int_0^1 \omega(\tau)\, d\tau \right) \\
&\quad \times \left| \int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right| .
\end{aligned} \tag{31}
$$

By mean value theorem, we obtain that

$$
\begin{aligned}
&\int_0^{t_1} \left((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) ds \\
&\qquad \le \int_0^{t_1} (\alpha - 1)(1 - s)^{\alpha-2} (t_2 - t_1)\, ds \\
&\qquad \le \int_0^1 (\alpha - 1)(1 - s)^{\alpha-2} (t_2 - t_1)\, ds \\
&\qquad = t_2 - t_1.
\end{aligned} \tag{32}
$$

Thus

$$
\begin{aligned}
&|Tx(t_2) - Tx(t_1)| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \phi_q \left(M \int_0^1 \omega(\tau)\, d\tau \right) \left[(t_2 - t_1) + \frac{1}{\alpha} (t_2 - t_1)^{\alpha} \right].
\end{aligned} \tag{33}
$$

This shows that $|Tx(t_2) - Tx(t_1)| \to 0$, as $t_2 - t_1 \to 0$, so $\{Tx : x \in B_r\}$ is equicontinuous. Therefore, the operator $T : P \to P$ is completely continuous by the Arzela-Ascoli theorem. □

Now for convenience we introduce the following notations. Let

$$
\begin{aligned}
&\int_0^1 (1 - s)^{\alpha-1} \phi_q \left(\int_0^s \omega(\tau)\, d\tau \right) ds = \gamma_1, \\
&\phi_q \left(\int_0^1 \omega(\tau)\, d\tau \right) = \gamma_2, \\
&m(\tilde{r}) = \min \left\{ \min_t \frac{f(t, x)}{\phi_p(\tilde{r})}, x \in [\sigma \tilde{r}, \tilde{r}] \right\}.
\end{aligned} \tag{34}
$$

Theorem 9. *Assume that (H1)–(H5) hold.*

(i) *If* $0 < f_\phi^0 < +\infty$, *then there exists* $\xi_0 > 0$ *such that, for every* $0 < r < \xi_0$, *problem* (1) *has a positive solution* $x_r(t)$ *satisfying* $\|x_r\| = r$ *associated with*

$$
\lambda = \lambda_r \in \left[\overline{\lambda}_0, \lambda_0 \right], \tag{35}
$$

where $\overline{\lambda}_0$ *and* λ_0 *are two positive finite numbers.*

(ii) *If* $0 < f_\phi^\infty < +\infty$, *then there exists* $\xi_0^* > 0$ *such that, for every* $R > \xi_0^*$, *problem* (1) *has a positive solution* $x_R(t)$ *satisfying* $\|x_R\| = R$ *for any*

$$
\lambda = \lambda_R \in \left[\overline{\lambda}_0^*, \lambda_0^* \right], \tag{36}
$$

where $\overline{\lambda}_0^*$ *and* λ_0^* *are two positive finite numbers.*

(iii) *If* $f_\phi^0 = +\infty$, *then there exists* $\xi_1 > 0$ *such that, for any* $0 < r^* < \xi_1$, *problem* (1) *has a positive solution* $x_{r^*}(t)$ *satisfying* $\|x_{r^*}\| = r^*$ *for any*

$$
\lambda = \lambda_{r^*} \in (0, \lambda^*], \tag{37}
$$

where λ^* *is a positive finite number.*

(iv) *If* $f_\phi^\infty = +\infty$, *then there exists* $\overline{\xi}_1 > 0$ *such that, for every* $R_* > \overline{\xi}_1$, *problem* (1) *has a positive solution* $x_{R_*}(t)$ *satisfying* $\|x_{R_*}\| = R_*$ *for any*

$$
\lambda = \lambda_{R_*} \in (0, \lambda_*], \tag{38}
$$

where λ_* *is a positive finite number.*

(v) *If there exists* $\xi > 0$ *such that* $m(\tilde{r}) > \xi$, *then problem* (1) *has positive solution* $x_{\tilde{r}}(t)$ *satisfying* $\|x_{\tilde{r}}\| = \tilde{r}$ *for any*

$$
\lambda = \lambda_{\tilde{r}} \in (0, \tilde{\lambda}], \tag{39}
$$

where $\tilde{\lambda}$ *is a positive finite number.*

Proof. (i) It follows from $0 < f_\phi^0 < +\infty$ that there exists $l_1, l_2, \rho > 0$, such that

$$
\begin{aligned}
&l_1 \phi_p(x) \le f(t, x) \le l_2 \phi_p(x), \\
&(\forall t \in [0, 1], 0 \le x \le \rho).
\end{aligned} \tag{40}
$$

Let $\xi_0 = \rho / \sigma$; we show that ξ_0 is required. When $x \in P \cap \partial\Omega_r$, we have

$$
x(t) \ge \sigma \|x\| = \sigma r; \tag{41}
$$

we need $x(t) < \rho$; this implies that $\sigma r < \rho$, as $r < \xi_0$, so $\sigma r < \sigma \xi_0 = \rho$.

Let $\lambda_0 = (1/l_1)(\Gamma(\alpha)/\sigma\gamma_1)^{p-1}$. Then we may assume that

$$
x - \phi_q(\lambda_0) Tx \ne 0, \quad (\forall x \in P \cap \partial\Omega_r); \tag{42}
$$

if not, then there exists $x_r \in P \cap \partial\Omega_r$ such that $\phi_q(\lambda_0)Tx_r = x_r$, and then (35) already holds for $\lambda_r = \lambda_0$.

Define $\psi(t) \equiv 1, \forall t \in [0, 1]$; then $\psi(t) \in P$, and $\|\psi\| \equiv 1$. We now show that

$$
x - \phi_q(\lambda_0) Tx \ne \kappa\psi \quad (\forall x \in P \cap \partial\Omega_r, \kappa \ge 0). \tag{43}
$$

In fact, if there exists $x_1 \in P \cap \partial\Omega_r, \kappa_1 \ge 0$ such that $x_1 - \phi_q(\lambda_0)Tx_1 = \kappa_1\psi$, then (42) implies that $\kappa_1 > 0$. On the other hand $x_1 = \phi_q(\lambda_0)Tx_1 + \kappa_1\psi \ge \kappa_1\psi$; we may choose $\kappa^* = \max\{\kappa \mid x_1 \ge \kappa\psi\}\}$; then $\kappa_1 \le \kappa^* < +\infty, x_1 \ge \kappa^*\psi$. Therefore

$$
\kappa^* = \kappa^* \|\psi\| \le \|x_1\| = r. \tag{44}
$$

Consequently, for any $t \in [0,1]$, (26) and (44) imply that

$$
\begin{aligned}
x_1(t) &= \phi_q(\lambda_0)\frac{1}{\Gamma(\alpha)} \\
&\quad \times \left\{ \int_0^1 (1-s)^{\alpha-1} \right. \\
&\qquad \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_1(\tau))\, d\tau\right) ds \\
&\qquad - \int_0^t (t-s)^{\alpha-1} \\
&\qquad \left. \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_1(\tau))\, d\tau\right) ds \right\} \\
&\quad + \frac{\eta\phi_q(\lambda_0)}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \\
&\quad \times \left\{ \int_0^1 g(s) \right. \\
&\qquad \times \left(\int_0^1 (1-\tau)^{\alpha-1} \right. \\
&\qquad\quad \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_1(\mu))\, d\mu\right) d\tau \\
&\qquad\quad - \int_0^s (s-\tau)^{\alpha-1} \\
&\qquad\quad \left.\left. \times \phi_q\left(\left(\int_0^\tau \omega(\mu) f(\mu, x_1(\mu))\, d\mu\right) d\tau\right) ds \right\} \right. \\
&\quad + \kappa_1\psi \\
&\geq \frac{\eta\sigma\phi_q(\lambda_0)}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \\
&\quad \times \left\{ \int_0^1 g(s) \right. \\
&\qquad \times \left(\int_0^1 (1-\tau)^{\alpha-1} \right. \\
&\qquad\quad \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_1(\mu))\, d\mu\right) d\tau \\
&\qquad\quad - \int_0^s (s-\tau)^{\alpha-1} \phi_q \\
&\qquad\quad \left.\left. \times \left(\left(\int_0^\tau \omega(\mu) f(\mu, x_1(\mu))\, d\mu\right) d\tau\right) ds \right\} \right. \\
&\quad + \frac{\sigma\phi_q(\lambda_0)}{\Gamma(\alpha)} \\
&\quad \times \left\{ \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_1(\tau))\, d\tau\right) ds \right\} \\
&\quad + \kappa_1\psi \\
&\geq \frac{\sigma\phi_q(\lambda_0)}{\Gamma(\alpha)} \\
&\quad \times \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_1(\tau))\, d\tau\right) ds \\
&\quad + \kappa_1\psi \\
&\geq \frac{\sigma\phi_q(\lambda_0)}{\Gamma(\alpha)} \\
&\quad \times \int_0^1 (1-s)^{\alpha-1} \phi_q\left(l_1\int_0^s \omega(\tau) \phi_p(x_1(\tau))\, d\tau\right) ds \\
&\quad + \kappa_1\psi \\
&\geq \frac{\sigma\phi_q(\lambda_0)}{\Gamma(\alpha)} \\
&\quad \times \int_0^1 (1-s)^{\alpha-1} \phi_q\left(l_1\int_0^s \omega(\tau) \phi_p(\kappa^*\psi(\tau))\, d\tau\right) ds \\
&\quad + \kappa_1\psi \\
&= \frac{\sigma\kappa^* l_1^{q-1}\lambda_0^{q-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s \omega(\tau)\, d\tau\right) ds + \kappa_1\psi \\
&= \frac{\sigma\kappa^* l_1^{q-1}\lambda_0^{q-1}\gamma_1}{\Gamma(\alpha)} + \kappa_1\psi \\
&= \kappa^* + \kappa\psi,
\end{aligned}
\tag{45}
$$

which implies that $x_1(t) \geq (\kappa^* + \kappa)\psi(t)$, $t \in [0,1]$, which is a contradiction to the definition of κ^*. Thus, (42) holds and, by Lemma 5, the fixed point index

$$i\left(\phi_q(\lambda_0)T, P \cap \partial\Omega_r, P\right) = 0. \tag{46}$$

On the other hand, by the fact that the fixed point index of constants operator is 1, so

$$i(\vartheta, P \cap \partial\Omega_r, P) = 1, \tag{47}$$

where ϑ is the zero operator. It follows therefore from (46) and (47) and the homotopy invariance property that there exist $x_r \in P \cap \partial\Omega_r$ and $0 < \nu_0 < 1$ such that $\nu_0\phi_q(\lambda_0)Tx_r = x_r$, but $x_r = \phi_q(\lambda_r)Tx_r$, which implies that $\nu_0\phi_q(\lambda_0)Tx_r = \phi_q(\lambda_r)Tx_r$; it follows that we get $\nu_0\phi_q(\lambda_0) = \phi_q(\lambda_r)$); that is,

$$\nu_0\lambda_0^{q-1} = \lambda_r^{q-1}, \tag{48}$$

and then

$$\lambda_r = \nu_0^{1/(q-1)}\lambda_0 = \nu_0^{p-1}\lambda_0 < \lambda_0. \tag{49}$$

From the proof above, for any $r < \xi_0$, there exists a positive solution $x_r \in P \cap \partial\Omega_r$ associated with $\lambda = \lambda_r > 0$.

Thus

$$\begin{aligned} x_r(t) &= \frac{\phi_q(\lambda_r)}{\Gamma(\alpha)} \left\{ \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_r(\tau))\, d\tau \right) ds - \int_0^t (t-s)^{\alpha-1} \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_r(\tau))\, d\tau \right) ds \right\} \\ &+ \frac{\eta \phi_q(\lambda_r)}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \times \left\{ \int_0^1 g(s) \times \left(\int_0^1 (1-\tau)^{\alpha-1} \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_r(\mu))\, d\mu \right) d\tau - \int_0^s (s-\tau)^{\alpha-1} \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_r(\mu))\, d\mu \right) d\tau \right) ds \right\} \end{aligned} \tag{50}$$

with $\|x_r\| = r$.

Next, we show that $\lambda_r \geq \overline{\lambda}_0$. In fact,

$$\begin{aligned} x_r(t) &= \frac{\phi_q(\lambda_r)}{\Gamma(\alpha)} \left\{ \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_r(\tau))\, d\tau \right) ds - \int_0^t (t-s)^{\alpha-1} \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_r(\tau))\, d\tau \right) ds \right\} \\ &+ \frac{\eta \phi_q(\lambda_r)}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \times \left\{ \int_0^1 g(s) \times \left(\int_0^1 (1-\tau)^{\alpha-1} \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_r(\mu))\, d\mu \right) d\tau - \int_0^s (s-\tau)^{\alpha-1} \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_r(\mu))\, d\mu \right) d\tau \right) ds \right\} \\ &\leq \frac{\eta \phi_q(\lambda_r)}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \times \int_0^1 g(s) \int_0^1 (1-\tau)^{\alpha-1} \times \phi_q\left(\int_0^\tau \omega(\mu) f(\mu, x_r(\mu))\, d\mu \right) d\tau\, ds \\ &+ \frac{\phi_q(\lambda_r)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_r(\tau))\, d\tau \right) ds \\ &= \frac{\phi_q(\lambda_r)\eta\int_0^1 g(s)\, ds}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \times \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_r(\tau))\, d\tau \right) ds \\ &+ \frac{\phi_q(\lambda_r)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\int_0^s \omega(\tau) f(\tau, x_r(\tau))\, d\tau \right) ds \\ &\leq \frac{\phi_q(\lambda_r)\eta\int_0^1 g(s)\, ds}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \times \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(l_2 \int_0^s \omega(\tau) x_r(\tau)\, d\tau \right) ds \\ &+ \frac{\phi_q(\lambda_r)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(l_2 \lambda_r \int_0^s \omega(\tau) x_r(\tau)\, d\tau \right) ds \\ &= \frac{\phi_q(\lambda_r)}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\, ds\right)} \times \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(l_2 \int_0^s \omega(\tau) x_r(\tau)\, d\tau \right) ds \end{aligned}$$

$$\begin{aligned}&\leq \frac{\|x_r\| \phi_q(\lambda_r l_2)}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\\&\quad\times \int_0^1 (1-s)^{\alpha-1}\\&\quad\times \phi_q\left(\int_0^1 \omega(\tau)\,d\tau\right)ds\\&= \frac{\|x_r\| \phi_q(\lambda_r l_2)\gamma_2}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)},\end{aligned} \quad (51)$$

which implies that

$$\|x_r\| \leq \frac{\|x_r\| \phi_q(\lambda_r l_2)\gamma_2}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)}; \quad (52)$$

that is,

$$\frac{\phi_q(\lambda_r l_2)\gamma_2}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)} \geq 1; \quad (53)$$

it follows that we get

$$\lambda_r \geq \frac{1}{l_2}\left(\frac{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)}{\gamma_2}\right)^{p-1} := \overline{\lambda}_0. \quad (54)$$

In conclusion, $\lambda_r \in [\overline{\lambda}_0, \lambda_0]$.

(ii) It follows from $0 < f_\phi^\infty < \infty$ that there exist $l_1^*,\ l_2^*,\ \rho^* > 0$ such that

$$\begin{aligned}&l_1^*\phi_p(x) \leq f(t,x) \leq l_2^*\phi_p(x),\\&\quad(\forall t \in [0,1], x \geq \rho^*).\end{aligned} \quad (55)$$

Let $\xi_0^* = \rho^*/\sigma$, the following proof are similar to (i).

(iii) It follows from $f_\phi^0 = +\infty$, there exists $l^* > 0, \rho^* > 0$, such that

$$f_\phi^0 \geq l^*\phi_p(x), \quad (\forall t \in [0,1], 0 \leq x \leq \rho^*). \quad (56)$$

Let $\xi_1 = \rho^*/\sigma$; we show that ξ_1 is required. When $x \in P \cap \partial\Omega_{r^*}$, we have $x(t) \geq \sigma\|x\| = \sigma r^*$; since $x \leq \rho^*$, then we need $\sigma r^* < \rho^*$, as $r^* < \xi_1$, so $\sigma r^* < \rho^*$ holds.

Let $\lambda^* = (1/l^*)(\Gamma(\alpha)/\sigma\gamma_1)^{1/(q-1)}$; we proceed in the same way as in the proof of (i): replacing (42) we may assume that $x - \phi_q(\lambda^*)Tx \neq 0, (\forall x \in P \cap \partial\Omega_{r^*})$, and replacing (43) we can prove $x - \phi_q(\lambda^*)Tx \neq \kappa\psi, (\forall x \in P \cap \partial\Omega_{r^*}, \kappa \geq 0)$. It follows from Lemma 5 that $i(\phi_q(\lambda^*)T, P \cap \Omega_{r^*}, P) = 0$. Note that $i(\theta, P \cap \Omega_{r^*}, P) = 1$; we can easily show that there exists $x_{r^*} \in P \cap \partial\Omega_{r^*}$ and $0 < \nu_{r^*} < 1$ such that $\nu_{r^*}\phi_q(\lambda^*)Tx_{r^*} = x_{r^*}$. Hence (37) holds for $\lambda_{r^*} = \lambda^*\nu_{r^*}^{1-p} < \lambda^*$.

The proof of Theorem 9(iv) follows by the method similar to Theorem 9(iii); we omit it here.

(v) It follows that $m(\widetilde{r}) > \xi$; for any $x \in P \cap \partial\Omega_{\widetilde{r}}$, we have $\sigma\widetilde{r} \leq x \leq \widetilde{r}$ and $f(t,x)/\phi_p(x) \geq f(t,x)/\phi_p(\widetilde{r}) \geq m(\widetilde{r}) \geq \xi$; then

$$f(t,x) \geq \xi\phi_p(x), \quad \forall x \in [\sigma\widetilde{r}, \widetilde{r}],\ t \in [0,1]. \quad (57)$$

Let $\widetilde{\lambda} = (1/\xi)(\Gamma(\alpha)/\sigma\gamma_1)^{p-1}$. The following proof is similar to that of (iv). This finished the proof of (v). □

Remark 10. In Theorem 9, all the criteria obtained depend on one of f_ϕ^0 and f_ϕ^∞.

Let $\Psi(x) = \phi_q(\lambda)Tx$; the following theorems give out the multiply, nonexistence, and the dependence of parameter.

Theorem 11. *Assume that (H1)–(H5) hold.*

(i) *If $f_\phi^0 = 0$ and $f_\phi^\infty = 0$, $\underline{\lim}_{x\to\infty} f(t,x) = \widetilde{f}_\infty \in (0,+\infty]$ uniformly for $t \in [0,1]$, then there exists $\lambda^* > 0$ such that the problem* (1) *has two solutions for any $\lambda > \lambda^*$.*

(ii) *If $f_\phi^0 = 0$ and $f_\phi^\infty = 0$, then there exists $\underline{\lambda} > 0$ such that, for any $\lambda < \underline{\lambda}$, problem* (1) *has no solution.*

Proof. (i) It follows from $\widetilde{f}_\infty \in (0,+\infty]$ that, for any $\widetilde{\zeta} > 0$, there exists $\widetilde{\rho} > 0$, such that

$$f(t,x) > \widetilde{\zeta}, \quad (\forall t \in [0,1], x \geq \widetilde{\rho}). \quad (58)$$

Since $f_\phi^0 = 0$, there exists $\varepsilon_1 > 0$, $\overline{\rho}_1 > 0$ such that

$$f(t,x) \leq \varepsilon_1\phi_p(x), \quad (\forall t \in [0,1], 0 \leq x \leq \overline{\rho}_1), \quad (59)$$

where ε_1 satisfied $\phi_q(\lambda\varepsilon_1)\gamma_2/\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds) \leq 1$.

For $\forall x \in P \cap \partial\Omega_{\overline{\rho}_1}$, we have

$$\begin{aligned}\Psi(x) &= \left(\phi_q(\lambda)Tx\right)(t)\\&= \frac{1}{\Gamma(\alpha)}\left\{\int_0^1 (1-s)^{\alpha-1}\right.\\&\qquad\times\phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right)ds\\&\qquad-\int_0^t (t-s)^{\alpha-1}\\&\qquad\left.\times\phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right)ds\right\}\\&\quad+\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\\&\quad\times\left\{\int_0^1 g(s)\right.\end{aligned}$$

$$\times\left(\int_0^1(1-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^\tau\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau-\int_0^s(s-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^\tau\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\right)ds\Bigg\}$$
$$\le\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1g(s)\,ds\right)}\times\int_0^1g(s)\times\left(\int_0^1(1-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^\tau\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\right)ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\times\phi_q\left(\lambda\int_0^s\omega(\tau)\varepsilon_1f(\tau,x(\tau))\,d\tau\right)ds$$
$$\le\frac{\eta\int_0^1g(s)\,ds}{\Gamma(\alpha)\left(1-\eta\int_0^1g(s)\,ds\right)}\times\int_0^1(1-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^1\omega(\mu)\varepsilon_1\phi_p(x(\mu))\,d\mu\right)d\tau+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\times\phi_q\left(\lambda\int_0^1\omega(\tau)\varepsilon_1\phi_p(x(\tau))\,d\tau\right)ds$$
$$\le\frac{1}{\Gamma(\alpha+1)\left(1-\eta\int_0^1g(s)\,ds\right)}\times\phi_q\left(\lambda\int_0^1\omega(\tau)\varepsilon_1\phi_p(\|x\|)\,d\tau\right)$$
$$=\frac{\phi_q(\lambda\varepsilon_1)\gamma_2}{\Gamma(\alpha+1)\left(1-\eta\int_0^1g(s)\,ds\right)}\|x\|\le\|x\|. \quad (60)$$

This implied

$$\|\Psi x\|\le\|x\|,\quad\forall x\in P\cap\partial\Omega_{\overline{\rho}_1}. \quad (61)$$

Let $\overline{\rho}_3>\max\{\widetilde{\rho}/\sigma,\ \overline{\rho}_1\}$ and $\lambda^*=(1/\widetilde{\zeta})(\overline{\rho}_3\Gamma(\alpha)/\sigma\gamma_1)^{p-1}$; then $\forall x\in P\cap\partial\Omega_{\overline{\rho}_3}$; we have $\sigma\|x\|\le x\le\overline{\rho}_3$ and

$$(\Psi x)(t)=\frac{1}{\Gamma(\alpha)}\left\{\int_0^1(1-s)^{\alpha-1}\times\phi_q\left(\lambda\int_0^s\omega(\tau)f(\tau,x(\tau))\,d\tau\right)ds-\int_0^t(t-s)^{\alpha-1}\times\phi_q\left(\lambda\int_0^s\omega(\tau)f(\tau,x(\tau))\,d\tau\right)ds\right\}$$
$$+\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1g(s)\,ds\right)}\times\left\{\int_0^1g(s)\times\left(\int_0^1(1-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^\tau\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau-\int_0^s(s-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^\tau\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\right)ds\right\}$$
$$\ge\frac{\sigma\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1g(s)\,ds\right)}\times\left\{\int_0^1g(s)\times\left(\int_0^1(1-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^\tau\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau-\int_0^s(s-\tau)^{\alpha-1}\times\phi_q\left(\lambda\int_0^\tau\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\right)ds\right\}$$
$$+\frac{\sigma}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\times\phi_q\left(\lambda\int_0^s\omega(\tau)f(\tau,x(\tau))\,d\tau\right)ds$$
$$\ge\frac{\sigma}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}$$

$$\times \phi_q \left(\lambda \int_0^s \omega(\tau) f(\tau, x(\tau)) \, d\tau \right) ds$$
$$> \frac{\sigma}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}$$
$$\times \phi_q \left(\lambda \int_0^s \omega(\tau) \widetilde{\zeta} \, d\tau \right) ds$$
$$> \frac{\sigma \phi_q (\lambda \widetilde{\zeta})}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}$$
$$\times \phi_q \left(\int_0^s \omega(\tau) \, d\tau \right) ds$$
$$= \frac{\sigma \phi_q (\lambda \widetilde{\zeta}) \gamma_1}{\Gamma(\alpha)} = \overline{\rho}_3, \tag{62}$$

which implies

$$\|\Psi x\| > \|x\|, \quad \forall x \in P \cap \partial \Omega_{\overline{\rho}_3}. \tag{63}$$

Next, for $f_\phi^\infty = 0$, there exist $\varepsilon_2 > 0$, $\overline{\rho}_2 > \overline{\rho}_3 > 0$ such that

$$f(t,x) \le \varepsilon_2 \phi_p(x), \quad (\forall t \in [0,1], x \ge \overline{\rho}_2), \tag{64}$$

where ε_2 satisfied $\phi_q(\lambda\varepsilon_2)\gamma_2/\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds) \le 1$.

Similar to the above proof, we get

$$\|\Psi x\| \le \|x\|, \quad \forall x \in P \cap \partial \Omega_{\overline{\rho}_2}. \tag{65}$$

Applying Lemma 7 to (61), (65), and (63) yields that Ψ has two fixed points x_1, x_2 such that $x_1 \in P \cap (\overline{\Omega}_{\overline{\rho}_3} \setminus \Omega_{\overline{\rho}_1})$ and $x_2 \in P \cap (\overline{\Omega}_{\overline{\rho}_2} \setminus \Omega_{\overline{\rho}_3})$.

(ii) It follows from $f_\phi^0 = 0$ and $f_\phi^\infty = 0$ that there exists $\varepsilon_1, \varepsilon_2, \overline{\rho}_1, \overline{\rho}_2 > 0$, such that

$$\begin{gathered} f(t,x) \le \varepsilon_1 \phi_p(x), \quad (\forall t \in [0,1], 0 \le x \le \overline{\rho}_1), \\ f(t,x) \le \varepsilon_2 \phi_p(x), \quad (\forall t \in [0,1], x \ge \overline{\rho}_2); \end{gathered} \tag{66}$$

then for $x \in [\overline{\rho}_1, \overline{\rho}_2]$, by the continuous of $f(t,x(t))$, there exists $\overline{x} \in [\overline{\rho}_1, \overline{\rho}_2]$ such that

$$\max_{\overline{\rho}_1 \le x \le \overline{\rho}_2} \frac{f(t,x)}{\phi_p(x)} = \frac{f(t,\overline{x})}{\phi_p(\overline{x})}. \tag{67}$$

Let $\underline{\Upsilon} = \max\{\varepsilon_1, \varepsilon_2, f(t,\overline{x})/\phi_p(\overline{x})\} > 0$; then

$$f(t,x) \le \underline{\Upsilon} \phi_p(x), \quad (\forall 0 \le t \le 1, x \ge 0). \tag{68}$$

Assuming $x(t)$ is a positive solution of problem (1), we will show that this leads to a contradiction for $\lambda < \underline{\lambda}$, where $\underline{\lambda} = (1/\underline{\Upsilon})(\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds)/\gamma_2)^{p-1}$. It follows from (26) that

$$\|x\| = x(0)$$
$$= \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}$$
$$\times \left\{ \int_0^1 g(s) \right.$$
$$\times \left(\int_0^1 (1-\tau)^{\alpha-1} \right.$$
$$\times \phi_q \left(\lambda \int_0^\tau \omega(\mu) f(\mu, x(\mu)) \, d\mu \right) d\tau$$
$$- \int_0^s (s-\tau)^{\alpha-1}$$
$$\left. \left. \times \phi_q (\lambda \int_0^\tau \omega(\mu) f(\mu, x(\mu)) \, d\mu \right) d\tau \right) ds \right\}$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}$$
$$\times \phi_q \left(\lambda \int_0^s \omega(\tau) f(\tau, x(\tau)) \, d\tau \right) ds$$
$$\le \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}$$
$$\times \int_0^1 g(s)$$
$$\times \int_0^1 (1-\tau)^{\alpha-1}$$
$$\times \phi_q (\lambda \int_0^\tau \omega(\mu) f(\mu, x(\mu)) \, d\mu \Big) d\tau \, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}$$
$$\times \phi_q \left(\lambda \int_0^s \omega(\tau) f(\tau, x(\tau)) \, d\tau \right) ds$$
$$= \frac{1}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}$$
$$\times \int_0^1 (1-s)^{\alpha-1}$$
$$\times \phi_q \left(\lambda \int_0^s \omega(\tau) f(\tau, x(\tau)) \, d\tau \right) ds$$
$$\le \frac{1}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}$$
$$\times \int_0^1 (1-s)^{\alpha-1}$$

$$\times \phi_q\left(\lambda \int_0^1 \omega(\tau) \underline{\Upsilon}\phi_p(x(\tau))\,d\tau\right) ds$$
$$= \frac{\phi_q(\lambda\underline{\Upsilon})\gamma_2}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)}\|x\|. \tag{69}$$

This implies that $1 \le \phi_q(\lambda\underline{\Upsilon})\gamma_2/\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds) = (\lambda\underline{\Upsilon})^{q-1}\gamma_2/\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds) < (\underline{\lambda}\ \underline{\Upsilon})^{q-1}\gamma_2/\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds) = 1$, which is a contradiction. This finishes the proof of (ii). □

Remark 12. In (i), the condition $\underline{\lim}_{x\to\infty} f(t,x) = \widetilde{f}_\infty \in (0,+\infty]$ uniformly for $t \in [0,1]$ can be replaced with condition (H6). The same methods can be used to prove the results.

Remark 13. From the above proof, the condition $\widetilde{f}_\infty \in (0,+\infty]$ is important to keep the problem (1) having at least two positive solutions, if this condition is not satisfied, then there exists $\underline{\lambda} > 0$ small enough such that the problem (1) has no positive solutions for all $\lambda < \underline{\lambda}$.

Theorem 14. *Assume that (H1)–(H6) hold.*

(i) *If $f_\phi^0 = +\infty$ and $f_\phi^\infty = +\infty$, then there exists $\lambda_{*0} > 0$ such that problem (1) has at least two solutions for $\lambda \in (0, \lambda_{*0})$.*

(ii) *If $f_\phi^0 = +\infty$ and $f_\phi^\infty = +\infty$, then there exists $\overline{\lambda} > 0$ such that problem (1) has no solution for all $\lambda > \overline{\lambda}$.*

Proof. (i) It follows from $f_\phi^0 = +\infty$ that there exists $r_1 > 0$, such that

$$f(t,x) \ge L_{*1}\phi_p(x), \quad (\forall t \in [0,1], 0 \le x \le r_1), \tag{70}$$

where L_{*1} satisfies $\sigma^2\phi_q(\lambda L_{*1})\gamma_1/\Gamma(\alpha) \ge 1$.

Then for $x \in P\cap\partial\Omega_{r_1}$, we have $\sigma\|x\| \le x \le r_1$ for $t \in [0,1]$ and

$$\begin{aligned}
&(\Psi x)(t) \\
&= \frac{1}{\Gamma(\alpha)} \\
&\quad\times\left\{\int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right) ds - \int_0^t (t-s)^{\alpha-1} \times \phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right) ds\right\} \\
&\quad+ \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\
&\quad\times\left\{\int_0^1 g(s) \times\left(\int_0^1 (1-\tau)^{\alpha-1} \times\phi_q\left(\lambda\int_0^\tau \omega(\mu) f(\mu,x(\mu))\,d\mu\right) d\tau - \int_0^s (s-\tau)^{\alpha-1} \times\phi_q\left(\lambda\int_0^\tau \omega(\mu) f(\mu,x(\mu))\,d\mu\right) d\tau\right) ds\right\} \\
&\ge \frac{\sigma\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\
&\quad\times\left\{\int_0^1 g(s) \times\left(\int_0^1 (1-\tau)^{\alpha-1} \times\phi_q\left(\lambda\int_0^\tau \omega(\mu) f(\mu,x(\mu))\,d\mu\right) d\tau - \int_0^s (s-\tau)^{\alpha-1} \times\phi_q\left(\lambda\int_0^\tau \omega(\mu) f(\mu,x(\mu))\,d\mu\right) d\tau\right) ds\right\} \\
&\quad+ \frac{\sigma}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \times\phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right) ds \\
&\ge \frac{\sigma}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \times\phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right) ds \\
&\ge \frac{\sigma}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \times\phi_q\left(\lambda\int_0^s \omega(\tau) L_{*1}\phi_p(x(\tau))\,d\tau\right) ds \\
&\ge \frac{\sigma^2\phi_q(\lambda L_{*1})\|x\|}{\Gamma(\alpha)} \times\int_0^1 (1-s)^{\alpha-1} \times\phi_q\left(\int_0^s \omega(\tau)\,d\tau\right) ds \\
&= \frac{\sigma^2\phi_q(\lambda L_{*1})\gamma_2}{\Gamma(\alpha)}\|x\|,
\end{aligned} \tag{71}$$

which implies that

$$\|\Psi x\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_{r_1}. \tag{72}$$

Next, from $f_{\phi}^{\infty} = +\infty$, there exists $\overline{r}_2 > 0$, such that

$$f(t,x) \geq L_{*2}\phi_p(x), \quad \forall x \geq \overline{r}_2, \tag{73}$$

where L_{*2} satisfies

$$\frac{\sigma^2 \phi_q(\lambda L_{*2})\gamma_2}{\Gamma(\alpha)} \geq 1. \tag{74}$$

Let $r_2 > \max\{r_1, \overline{r}_2/\sigma\}$; then for $x \in P \cap \partial\Omega_{r_2}$ we have

$$x(t) \geq \sigma \|x\| = \sigma r_2 > \overline{r}_2. \tag{75}$$

Similar to the above proof, we can get

$$\|\Psi x\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_{r_2}. \tag{76}$$

Let $r_1 < r_3 < r_2$, $M_{r_3} = \max\{f(t,x) \mid t \in [0,1], \|x\| \leq r_3\}+1 > 0$, and $\lambda_{*0} = (1/M_{r_3})(\Gamma(\alpha+1)r_3(1-\eta\int_0^1 g(s)ds)/\gamma_2)^{p-1}$. Then for $x \in P \cap \partial\Omega_{r_3}$, we have

$$f(t,x) \leq M_{r_3}, \quad \forall t \in [0,1]. \tag{77}$$

Hence,

$$\begin{aligned}
(\Psi x)(t) &= \frac{1}{\Gamma(\alpha)} \left\{ \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\lambda \int_0^s \omega(\tau) f(\tau, x(\tau))\, d\tau\right) ds - \int_0^t (t-s)^{\alpha-1} \times \phi_q\left(\lambda \int_0^s \omega(\tau) f(\tau, x(\tau))\, d\tau\right) ds \right\} \\
&\quad + \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \times \left\{ \int_0^1 g(s) \times \left(\int_0^1 (1-\tau)^{\alpha-1} \times \phi_q\left(\lambda \int_0^\tau \omega(\mu) f(\mu, x(\mu))\, d\mu\right) d\tau - \int_0^s (s-\tau)^{\alpha-1} \times \phi_q\left(\lambda \int_0^\tau \omega(\mu) f(\mu, x(\mu))\, d\mu\right) d\tau\right) ds \right\} \\
&\leq \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \times \int_0^1 g(s) \times \int_0^1 (1-\tau)^{\alpha-1}\phi_q \times \left(\lambda \int_0^\tau \omega(\mu) f(\mu, x(\mu))\, d\mu\right) d\tau\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\lambda \int_0^s \omega(\tau) f(\tau, x(\tau))\, d\tau\right) ds \\
&\leq \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \times \int_0^1 g(s) \times \int_0^1 (1-\tau)^{\alpha-1}\phi_q \times \left(\lambda \int_0^\tau \omega(\mu) M_{r_3} d\mu\right) d\tau ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\lambda \int_0^s \omega(\tau) M_{r_3} d\tau\right) ds \\
&\leq \frac{\eta\phi_q(\lambda M_{r_3})\int_0^1 g(s)\,ds}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \times \int_0^1 (1-\tau)^{\alpha-1} \times \phi_q\left(\int_0^1 \omega(\mu)\, d\mu\right) d\tau \\
&\quad + \frac{\phi_q(\lambda M_{r_3})}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \times \phi_q\left(\int_0^1 \omega(\tau)\, d\tau\right) ds \\
&= \frac{\gamma_2 \phi_q(\lambda M_{r_3})}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\
&< \frac{\gamma_2 \phi_q(\lambda_{*0} M_{r_3})}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)} = r_3.
\end{aligned} \tag{78}$$

This implies that

$$\|\Psi x\| < \|x\|, \quad \forall x \in P \cap \partial\Omega_{r_3}. \tag{79}$$

Applying Lemma 7 to (72), (76), and (79) yields that Ψ has two fixed points x_1, x_2 such that $x_1 \in P \cap (\overline{\Omega}_{r_3} \setminus \Omega_{r_1})$ and $x_2 \in P \cap (\overline{\Omega}_{r_2} \setminus \Omega_{r_3})$.

(ii) By the proof of (i) and the continuity of $f(t,x)$, there exists $\tilde{x} \in [r_1, \overline{r}_2]$ such that $\min_{x\in[r_1,\overline{r}_2]}(f(t,x)/\phi_p(x)) = f(t,\tilde{x})/\phi_p(\tilde{x})$.

Let $\overline{\Upsilon} = \min\{L_{*1}, L_{*2}, f(t,\tilde{x})/\phi_p(\tilde{x})\}$; since (H6) holds, then $\overline{\Upsilon} > 0$ and

$$f(t,x) \geq \overline{\Upsilon}\phi_p(x), \quad \forall x \geq 0,\ t \in [0,1]. \tag{80}$$

Assuming $x(t)$ is a positive solution of problem (1), we will show that this leads to contradiction for all $\lambda > \overline{\lambda}$, where $\overline{\lambda} = (1/\overline{\Upsilon})(\Gamma(\alpha)/\gamma_1\sigma)^{p-1}$. It follows form (26) that

$$\begin{aligned}
\|x\| &= x(0) \\
&= \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\
&\quad\times\left\{\int_0^1 g(s)\right. \\
&\qquad\times\left(\int_0^1 (1-\tau)^{\alpha-1}\right. \\
&\qquad\quad\times\phi_q\left(\lambda\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right)d\tau \\
&\qquad\quad-\int_0^s (s-\tau)^{\alpha-1} \\
&\qquad\quad\left.\left.\times\phi_q\left(\lambda\int_0^\tau \omega(\mu) f(\mu, x(\mu))\,d\mu\right)d\tau\right)ds\right\} \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\qquad\times\phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right)ds \\
&\geq \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\qquad\times\phi_q\left(\lambda\int_0^s \omega(\tau) f(\tau, x(\tau))\,d\tau\right)ds \\
&\geq \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\qquad\times\phi_q\left(\lambda\int_0^s \omega(\tau)\overline{\Upsilon}\phi_p(x(\tau))\,d\tau\right)ds \\
&\geq \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\qquad\times\phi_q\left(\lambda\int_0^s \omega(\tau)\overline{\Upsilon}\phi_p(\sigma\|x\|\,d\tau)\right)ds \\
&\geq \frac{\phi_q(\lambda\overline{\Upsilon})\sigma\|x\|}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\qquad\times\phi_q\left(\int_0^s \omega(\tau)\,d\tau\right)ds \\
&= \frac{\phi_q(\lambda\overline{\Upsilon})\gamma_1\sigma}{\Gamma(\alpha)}\|x\|.
\end{aligned} \tag{81}$$

This implies that $1 \geq \phi_q(\lambda\overline{\Upsilon})\gamma_1\sigma/\Gamma(\alpha) = (\lambda\overline{\Upsilon})^{q-1}\gamma_1\sigma/\Gamma(\alpha) > (\overline{\lambda}\overline{\Upsilon})^{q-1}\gamma_1\sigma/\Gamma(\alpha) = 1$, which is a contradiction. This completes the proof of (ii). □

Corollary 15. *Assume that (H1)–(H5) holds.*

(i) *If* $\underline{\lim}_{x\to\infty} f(t,x) = \tilde{f}_\infty \in (0,+\infty]$ *uniformly for* $t \in [0,1]$, *and one of* $f_\phi^0 = 0$ *and* $f_\phi^\infty = 0$ *is satisfied, then there exists* $\lambda^* > 0$ *such that the problem* (1) *has at least one positive solution for any* $\lambda > \lambda^*$.

(ii) *If one of* $f_\phi^0 = +\infty$ *and* $f_\phi^\infty = +\infty$ *holds, then there exists* $\lambda_{*0} > 0$ *such that problem* (1) *has at least one positive solutions for any* $\lambda \in (0, \lambda_{*0})$.

Proof. (i) The conclusion is a direct consequence of Theorem 11(i).

(ii) The conclusion is a direct consequence of Theorem 14(i). □

Theorem 16. *Assume that (H1)–(H5) hold. Then the following conclusions hold.*

(i) *If* $f_\phi^0 = 0$ *and* $f_\phi^\infty = \infty$, *then the problem* (1) *has a positive solution* $x_\lambda(t)$ *for all* $\lambda > 0$.

(ii) *If* $f_\phi^0 = \infty$ *and* $f_\phi^\infty = 0$, *then the problem* (1) *has a positive solution* $x_\lambda(t)$ *for all* $\lambda > 0$.

Proof. We only prove (i); the proof of (ii) is similar, so we omit it here.

Let $\lambda > 0$; since $f_\phi^0 = 0$, there exists $r > 0$, $l > 0$ such that

$$f(t,x) \leq l\phi_p(x), \quad (\forall t \in [0,1], x \in [0,r]), \tag{82}$$

where l satisfied

$$\frac{\gamma_2\lambda^{q-1}l^{q-1}}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)} \leq 1. \tag{83}$$

Similar to the proof of Theorem 14, we obtain that

$$\|\Psi x\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_r. \tag{84}$$

From $f_{\phi}^{\infty}=\infty$, there exists $R>r>0,\ L>0$ such that

$$f(t,x)\geq L\phi_p(x),\quad (\forall t\in[0,1],x\geq R),\tag{85}$$

where L satisfied

$$\frac{\sigma^2\lambda^{q-1}L^{q-1}\gamma_1}{\Gamma(\alpha)}\geq 1.\tag{86}$$

We get that

$$\|\Psi x\|\geq\|x\|,\quad \forall x\in P\cap\partial\Omega_R.\tag{87}$$

Applying Lemma 7 to (84), (87) yields that Ψ has a fixed point x such that $x\in P\cap(\overline{\Omega}_R\setminus\Omega_r)$. □

Theorem 17. *Assume that (H1)–(H6) hold. Then the following conclusions hold.*

If $0<f_{\phi}^{0}<\infty$ *and* $0<f_{\phi}^{\infty}<\infty$, *then there exists* $\widehat{\lambda}_1,\widehat{\lambda}_2>0$ *such that problem* (1) *has no positive solution for any* $\lambda>\widehat{\lambda}_1$ *and* $0<\lambda<\widehat{\lambda}_2$.

Proof. It follows from $0<f_{\phi}^{0}<\infty$ and $0<f_{\phi}^{\infty}<\infty$ that there exist $\widehat{L}_1>0,\ \widehat{L}_2>0,\ \widehat{l}_1>0,\ \widehat{l}_2>0$, and $\widehat{r}_2>\widehat{r}_1>0$ such that

$$\begin{gathered}\widehat{l}_1\phi_p(x)\leq f(t,x)\leq\widehat{L}_1\phi_p(x),\\ (\forall t\in[0,1],\ 0\leq x\leq\widehat{r}_1),\\ \widehat{l}_2\phi_p(x)\leq f(t,x)\leq\widehat{L}_2\phi_p(x),\\ (\forall t\in[0,1],\ x\geq\widehat{r}_2).\end{gathered}\tag{88}$$

Let $\widehat{M}=\max\{f(t,x)/\phi_p(x),\widehat{r}_1\leq x\leq\widehat{r}_2,0\leq t\leq 1\}$, $\widehat{m}=\min\{f(t,x)/\phi_p(x),\widehat{r}_1\leq x\leq\widehat{r}_2,0\leq t\leq 1\}$. By the condition (H6), $\widehat{m}>0$. Then $\forall x\geq 0$; we have

$$\widehat{l}\phi_p(x)\leq f(t,x)\leq\widehat{L}\phi_p(x),\tag{89}$$

where $\widehat{L}=\max\{\widehat{L}_1,\widehat{L}_2,\widehat{M}\}>0$ and $\widehat{l}=\min\{\widehat{l}_1,\widehat{l}_2,\widehat{m}\}>0$.

Let $\widehat{\lambda}_1=(1/\widehat{l})(\Gamma(\alpha)/\gamma_1\sigma)^{p-1}$ and $\widehat{\lambda}_2=(1/\widehat{L})(\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds)/\gamma_2)^{p-1}$. If problem (1) has positive solution $x(t)$ in P, then

$$\begin{aligned}\|x\|&=x(0)\\ &=\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\\ &\quad\times\Bigg\{\int_0^1 g(s)\\ &\quad\times\left(\int_0^1(1-\tau)^{\alpha-1}\right.\\ &\quad\times\phi_q\left(\lambda\int_0^{\tau}\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\\ &\quad-\int_0^s(s-\tau)^{\alpha-1}\\ &\quad\left.\times\phi_q\left(\lambda\int_0^{\tau}\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\right)ds\Bigg\}\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\ &\quad\times\phi_q\left(\lambda\int_0^s\omega(\tau)f(\tau,x(\tau))\,d\tau\right)ds\\ &\geq\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\ &\quad\times\phi_q\left(\lambda\int_0^s\omega(\tau)f(\tau,x(\tau))\,d\tau\right)ds\\ &\geq\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\ &\quad\times\phi_q\left(\lambda\int_0^s\omega(\tau)\widehat{l}\phi_p(x(\tau))\,d\tau\right)ds\\ &\geq\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\ &\quad\times\phi_q\left(\lambda\int_0^s\omega(\tau)\widehat{l}\phi_p(\sigma\|x\|\,d\tau)\right)ds\\ &\geq\frac{\phi_q(\lambda\widehat{l})\sigma\|x\|}{\Gamma(\alpha)}\\ &\quad\times\int_0^1(1-s)^{\alpha-1}\phi_q\left(\int_0^s\omega(\tau)\,d\tau\right)ds\\ &=\frac{\phi_q(\lambda\widehat{l})\gamma_1\sigma}{\Gamma(\alpha)}\|x\|\\ &>\frac{\phi_q(\widehat{\lambda}_1\widehat{l})\gamma_1\sigma}{\Gamma(\alpha)}\|x\|=\|x\|;\end{aligned}$$

$$\begin{aligned}\|x\|&=x(0)\\ &=\frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)}\\ &\quad\times\Bigg\{\int_0^1 g(s)\\ &\quad\times\left(\int_0^1(1-\tau)^{\alpha-1}\phi_q\right.\\ &\quad\times\left(\lambda\int_0^{\tau}\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\\ &\quad-\int_0^s(s-\tau)^{\alpha-1}\phi_q\\ &\quad\left.\times\left(\lambda\int_0^{\tau}\omega(\mu)f(\mu,x(\mu))\,d\mu\right)d\tau\right)ds\Bigg\}\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\phi_q\\ &\quad\times\left(\lambda\int_0^s\omega(\tau)f(\tau,x(\tau))\,d\tau\right)ds\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\eta}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\
&\quad\times\int_0^1 g(s) \\
&\quad\quad\times\int_0^1 (1-\tau)^{\alpha-1} \\
&\quad\quad\quad\times\phi_q\left(\lambda\int_0^\tau \omega(\mu) f(\mu,x(\mu))\,d\mu\right)d\tau\,ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q \\
&\quad\quad\quad\times\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right)ds \\
&= \frac{1}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\
&\quad\times\int_0^1 (1-s)^{\alpha-1}\phi_q \\
&\quad\quad\times\left(\lambda\int_0^s \omega(\tau) f(\tau,x(\tau))\,d\tau\right)ds \\
&\leq \frac{1}{\Gamma(\alpha)\left(1-\eta\int_0^1 g(s)\,ds\right)} \\
&\quad\times\int_0^1 (1-s)^{\alpha-1}\phi_q \\
&\quad\quad\times\left(\lambda\int_0^1 \omega(\tau)\widehat{L}\phi_p(x(\tau))\,d\tau\right)ds \\
&= \frac{\phi_q(\lambda\widehat{L})\gamma_2}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)}\|x\| \\
&< \frac{\phi_q(\widehat{\lambda}_2\widehat{L})\gamma_2}{\Gamma(\alpha+1)\left(1-\eta\int_0^1 g(s)\,ds\right)}\|x\| = \|x\|,
\end{aligned} \tag{90}$$

which is a contradiction. This finishes the proof of Theorem 17. □

4. Examples

In this section, we give out same examples to illustrate our main results can be used in practice.

Example 1. Consider the following fractional differential equation:

$$\begin{gathered}
\left(\phi_p\left(D_{0+}^{5/2}x(t)\right)\right)' + \lambda t\left(1+t^2\right)^{1/2}x^{p-1}\arctan x = 0, \\
0<t<1, \\
x'(0) = x''(0) = 0, \\
x(1) = \frac{1}{2}\int_0^1 \frac{1}{2}x(s)\,ds,
\end{gathered} \tag{91}$$

where $\alpha = 5/2$, $p = 3/2$, $q = 3$, $\omega(t) = t$, $f(t,x) = (1+t^2)^{1/2}x^{p-1}\arctan x$, $\eta = 1/2$, and $g(t) = 1/2$.

By simple computation, $l_1^* = \pi/2$, $l_2^* = \sqrt{2}\pi/2$, $\gamma_1 = \int_0^1(1-s)^{3/2}\phi_3(\int_0^s \tau\,d\tau)ds \approx 0.0043$, $\gamma_2 = \phi_3(\int_0^1 \tau\,d\tau) = 0.25$, $\sigma \approx 0.1429$, $\lambda_0^* = (1/l_1^*)(\Gamma(\alpha)/\sigma\gamma_1)^{p-1} \approx 2.5149$, $\overline{\lambda}_0^* = (1/l_2^*)(\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds)/\gamma_2)^{p-1} \approx 1.4222$. Then, it is easy to see all the conditions of Theorem 9(ii) are satisfied; thus $\forall\lambda \in [1.4222, 2.5149]$]; the problem (91) has at least one positive solution.

Example 2. Consider the boundary value problem

$$\begin{gathered}
\left(\phi_p\left(D_{0+}^{5/2}x(t)\right)\right)' + \lambda t\left(1+t^2\right)^{1/2}x^{p}e^{-x} = 0, \\
0<t<1, \\
x'(0) = x''(0) = 0, \\
x(1) = \frac{1}{2}\int_0^1 \frac{1}{2}x(s)\,ds,
\end{gathered} \tag{92}$$

where $\alpha = 5/2$, $p = 3/2$, $q = 3$, $\omega(t) = t$, $f(t,x) = (1+t^2)^{1/2}x^pe^{-x}$, $\eta = 1/2$, and $g(t) = 1/2$:

$$\begin{gathered}
f_\phi^0 = \lim_{x\to 0}\frac{f(t,x)}{\phi_p(x)} = \lim_{x\to 0}\frac{\left(1+t^2\right)^{1/2}x^pe^{-x}}{x^{p-1}} = 0, \\
\forall t\in[0,1], \\
f_\phi^0 = \lim_{x\to \infty}\frac{f(t,x)}{\phi_p(x)} = \lim_{x\to \infty}\frac{\left(1+t^2\right)^{1/2}x^pe^{-x}}{x^{p-1}} = 0, \\
\forall t\in[0,1].
\end{gathered} \tag{93}$$

Let $\overline{\rho}_1 = 1/2$, $\overline{\rho}_2 = 4$; then

$$\begin{gathered}
\frac{f(t,x)}{\phi_p(x)} \leq \left(1+t^2\right)^{1/2}\frac{1}{2\sqrt{e}} \leq \frac{1}{\sqrt{2e}}, \\
\forall t\in[0,1], \quad x\in\left[0,\frac{1}{2}\right], \\
\frac{f(t,x)}{\phi_p(x)} \leq \left(1+t^2\right)^{1/2}\frac{4}{e^4} \leq \frac{4\sqrt{2}}{e^4}, \\
\forall t\in[0,1], \quad x\in[4,\infty),
\end{gathered} \tag{94}$$

so $\varepsilon_1 = 1/\sqrt{2e}$, $\varepsilon_2 = 4\sqrt{2}/e^4$, and $\forall x \in [1/2,4]$, $\max_{1/2\leq x\leq 4}(f(t,x)/\phi_p(x)) = \sqrt{2}e^{-1}$. We have $\underline{\Upsilon} = \max\{1/\sqrt{2e}, 4\sqrt{2}/e^4, \sqrt{2}e^{-1}\} = \sqrt{2}e^{-1}$, $\gamma_2 = 0.25$, and $\underline{\lambda} = (1/\underline{\Upsilon})(\Gamma(\alpha+1)(1-\eta\int_0^1 g(s)ds)/\gamma_2)^{p-1} = 3\sqrt{5}\pi^{1/4}e/4$. By Theorem 11(ii), for any $\lambda < \underline{\lambda} = 3\sqrt{5}\pi^{1/4}e/4$, the problem has no positive solution.

5. Conclusions

Recently, differential equations with p-Laplacian operator were widely discussed by several authors. In this paper, by using fixed point techniques combining with partially ordered structure of Banach space, we obtained the existence, multiply, and the dependence of parameter. These new results we presented can be used in numerical computation and analyze mathematical models of physical phenomena, mechanics, nonlinear dynamics, and many other related fields.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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