

Research Article

Quasilinear Elliptic Equations with Hardy-Sobolev Critical Exponents: Existence and Multiplicity of Nontrivial Solutions

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We study the existence of positive solutions and multiplicity of nontrivial solutions for a class of quasilinear elliptic equations by using variational methods. Our obtained results extend some existing ones.

1. Introduction and Main Results

Let us consider the following problem:

$$-\Delta_p u - \mu \frac{|u|^{p^*(s)-2}}{|x|^s} u = f(x, u), \quad x \in \Omega \setminus \{0\}, \quad (1)$$

$$u = 0, \quad x \in \partial\Omega,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ denotes the p -Laplacian differential operator, Ω is an open bounded domain in \mathbb{R}^N ($N \geq 3$) with smooth boundary $\partial\Omega$ and $0 \in \Omega$, $0 < s < p$, $p^*(s) = p(N-s)/(N-p)$ is the Hardy-Sobolev critical exponent, and $p^* = p^*(0) = Np/(N-p)$ is the Sobolev critical exponent. Here, we let

$$\|u\| := \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p}, \quad (2)$$

which is equivalent to the usual norm of Sobolev space $W_0^{1,p}(\Omega)$ due to the Poincaré inequality. Let

$$A_s(\Omega) := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\Omega} (|u|^{p^*(s)}/|x|^s) dx \right)^{p/p^*(s)}}, \quad (3)$$

which is the best Hardy-Sobolev constant.

In the case where $s = 0$ and $\mu = 1$ hold, then (1) reduces to the quasilinear elliptic problem:

$$-\Delta_p u - |u|^{p^*-2} u = f(x, u), \quad x \in \Omega, \quad (4)$$

$$u = 0, \quad x \in \partial\Omega.$$

Gonçalves and Alves [1] have studied (4) in \mathbb{R}^N involving $f(x, u) = h(x)u^q$, $u \geq 0$ and $u \neq 0$ to obtain existence of positive solutions where $2 \leq p < N$, $0 < q < p-1$, or $p-1 < q < p^*-1$ and a suitable h . We should mention that problem (4) with $p = 2$ has been widely studied since Brézis and Nirenberg; see [2–4] and the references therein.

Ghoussoub and Yuan [5] have studied (1) with $f(x, u) = \lambda|u|^{r-2}u$, where $p < r < p^*$. They obtained a positive solution in the case where $\lambda > 0$, $\mu > 0$, and $N > [p(p-1)r + p^2]/[p + (p-1)(r-p)]$ (in particular if $N \geq p^2$) hold. They also obtained a sign-changing solution in the case where $\lambda > 0$, $\mu > 0$, and $N > [p(p-1)r + p]/[1 + (p-1)(r-p)]$ (in particular if $N > p^3 - p^2 + p$) hold. For other relevant papers, see [6–12] and the references therein.

We should mention that the energy functional associated with (1) is defined on $W_0^{1,p}(\Omega)$, which is not a Hilbert space for $p \neq 2$. Due to the lack of compactness of the embedding in $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*(s)}(\Omega, |x|^{-s} dx)$, we cannot use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais-Smale ((PS) for short) condition in $W_0^{1,p}(\Omega)$. However,

a local (PS) condition can be established in a suitable range. Then the existence result is obtained via constructing a minimax level within this range and the Mountain Pass Lemma due to Ambrosetti et al. [2] and Rabinowitz [13].

In this paper, we study (1) with a general nonlinearity by using a variational method; besides, we also considerably generalize the results obtained in [5]. In what follows, we always assume that the nonlinearity f satisfies $f(x, 0) \equiv 0$. Let $F(x, t) := \int_0^t f(x, s)ds$, $x \in \Omega$. To state our main results, we still need the following assumptions.

$$(A_1) \quad f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}), \lim_{t \rightarrow 0^+} (f(x, t)/t^{p-1}) = 0, \text{ and } \lim_{t \rightarrow \infty} (f(x, t)/t^{p^*-1}) = 0 \text{ uniformly for } x \in \overline{\Omega}.$$

$$(A_2) \quad \text{There exists a constant } \rho \text{ with } \rho > p \text{ such that } 0 < \rho F(x, t) \leq f(x, t)t, \quad \forall x \in \overline{\Omega}, \forall t \in \mathbb{R}^+ \setminus \{0\}. \quad (5)$$

$$(A_3) \quad f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}), \lim_{t \rightarrow 0} (f(x, t)/|t|^{p-1}) = 0, \text{ and } \lim_{|t| \rightarrow \infty} (f(x, t)/|t|^{p^*-1}) = 0 \text{ uniformly for } x \in \overline{\Omega}.$$

$$(A_4) \quad \text{There exists a constant } \rho \text{ with } \rho > p \text{ such that } 0 < \rho F(x, t) \leq f(x, t)t, \quad \forall x \in \overline{\Omega}, \forall t \in \mathbb{R} \setminus \{0\}. \quad (6)$$

Now, our main results read as follows.

Theorem 1. *Suppose that $N \geq 3$, $0 < \mu < \infty$, $0 \leq s < p$, and $1 < p < N$ hold. If (A_1) , (A_2) , and*

$$\rho > \max \left\{ p, p^* \left(1 - \frac{1}{p} \right), p^* - \frac{p}{p-1} \right\} \quad (7)$$

hold, then (1) has at least one positive solution.

Theorem 2. *Suppose that $N \geq 3$, $0 < \mu < \infty$, $0 \leq s < p$, and $1 < p < N$ hold. If (A_3) , (A_4) , and (7) hold, then (1) has at least two distinct nontrivial solutions.*

Noting that $\rho > p$ and $p^2 \leq N$ imply that (7) holds, therefore, we have the following corollaries.

Corollary 3. *Suppose that $0 < \mu < \infty$, $0 \leq s < p$, $p^2 \leq N$, and $1 < p < N$. Moreover, (A_1) and (A_2) hold; then (1) has at least one positive solution.*

Corollary 4. *Suppose that $0 < \mu < \infty$, $0 \leq s < p$, $p^2 \leq N$, and $1 < p < N$. Moreover, (A_3) and (A_4) hold; then (1) has at least two distinct nontrivial solutions.*

Remark 5. Theorem 1 generalizes Theorem 1.3 in [5], where the author only studied the special situation that $f(x, u) = \lambda|u|^{r-2}u$, $p < r < p^*$. There are functions f satisfying the assumptions of our Theorem 1 and not satisfying those in [5]. Let

$$f(x, t) := g(x) |t|^{k-2}t + a|t|^{l-2}t, \quad (x, t) \in \overline{\Omega} \times \mathbb{R}, \quad (8)$$

where $g(x) > 0$, $g \in L^\infty(\Omega)$, $a > 0$, and $p < k < l < p^*$. Obviously, f satisfies all the conditions of Theorem 1 in this paper, while it does not satisfy the conditions of Theorem 1.3 in [5].

The rest of this paper is organized as follows. In Section 2, we give some preliminary lemmas, which are useful in the proofs of our main results. In Section 3, we give the detailed proofs of our main results.

2. Preliminaries

In what follows, we let $\|\cdot\|_p$ denote the norm in $L^p(\Omega)$. It is obvious that the values of $f(x, t)$ for $t < 0$ are irrelevant in Theorem 1, so we may define

$$f(x, t) \equiv 0 \quad \text{for } x \in \Omega, t \leq 0. \quad (9)$$

We firstly consider the existence of nontrivial solutions to the problem:

$$-\Delta_p u = \mu \frac{(u^+)^{p^*(s)-1}}{|x|^s} + f(x, u), \quad x \in \Omega \setminus \{0\}, \quad (10)$$

$$u = 0, \quad x \in \partial\Omega.$$

The energy functional corresponding to (10) is given by

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\mu}{p^*(s)} \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s} dx - \int_\Omega F(x, u) dx, \quad u \in W_0^{1,p}(\Omega). \quad (11)$$

By Hardy-Sobolev inequalities (see [5, 14]) and (A_1) , we know $I \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Now it is well known that there exists a one-to-one correspondence between the weak solutions of (10) and the critical points of I on $W_0^{1,p}(\Omega)$. More precisely we say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (10), if, for any $v \in W_0^{1,p}(\Omega)$, there holds

$$\langle I'(u), v \rangle = \int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla v) dx - \mu \int_\Omega \frac{(u^+)^{p^*(s)-1}}{|x|^s} v dx - \int_\Omega f(x, u) v dx = 0. \quad (12)$$

Lemma 6 (see [15]). *If $f_n \rightarrow f$ a.e. in Ω and $\|f_n\|_p \leq C < \infty$ for all n and some $0 < p < \infty$, then*

$$\lim_{n \rightarrow \infty} (\|f_n\|_p^p - \|f_n - f\|_p^p) = \|f\|_p^p. \quad (13)$$

Lemma 7. *For any $a > 0$, $0 \leq b \leq 1$, and $\lambda \geq 1$, we have $(a+b)^\lambda \leq a^\lambda + \lambda(a+1)^{\lambda-1}b$.*

Proof. Let

$$h(x) := (a+x)^\lambda - a^\lambda - \lambda(a+1)^{\lambda-1}x. \quad (14)$$

Clearly, $h'(x) < 0$, $x \in (0, 1)$, so $h(b) \leq h(0) = 0$. □

Lemma 8 (see [5]). *If $0 \leq s < p$ hold, then we have*

(i) $A_s(\Omega)$ is independent of Ω , and will henceforth be denoted by A_s ;

(ii) $A_s(\Omega)$ is attained when $\Omega = \mathbb{R}^N$ by the functions

$$l_\varepsilon(x) = \left(\varepsilon(N-s) \left(\frac{N-p}{p-1} \right)^{p-1} \right)^{(N-p)/p(p-s)} \times (\varepsilon + |x|^{(p-s)/(p-1)})^{(p-N)/(p-s)} \quad (15)$$

for some $\varepsilon > 0$. Moreover, the functions $l_\varepsilon(x)$ are the only positive radial solutions of

$$-\Delta_p u = \frac{u^{p^*(s)-1}}{|x|^s} \quad (16)$$

in \mathbb{R}^N , and satisfy

$$\int_{\mathbb{R}^N} |\nabla l_\varepsilon|^p dx = \int_{\mathbb{R}^N} \frac{|l_\varepsilon|^{p^*(s)}}{|x|^s} dx = (A_s(\Omega))^{(N-s)/(p-s)}. \quad (17)$$

Lemma 9. *If (A_1) , (A_2) , and $0 < c < ((p-s)/p(N-s))A_s^{(N-s)/(p-s)}\mu^{(p-N)/(p-s)}$ hold, then I satisfies $(PS)_c$ condition.*

Proof. Suppose that $\{u_n\}$ is a $(PS)_c$ sequence in $W_0^{1,p}(\Omega)$. By (A_2) , we have

$$\begin{aligned} c + 1 + o(1) \|u_n\| &\geq I(u_n) - \frac{1}{\theta} \langle I'(u_n), u_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{\theta} \right) \|u_n\|^p + \left(\frac{1}{\theta} - \frac{1}{p^*(s)} \right) \mu \int_{\Omega} \frac{(u_n^+)^{p^*(s)}}{|x|^s} dx \\ &\quad - \int_{\Omega} \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) dx \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta} \right) \|u_n\|^p, \end{aligned} \quad (18)$$

where $\theta = \min\{p, p^*(s)\}$. Hence we conclude that $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. So there exists $u \in W_0^{1,p}(\Omega)$; going if necessary to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^\gamma(\Omega) \quad (1 < \gamma < p^*), \\ u_n &\rightarrow u \quad \text{a.e. in } \Omega, \\ &n \rightarrow \infty. \end{aligned} \quad (19)$$

By the continuity of embedding, we have $\|u_n\|_{p^*}^{p^*} \leq C_1 < \infty$. From [5], going if necessary to a subsequence, one can get that

$$\begin{aligned} \nabla u_n &\rightarrow \nabla u \quad \text{a.e. in } \Omega, \\ \frac{u_n}{x} &\rightharpoonup \frac{u}{x} \quad \text{weakly in } L^p(\Omega), \\ \int_{\Omega} \frac{|u_n|^{p^*(s)-2} u_n}{|x|^s} v dx &\rightarrow \int_{\Omega} \frac{|u|^{p^*(s)-2} u}{|x|^s} v dx, \quad \forall v \in W_0^{1,p}(\Omega) \end{aligned} \quad (20)$$

as $n \rightarrow \infty$. By (A_1) , we know that for any $\varepsilon > 0$ there exists $a(\varepsilon) > 0$ such that

$$|f(x, t) t| \leq a(\varepsilon) + \frac{1}{2C_1} \varepsilon |t|^{p^*} \quad \text{for } (x, t) \in \bar{\Omega} \times (0, +\infty). \quad (21)$$

Set $\delta := \varepsilon/2a(\varepsilon) > 0$. When $E \subset \Omega$, $\text{meas}(E) < \delta$, we get

$$\begin{aligned} \left| \int_E f(x, u_n) u_n dx \right| &\leq \int_E a(\varepsilon) dx + \frac{\varepsilon}{2C_1} \int_E |u_n|^{p^*} dx \\ &< \varepsilon \rightarrow 0 \quad \text{as } \text{meas}(E) \rightarrow 0. \end{aligned} \quad (22)$$

It follows from Vitali's theorem that

$$\int_{\Omega} f(x, u_n) u_n dx \rightarrow \int_{\Omega} f(x, u) u dx \quad \text{as } n \rightarrow \infty. \quad (23)$$

Similarly, we can also get

$$\int_{\Omega} F(x, u_n) dx \rightarrow \int_{\Omega} F(x, u) dx \quad \text{as } n \rightarrow \infty. \quad (24)$$

Since $I'(u_n) \rightarrow 0$, we have

$$\begin{aligned} \langle I'(u_n), u_n \rangle &= \|u_n\|^p - \mu \int_{\Omega} \frac{(u_n^+)^{p^*(s)}}{|x|^s} dx - \int_{\Omega} f(x, u_n) u_n dx = o(1). \end{aligned} \quad (25)$$

Let $v_n := u_n - u$, which together with Lemma 6 implies

$$\begin{aligned} \|v_n\|^p + \|u\|^p - \mu \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx - \mu \int_{\Omega} \frac{(u^+)^{p^*(s)}}{|x|^s} dx \\ - \int_{\Omega} f(x, u) u dx = o(1). \end{aligned} \quad (26)$$

From (20), we can obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle I'(u_n), u \rangle &= \|u\|^p - \mu \int_{\Omega} \frac{(u^+)^{p^*(s)}}{|x|^s} dx \\ &\quad - \int_{\Omega} f(x, u) u dx = 0. \end{aligned} \quad (27)$$

Note that $I(u_n) \rightarrow c$ as $n \rightarrow \infty$, which together with Lemma 6 implies

$$\begin{aligned} I(u_n) &= \frac{1}{p} \|v_n\|^p + \frac{1}{p} \|u\|^p - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx \\ &\quad - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{(u^+)^{p^*(s)}}{|x|^s} dx - \int_{\Omega} F(x, u) dx + o(1) \\ &= I(u) + \frac{1}{p} \|v_n\|^p - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx + o(1) \\ &= c + o(1). \end{aligned} \tag{28}$$

Therefore, one gets that

$$I(u) + \frac{1}{p} \|v_n\|^p - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx = c + o(1). \tag{29}$$

From (26) and (27), we have

$$\|v_n\|^p - \mu \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx = o(1); \tag{30}$$

then $\|v_n\|^p \rightarrow 0$ as $n \rightarrow \infty$. Otherwise, there exists a subsequence (still denoted by v_n) such that

$$\lim_{n \rightarrow \infty} \|v_n\|^p = k, \quad \lim_{n \rightarrow \infty} \mu \int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx = k, \quad k > 0. \tag{31}$$

By (3), we have

$$\|v_n\|^p \geq A_s \left(\int_{\Omega} \frac{(v_n^+)^{p^*(s)}}{|x|^s} dx \right)^{p/p^*(s)}, \quad \forall n \in \mathbb{N}; \tag{32}$$

then $k \geq A_s (k/\mu)^{p/p^*(s)}$. That is, $k \geq A_s^{(N-s)/(p-s)} \mu^{(p-N)/(p-s)}$. It follows from (29) and $c < ((p-s)/p(N-s)) A_s^{(N-s)/(p-s)} \mu^{(p-N)/(p-s)}$ that $I(u) < 0$. However, we have $I(u) \geq 0$ by (27) and (A_2) . We get a contradiction. Therefore, we can obtain

$$\|v_n\|^p \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{33}$$

From the discussion above, I satisfies $(PS)_c$ condition. \square

In the following, we shall give some estimates for the extremal functions. Let

$$\begin{aligned} C_\varepsilon &:= \left[\varepsilon(N-s) \left(\frac{N-p}{p-1} \right)^{p-1} \right]^{(N-p)/p(p-s)}, \\ U_\varepsilon(x) &:= \frac{l_\varepsilon(x)}{C_\varepsilon}. \end{aligned} \tag{34}$$

Define a function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $|x| \leq R$, $\varphi(x) = 0$ for $|x| \geq 2R$, $0 \leq \varphi(x) \leq 1$, where $B_{2R}(0) \subset \Omega$. Set

$$\begin{aligned} u_\varepsilon(x) &= \varphi(x) U_\varepsilon(x), \\ v_\varepsilon(x) &= \frac{u_\varepsilon(x)}{\left(\int_{\Omega} (|u_\varepsilon|^{p^*(s)}/|x|^s) \right)^{1/p^*(s)}}, \end{aligned} \tag{35}$$

so that $\int_{\Omega} (|v_\varepsilon|^{p^*(s)}/|x|^s) = 1$. Then, by using the argument as used in [5], we can get the following results:

$$A_s + C_2 \varepsilon^{(N-p)/(p-s)} \leq \|v_\varepsilon\|^p \leq A_s + C_3 \varepsilon^{(N-p)/(p-s)}, \tag{36}$$

$$C_4 \varepsilon^{(N-p)r/p(p-s)} \leq \int_{\Omega} |v_\varepsilon|^r dx \leq C_5 \varepsilon^{(N-p)r/p(p-s)},$$

$$1 \leq r < p^* \left(1 - \frac{1}{p} \right),$$

$$C_4 \varepsilon^{(N-p)r/p(p-s)} |\ln \varepsilon| \leq \int_{\Omega} |v_\varepsilon|^r dx \leq C_5 \varepsilon^{(N-p)r/p(p-s)} |\ln \varepsilon|,$$

$$r = p^* \left(1 - \frac{1}{p} \right),$$

$$C_4 \varepsilon^{((p-1)/(p-s))(N-r(N-p)/p)}$$

$$\leq \int_{\Omega} |v_\varepsilon|^r dx \leq C_5 \varepsilon^{((p-1)/(p-s))(N-r(N-p)/p)},$$

$$p^* \left(1 - \frac{1}{p} \right) < r < p^*.$$

(37)

Moreover, by using the Sobolev embedding theorem and (36), one can deduce

$$\int_{\Omega} |v_\varepsilon|^{p^*} dx \leq C_6 A_s^{N/(N-p)} \quad \text{for } \varepsilon \rightarrow 0^+. \tag{38}$$

Lemma 10. Suppose that $0 \leq s < p$. If (A_1) , (A_2) and (7) hold, then there exists $u_0 \in W_0^{1,p}(\Omega)$, $u_0 \not\equiv 0$, such that

$$\sup_{t \geq 0} I(tu_0) < \frac{p-s}{p(N-s)} A_s^{(N-s)/(p-s)} \mu^{(p-N)/(p-s)}. \tag{39}$$

Proof. We consider the functions

$$g(t) = I(tv_\varepsilon) = \frac{t^p}{p} \|v_\varepsilon\|^p - \mu \frac{t^{p^*(s)}}{p^*(s)} - \int_{\Omega} F(x, tv_\varepsilon) dx, \tag{40}$$

$$\bar{g}(t) = \frac{t^p}{p} \|v_\varepsilon\|^p - \mu \frac{t^{p^*(s)}}{p^*(s)}.$$

Since $\lim_{t \rightarrow \infty} g(t) = -\infty$, $g(0) = 0$, and $g(t) > 0$ for t small enough, $\sup_{t \geq 0} g(t)$ is attained for some $t_\varepsilon > 0$. Therefore, we have

$$\begin{aligned} 0 &= g'(t_\varepsilon) \\ &= t_\varepsilon^{p-1} \left(\|v_\varepsilon\|^p - \mu t_\varepsilon^{p^*(s)-p} - \frac{1}{t_\varepsilon^{p-1}} \int_{\Omega} f(x, t_\varepsilon v_\varepsilon) v_\varepsilon dx \right), \end{aligned} \tag{41}$$

and hence

$$\|v_\varepsilon\|^p = \mu t_\varepsilon^{p^*(s)-p} + \frac{1}{t_\varepsilon^{p-1}} \int_{\Omega} f(x, t_\varepsilon v_\varepsilon) v_\varepsilon dx \geq \mu t_\varepsilon^{p^*(s)-p}. \tag{42}$$

Therefore, we obtain

$$t_\varepsilon \leq \left(\frac{\|v_\varepsilon\|^p}{\mu} \right)^{1/(p^*(s)-p)} := t_\varepsilon^0. \quad (43)$$

By (A_1) , we can easily get

$$|f(x, t)| \leq \varepsilon t^{p^*-1} + d(\varepsilon) t^{p-1}, \quad d(\varepsilon) > 0. \quad (44)$$

Hence, we can get

$$\|v_\varepsilon\|^p \leq \mu t_\varepsilon^{p^*(s)-p} + \varepsilon \int_\Omega |t_\varepsilon|^{p^*-p} |v_\varepsilon|^{p^*} dx + d(\varepsilon) \int_\Omega |v_\varepsilon|^p dx. \quad (45)$$

By (36)–(38), when ε is small enough, we conclude that

$$t_\varepsilon^{p^*(s)-p} \geq \frac{A_s}{2}. \quad (46)$$

On the one hand, from Lemma 7 and (36), it follows that

$$\|v_\varepsilon\|^{p(N-s)/(p-s)} \leq A_s^{(N-s)/(p-s)} + C_7 \varepsilon^{(N-s)/(p-s)}. \quad (47)$$

On the other hand, the function $\bar{g}(t)$ attains its maximum at t_ε^0 and is increasing in the interval $[0, t_\varepsilon^0]$. Note that (A_2) implies $F(x, t) \geq C_8 |t|^\rho$, which together with (36), (46), and (47) implies that

$$\begin{aligned} g(t_\varepsilon) &\leq \bar{g}(t_\varepsilon^0) - \int_\Omega F(x, t_\varepsilon v_\varepsilon) dx \\ &= \frac{p-s}{p(N-s)} \|v_\varepsilon\|^{p(N-s)/(p-s)} \mu^{(p-N)/(p-s)} \\ &\quad - \int_\Omega F(x, t_\varepsilon v_\varepsilon) dx \\ &\leq \frac{p-s}{p(N-s)} A_s^{(N-s)/(p-s)} \mu^{(p-N)/(p-s)} \\ &\quad + C_9 \mu^{(p-N)/(p-s)} \varepsilon^{(N-p)/(p-s)} \\ &\quad - C_8 \left(\frac{A_s}{2} \right)^{\rho/(p^*(s)-p)} \int_\Omega |v_\varepsilon|^\rho dx. \end{aligned} \quad (48)$$

Furthermore, from (7) and (37), we get

$$\int_\Omega |v_\varepsilon|^\rho dx \geq C_4 \varepsilon^{((p-1)/(p-s))(N-\rho(N-p)/p)}. \quad (49)$$

By (7), we have $\rho > p^* - p/(p-1)$, which implies

$$\frac{N-p}{p-s} > \frac{p-1}{p-s} \left(N - \frac{\rho(N-p)}{p} \right). \quad (50)$$

Therefore, by choosing ε small enough, we have

$$\sup_{t \geq 0} I(tv_\varepsilon) = g(t_\varepsilon) < \frac{p-s}{p(N-s)} A_s^{(N-s)/(p-s)} \mu^{(p-N)/(p-s)}. \quad (51)$$

Hence, the proof of the lemma is completed by taking $u_0 = v_\varepsilon$. \square

3. Proofs of Main Results

Proof of Theorem 1. Let $X := W_0^{1,p}(\Omega)$. From the Sobolev and Hardy-Sobolev inequalities, we can easily get

$$\begin{aligned} \|u\|_p^p &\leq C \|u\|^p, \quad \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} dx \leq C \|u\|^{p^*(s)}, \\ \|u\|_{p^*}^{p^*} &\leq C \|u\|^{p^*}, \\ &\forall u \in X. \end{aligned} \quad (52)$$

It follows from (A_1) that

$$\begin{aligned} \exists \delta_1 > 0, \quad \text{s.t. } |f(t)| &< t^{p^*-1}, \quad \forall t > \delta_1; \\ \forall \varepsilon > 0, \quad \exists 0 < \delta_2 < \delta_1, \quad \text{s.t. } |f(t)| &< \varepsilon t^{p-1}, \\ &\forall t \in (0, \delta_2) \end{aligned} \quad (53)$$

$$\exists M > 0, \quad \text{s.t. } |f(t)| \leq M, \quad \forall t \in [\delta_2, \delta_1]$$

uniformly for all $x \in \bar{\Omega}$. Therefore, we deduce that

$$|f(x, t)| \leq t^{p^*-1} + \varepsilon t^{p-1} + M \leq \varepsilon t^{p-1} + (1 + M\delta_2^{1-p^*}) t^{p^*-1} \quad (54)$$

for all $t \in \mathbb{R}^+$ and for $x \in \bar{\Omega}$. Then one gets

$$|F(x, t)| \leq \frac{1}{p} \varepsilon |t|^p + C_{10} |t|^{p^*} \quad (55)$$

for all $t \in \mathbb{R}$ and for $x \in \bar{\Omega}$. By (52) and (55) we have

$$\begin{aligned} I(u) &= \frac{1}{p} \|u\|^p - \frac{\mu}{p^*(s)} \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s} dx - \int_\Omega F(x, u) dx \\ &\geq \frac{1}{p} \|u\|^p - \frac{C\mu}{p^*(s)} \|u^+\|^{p^*(s)} - \frac{C}{p} \varepsilon \|u\|^p - CC_{10} \|u\|^{p^*} \end{aligned} \quad (56)$$

for ε small enough. So there exists $\beta > 0$ such that

$$I(u) \geq \beta \quad \forall u \in \partial B_r = \{u \in W_0^{1,p}(\Omega), \|u\| = r\}, \quad r > 0 \text{ small enough.} \quad (57)$$

By Lemma 10, there exists $u_0 \in W_0^{1,p}(\Omega)$ with $u_0 \neq 0$ such that

$$\sup_{t \geq 0} I(tu_0) < \frac{p-s}{p(N-s)} A_s^{(N-s)/(p-s)} \mu^{(p-N)/(p-s)}. \quad (58)$$

It follows from the nonnegativity of $F(x, t)$ that

$$\begin{aligned} I(tu_0) &= \frac{1}{p} t^p \|u_0\|^p - \frac{\mu}{p^*(s)} t^{p^*(s)} \int_\Omega \frac{(u_0^+)^{p^*(s)}}{|x|^s} dx \\ &\quad - \int_\Omega F(x, tu_0) dx \\ &\leq \frac{1}{p} t^p \|u_0\|^p - \frac{\mu}{p^*(s)} t^{p^*(s)} \int_\Omega \frac{(u_0^+)^{p^*(s)}}{|x|^s} dx. \end{aligned} \quad (59)$$

Therefore, $\lim_{t \rightarrow +\infty} I(tu_0) \rightarrow -\infty$, so we can choose $t_0 > 0$ such that

$$\|t_0 u_0\| > r, \quad I(t_0 u_0) \leq 0. \quad (60)$$

By virtue of the Mountain Pass Lemma in [16], there is a sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \rightarrow c \geq \beta, \quad I'(u_n) \rightarrow 0, \quad (61)$$

where

$$c = \inf_{h \in \Gamma} \max_{t \in [0,1]} I(h(t)), \quad (62)$$

$$\Gamma = \{h \in C([0, 1], X) \mid h(0) = 0, h(1) = t_0 u_0\}.$$

Note that

$$\begin{aligned} 0 < \beta \leq c &= \inf_{h \in \Gamma} \max_{t \in [0,1]} I(h(t)) \leq \max_{t \in [0,1]} I(tt_0 u_0) \leq \sup_{t \geq 0} I(tu_0) \\ &< \frac{p-s}{p(N-s)} A_s^{(N-s)/(p-s)} \mu^{(p-N)/(p-s)}. \end{aligned} \quad (63)$$

By Lemma 9 we can assume that $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$. From the continuity of I' , we know that u is a weak solution of problem (10). Then $\langle I'(u), u^- \rangle = 0$, where $u^- = \min\{u, 0\}$. Thus $u \geq 0$. Therefore, u is a nonnegative solution of (1). By the Strong Maximum Principle [17], u is a positive solution of problem (1). Therefore, Theorem 1 holds. \square

Proof of Theorem 2. By Theorem 1, we know that (1) has a positive solution u_1 . Set

$$g(x, t) := -f(x, -t) \quad \text{for } t \in \mathbb{R}. \quad (64)$$

It follows from Theorem 1 that the equation

$$-\Delta_p u = \mu \frac{|u^+|^{p^*(s)-2}}{|x|^s} u + g(x, u) \quad (65)$$

has at least one positive solution v . Let $u_2 = -v$; then u_2 is a solution of

$$-\Delta_p u = \mu \frac{|u^+|^{p^*(s)-2}}{|x|^s} u + f(x, u). \quad (66)$$

It is obvious that $u_1 \neq 0$, $u_2 \neq 0$, and $u_1 \neq u_2$. So (1) has at least two nontrivial solutions. Therefore, Theorem 2 holds. \square

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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