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Research Article

Variational Iteration Method for a Fractional-Order Brusselator System

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This paper presents approximate analytical solutions for the fractional-order Brusselator system using the variational iteration method. The fractional derivatives are described in the Caputo sense. This method is based on the incorporation of the correction functional for the equation. Two examples are solved as illustrations, using symbolic computation. The numerical results show that the introduced approach is a promising tool for solving system of linear and nonlinear fractional differential equations.

1. Introduction

In recent years, fractional differential equations (FDEs) have been the focus of many studies due to their appearance in various fields such as physics, chemistry, and engineering [1–3]. On the other hand, much attention has been paid to the solutions of fractional differential equations. Since most fractional differential equations do not have exact analytic solutions and approximate and numerical techniques, therefore, they are used extensively. Recently, the Adomian decomposition method, homotopy perturbation method, homotopy analysis method, and differential transform method have been used for solving a wide range of problems [4–10].

Another powerful analytical method, called the variational iteration method (VIM), was first introduced in [11]. This technique has successfully been applied to many situations: for example, see [12–17]. Reference [18] was the first where the variational iteration method was applied to fractional differential equations. Odibat and Momani [19] implemented the variational iteration method to solve partial differential equations of fractional order.

In this paper, we introduce a new application of the variational iteration method to provide approximate solutions of the fractional-order Brusselator system in the following form:

$$\begin{aligned} D_t^{\alpha_1} x(t) &= a - (\mu + 1) x(t) + x(t)^2 y(t), \\ D_t^{\alpha_2} y(t) &= \mu x(t) - x(t)^2 y(t), \end{aligned} \tag{1}$$

subject to the initial conditions

$$x(0) = c_1, \qquad y(0) = c_2, \tag{2}$$

with $a > 0$, $\mu > 0$, $0 < \alpha_i \leq 1$ $(i = 1, 2)$, and c_1, c_2 are constants.

D^{α_i} is used to represent the Caputo-type fractional derivative of order α_i.

The Riemann-Liouville definition of the fractional integration [2] is given by

$$I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \zeta)^{\alpha - 1} x(\zeta)\, d\zeta \quad \alpha > 0,\ t > 0. \tag{3}$$

For our purpose in this paper, we adopt Caputo's fractional derivative [2]:

$$D_t^{\alpha} x(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} x^{(n)}(\zeta)\, d\zeta & n-1<\alpha<n, \\ \dfrac{d^n x(t)}{dt^n} & \alpha = n, \end{cases} \quad (4)$$

where n is a positive integer and $\Gamma(\cdot)$ is the Gamma function. In particular, $0<\alpha_i<1$, and we have

$$D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{-\alpha} x'(\zeta)\, d\zeta. \quad (5)$$

The fractional-order Brusselator system has been considered by several authors recently [20–22]. Gafiychuk and Datsko investigated its stability [20]. Wang and Li proved by numerical method that the solutions of the fractional-order Brusselator system have a limit cycle [22]. We used the variational iteration method to investigate the approximate solutions of the fractional-order Brusselator system.

2. Variational Iteration Method

The principles of the variational iteration method and its applicability for various kinds of differential equations are given in [23, 24]. In [18], it was shown that the variational iteration method is also valid for fractional differential equations. In this section, following the discussion presented in [18], we extend the application of the variational iteration method to solve the fractional Brusselator equation:

$$\begin{aligned} D_t^{\alpha_1} x(t) &= a - (\mu+1) x(t) + x(t)^2 y(t), \\ D_t^{\alpha_2} y(t) &= \mu x(t) - x(t)^2 y(t). \end{aligned} \quad (6)$$

According to the variational iteration method, we can construct the correction functional for (6) as

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + I_t^{\alpha_1} \left[\lambda_1 \left(D_t^{\alpha_1} x_n(t) - a + (\mu+1) x_n(t) - x_n^2(t) y_n(t) \right) \right] \\ &= x_n(t) + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\zeta)^{\alpha_1-1} \lambda_1(\zeta) \\ &\quad \times \left(D_\zeta^{\alpha_1} x_n(\zeta) - a + (\mu+1) x_n(\zeta) - x_n^2(\zeta) y_n(\zeta) \right) d\zeta, \\ y_{n+1}(t) &= y_n(t) + I_t^{\alpha_2} \left[\lambda_2 \left(D_t^{\alpha_2} y_n(t) - \mu x_n(t) + x_n^2(t) y_n(t) \right) \right] \\ &= y_n(t) + \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-\zeta)^{\alpha_2-1} \lambda_2(\zeta) \\ &\quad \times \left(D_\zeta^{\alpha_2} y_n(\zeta) - \mu x_n(\zeta) + x_n^2(\zeta) y_n(\zeta) \right) d\zeta, \end{aligned} \quad (7)$$

where λ_i, $(i=1,2)$ are the general Lagrange multiplier, which can be identified optimally via variational theory [25, 26].

To identify approximately Lagrange multiplier, some approximations must be made. The correction functional equation (7) can be approximately expressed as follows:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + \int_0^t \lambda_1(\zeta) \left(x_n'(\zeta) - a + (\mu+1) \tilde{x}_n(\zeta) - \tilde{x}_n^2(\zeta) \tilde{y}_n(\zeta) \right) d\zeta, \\ y_{n+1}(t) &= y_n(t) + \int_0^t \lambda_2(\zeta) \left(y_n'(\zeta) - \mu \tilde{x}_n(\zeta) + \tilde{x}_n^2(\zeta) \tilde{y}_n(\zeta) \right) d\zeta, \end{aligned} \quad (8)$$

where $\tilde{x}_n$ and $\tilde{y}_n$ are considered as restricted variations, in which $\delta\tilde{x}_n = \delta\tilde{y}_n = 0$. To find the optimal λ_1 and λ_2, we proceed as follows:

$$\begin{aligned} \delta x_{n+1}(t) &= \delta x_n(t) + \delta \int_0^t \lambda_1(\zeta) \left(x_n'(\zeta) - a + (\mu+1) \tilde{x}_n(\zeta) - \tilde{x}_n^2(\zeta) \tilde{y}_n(\zeta) \right) d\zeta = 0, \\ \delta y_{n+1}(t) &= \delta y_n(t) + \delta \int_0^t \lambda_2(\zeta) \left(y_n'(\zeta) - \mu \tilde{x}_n(\zeta) + \tilde{x}_n^2(\zeta) \tilde{y}_n(\zeta) \right) d\zeta = 0. \end{aligned} \quad (9)$$

The stationary conditions can be obtained as follows:

$$\begin{aligned} \lambda_1'(\zeta)\big|_{\zeta=t} &= 0, \qquad 1 + \lambda_1(\zeta)\big|_{\zeta=t} = 0 \Longrightarrow \lambda_1(\zeta) = -1, \\ \lambda_2'(\zeta)\big|_{\zeta=t} &= 0, \qquad 1 + \lambda_2(\zeta)\big|_{\zeta=t} = 0 \Longrightarrow \lambda_2(\zeta) = -1. \end{aligned} \quad (10)$$

We substitute $\lambda_i(\zeta) = -1$, $(i=1,2)$ into the functional equation (11) to obtain the following iteration formula:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - I_t^{\alpha_1} \left[D_t^{\alpha_1} x_n(t) - a + (\mu+1) x_n(t) - x_n^2(t) y_n(t) \right], \\ y_{n+1}(t) &= y_n(t) - I_t^{\alpha_2} \left[D_t^{\alpha_2} y_n(t) - \mu x_n(t) + x_n^2(t) y_n(t) \right]. \end{aligned} \quad (11)$$

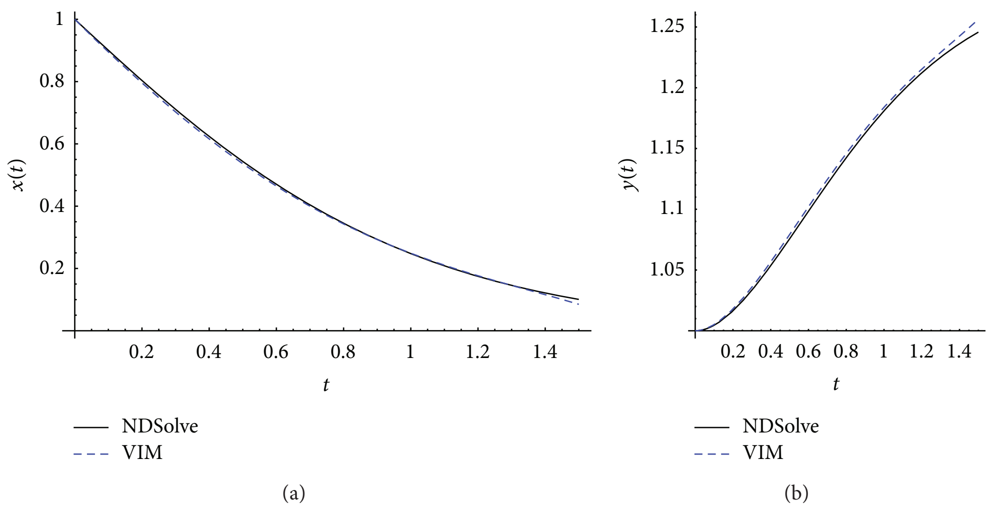


Figure 1

The initial approximations $x_0(t)$ and $y_0(t)$ can be freely chosen if they satisfy the initial conditions of the problem. Finally, we approximate the solutions $x(t) = \lim_{n\to 1} x_n(t)$ and $y(t) = \lim_{n\to 1} y_n(t)$ by the nth terms $x_n(t)$ and $y_n(t)$.

3. Illustrative Examples

For purposes of illustration of (VIM) for solving Brusselator equation, we present two examples.

Example 1. Consider the following fractional-order Brusselator system:

$$D_t^{\alpha_1} x(t) = -2x(t) + x(t)^2 y(t),$$
$$D_t^{\alpha_2} y(t) = x(t) - x(t)^2 y(t), \tag{12}$$

with the initial conditions:

$$x(0) = 1, \qquad y(0) = 1. \tag{13}$$

According to the variational iteration method and (11), the iteration formula for (12) is given by

$$x_{n+1}(t) = x_n(t) - I_t^{\alpha_1}\left[D_t^{\alpha_1} x_n(t) + 2x_n(t) - x_n^2(t)\, y_n(t)\right],$$
$$y_{n+1}(t) = y_n(t) - I_t^{\alpha_2}\left[D_t^{\alpha_2} y_n(t) - x_n(t) + x_n^2(t)\, y_n(t)\right]. \tag{14}$$

By using the above variational iteration formula, if we start with the initial approximations $x_0(t) = 1$ and $y_0(t) = 1$, we can obtain directly the other components as

$$x_1(t) = 1 - \frac{t^{\alpha_1}}{\Gamma[\alpha_1 + 1]},$$
$$y_1(t) = 1,$$
$$x_2(t) = 1 - \frac{t^{\alpha_1}}{\Gamma[\alpha_1 + 1]} + \frac{t^{3\alpha_1}\Gamma[2\alpha_1 + 1]}{\Gamma[1 + \alpha_1]^2\Gamma[1 + 3\alpha_1]},$$
$$y_2(t) = 1 + \frac{t^{\alpha_1+\alpha_2}}{\Gamma[\alpha_1 + \alpha_2 + 1]} - \frac{t^{2\alpha_1+\alpha_2}\Gamma[2\alpha_1 + 1]}{\Gamma[\alpha_1 + 1]^2\Gamma[1 + 2\alpha_1 + \alpha_2]},$$
$$\vdots \tag{15}$$

and so on; in the same way the rest of the components of the iteration formula can be obtained. Figures 1(a) and 1(b) show comparison between the approximate solutions $(x(t) \cong x_5(t))$, $((y(t) \cong y_5(t))$ of (12) obtained using VIM for the special case $\alpha_1 = \alpha_2 = 0.98$ and the numerical solutions for the special case $\alpha_1 = \alpha_2 = 1$, respectively. Figures 2(a) and 2(b) show the approximate solutions $(x(t) \cong x_5(t))$, $((y(t) \cong y_5(t))$ of (12) using VIM for the special case $\alpha_1 = \alpha_2 = 1$ and the numerical solutions, respectively.

Example 2. Consider the following fractional-order Brusselator system:

$$D^{\alpha_1} x(t) = 0.5 - 1.1x(t) + x(t)^2 y(t),$$
$$D^{\alpha_2} y(t) = 0.1x(t) - x(t)^2 y(t), \tag{16}$$

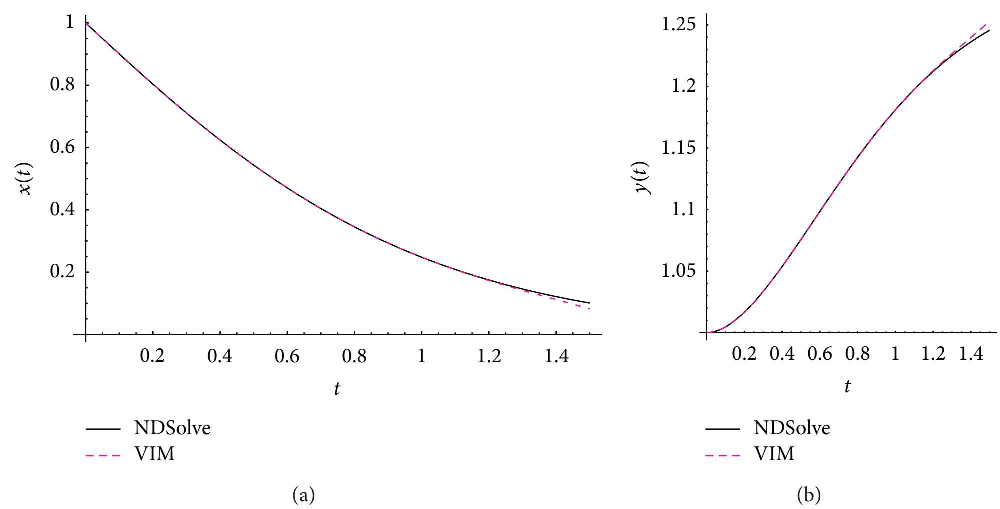


Figure 2

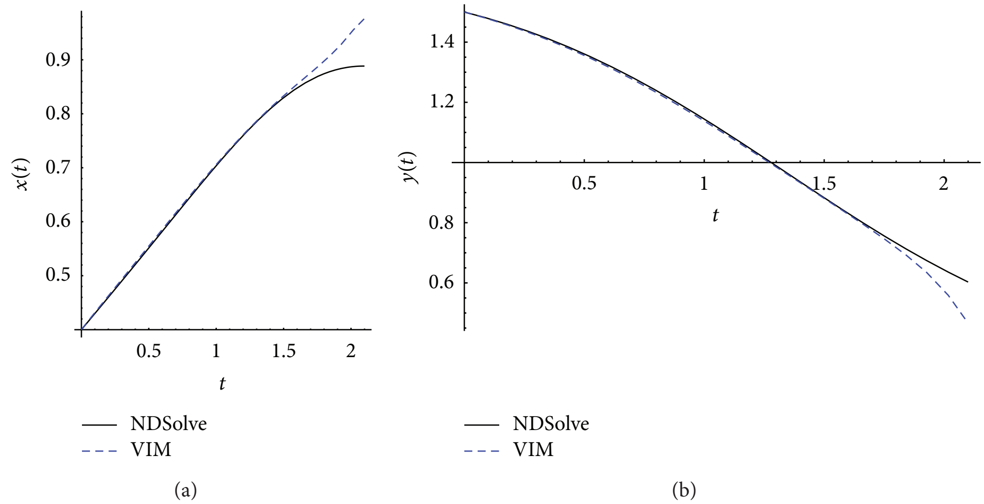


Figure 3

with the initial conditions:

$$x(0) = 0.4, \qquad y(0) = 1.5. \tag{17}$$

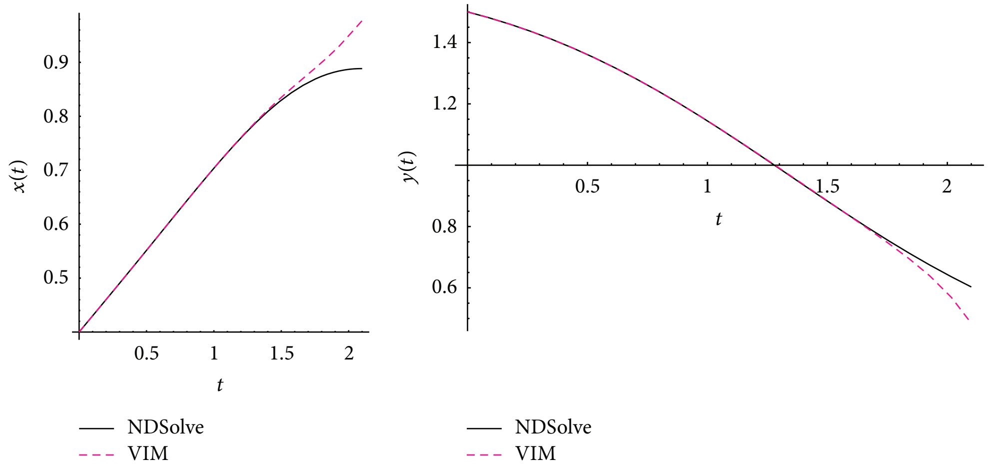
The correction functional for (16) turns out to be

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - I_t^{\alpha_1}\left[D_t^{\alpha_1} x_n(t) - 0.5 + 1.1 x_n(t) \right. \\ &\qquad \left. - x_n^2(t)\, y_n(t)\right], \\ y_{n+1}(t) &= y_n(t) - I_t^{\alpha_2}\left[D_t^{\alpha_2} y_n(t) - 0.1 x_n(t) \right. \\ &\qquad \left. + x_n^2(t)\, y_n(t)\right]. \end{aligned} \tag{18}$$

By the above variational iteration formula and beginning with the initial approximations $x_0(t) = 0.4 + (0.5 \times t^{\alpha_1}/\Gamma[\alpha_1 + 1])$ and $y_0(t) = 1.5$, we can obtain directly the other components as

$$\begin{aligned} x_1(t) &= 0.4 + \frac{0.5 t^{\alpha_1}}{\Gamma[\alpha_1+1]} + \frac{0.05 t^{2\alpha_1}}{\Gamma[2\alpha_1+1]} \\ &\quad + \frac{0.119316^{\alpha_1} t^{3\alpha_1}}{\Gamma[2\alpha_1+1]\,\Gamma[3\alpha_1+1]} - \frac{0.2 t^{\alpha_1}}{\Gamma[\alpha_1+1]}, \\ y_1(t) &= 1.5 - \frac{0.55 t^{\alpha_1+\alpha_2}}{\Gamma[\alpha_1+\alpha_2+1]} \\ &\quad - \frac{0.375 t^{2\alpha_1+\alpha_2}\Gamma[2\alpha_1+1]}{\Gamma[\alpha_1+1]^2\,\Gamma[2\alpha_1+\alpha_2+1]} - \frac{0.2 t^{\alpha_2}}{\Gamma[\alpha_2+1]}, \\ &\;\;\vdots \end{aligned} \tag{19}$$

and so on; in the same manner the remaining set of the components of the iteration formula can be obtained. Figures 3(a)



(a) (b)

Figure 4

and 3(b) show comparison between the approximate solutions $(x(t) \cong x_4(t))$, $((y(t) \cong y_4(t))$ of (16) obtained using VIM for the special case $\alpha_1 = \alpha_2 = .98$ and the numerical solutions for the special case $\alpha_1 = \alpha_2 = 1$, respectively. Figures 4(a) and 4(b) show the approximate solutions $(x(t) \cong x_4(t))$, $((y(t) \cong y_4(t))$ of (16) using VIM for the special case $\alpha_1 = \alpha_2 = 1$ and the numerical solutions, respectively.

4. Conclusions

The variational iteration method is a powerful method which is able to handle linear/nonlinear fractional differential equations. The method has been applied to fractional-order Brusselator system in order to find its approximate solutions. The results show that the applied method is suitable and inexpensive for obtaining the approximate solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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