Research Article

Strong and Total Lagrange Dualities for Quasiconvex Programming

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We consider the strong and total Lagrange dualities for infinite quasiconvex optimization problems. By using the epigraphs of the z-quasi-conjugates and the Greenberg-Pierskalla subdifferential of these functions, we introduce some new constraint qualifications. Under the new constraint qualifications, we provide some necessary and sufficient conditions for infinite quasiconvex optimization problems to have the strong and total Lagrange dualities.

1. Introduction

Consider the following infinite optimization problem:

$$\text{Minimize} \quad f(x),$$
$$\text{s.t.} \quad f_t(x) \le 0, \ t \in T, \tag{1}$$
$$x \in C,$$

where T is an arbitrary (possibly infinite) index set, C is a nonempty convex subset of a locally convex (Hausdorff topological vector) space X, and $f, f_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $t \in T$, are proper functions. This problem has been studied extensively under various degrees of restrictions imposed on the involved functions or on the underlying space and many problems in optimization and approximation theory such as linear semi-infinite optimization and the best approximation with restricted ranges can be recast into the form (1); see, for example, [1–12].

Observe that most works in the literature mentioned above were done under the assumptions that the involved functions are convex. Indeed, in mathematical programming, many of the problems naturally involve nonconvex functions. Recently, the quasiconvex programming, for which the involved functions are quasiconvex, has received much attention (cf. [13–18] and the references therein). Inspired by the works mentioned above, we continue to study the

optimization problem (1) but with f and f_t being quasiconvex functions. The present paper is centered around the strong Lagrange duality and the total Lagrange duality for this quasiconvex programming. Usually for the strong Lagrange duality, one finds conditions ensuring the following equality:

$$\inf_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \left\{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \right\}, \tag{2}$$

and for the total Lagrange duality, one seeks conditions ensuring that the following implication holds for $x_0 \in A := \{x \in C : f_t(x) \le 0, \ \forall t \in T\}$:

$$\left[f(x_0) = \min_{x \in A} f(x) \right]$$
$$\implies \left[\exists \lambda \in \mathbb{R}_+^{(T)}, \text{ s.t. } \inf_{x \in C} \left\{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \right\} \right. \tag{3}$$
$$\left. = f(x_0) \right],$$

where

$$\mathbb{R}_+^{(T)} := \left\{ \lambda = (\lambda_t) \in \mathbb{R}^T : \lambda_t \ge 0 \text{ for each } t \in T \right.$$
$$\left. \text{and only finitely many } \lambda_t \ne 0 \right\}. \tag{4}$$

To our knowledge, not many results are known to provide complete characterizations for the strong and total Lagrange dualities for quasiconvex programming.

Constraint qualifications involving epigraphs of the conjugate functions have been studied extensively. Our main aim in the present paper is to use these constraint qualifications (or their variations) to provide complete characterizations for the strong Lagrange duality and for the total Lagrange duality. It is well known that the Fenchel conjugate provides dual problems of convex minimization problems. In a similar way, different notions of conjugate for quasiconvex functions can be introduced in order to obtain dual problems of quasiconvex minimization problems. Note that the z-quasi-conjugate ($z \in \mathbb{R}$), defined by Greenberg and Pierskalla [13], plays in quasiconvex optimization the same role as the one Fenchel conjugate plays in convex optimization. Thus, by using the z-quasi-conjugate, we introduce a new constraint qualification which completely characterizes the strong Lagrange duality. Furthermore, many authors introduced some constraint qualifications involving the subdifferentials to establish the total Lagrange duality for convex programming. Similar to the convex case, we introduce the Greenberg-Pierskalla subdifferential to consider the total Lagrange duality for the quasiconvex programming.

The paper is organized as follows. The next section contains the necessary notations and preliminary results. In Section 3, some new constraint qualifications are provided and some relationships among them are given. In Section 4, we provide characterizations for the quasiconvex programming to have the strong Lagrange duality and the total Lagrange duality.

2. Notations and Preliminary Results

The notations used in this paper are standard (cf. [19]). In particular, we assume throughout the whole paper that X is a real locally convex space and let X^* denote the dual space of X. For $x \in X$ and $x^* \in X^*$, we write $\langle x^*, x \rangle$ for the value of x^* at x; that is, $\langle x^*, x \rangle := x^*(x)$. Let Z be a set in X. The indicator function δ_Z of Z is defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5}$$

The normal cone of Z at $z_0 \in Z$ is denoted by $N_Z(z_0)$ and is defined by

$$N_Z(z_0) := \left\{ x^* \in X^* : \langle x^*, z - z_0 \rangle \le 0 \ \forall z \in Z \right\}. \tag{6}$$

Following [2], we use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda = (\lambda_t)_{t \in T}$ with only finitely many $\lambda_t \neq 0$, and let $\mathbb{R}_+^{(T)}$ denote the nonnegative cone in $\mathbb{R}^{(T)}$; that is,

$$\mathbb{R}_+^{(T)} := \left\{ (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} : \lambda_t \ge 0 \ \text{for each} \ t \in T \right\}. \tag{7}$$

Let $f : X \to \overline{\mathbb{R}}$ be a proper function. The effective domain, convex conjugate function, and epigraph of f are

denoted by $\operatorname{dom} f$, f^*, and $\operatorname{epi} f$, respectively; they are defined by

$$\operatorname{dom} f := \left\{ x \in X : f(x) < +\infty \right\},$$

$$f^*(x^*) := \sup \left\{ \langle x^*, x \rangle - f(x) : x \in X \right\}$$
$$\text{for each } x^* \in X^*, \tag{8}$$

$$\operatorname{epi} f := \left\{ (x, r) \in X \times \mathbb{R} : f(x) \le r \right\}.$$

Recall that a function f is said to be quasiconvex if, for all $x, y \in \mathbb{R}$ and $\alpha \in (0, 1)$, the following inequality holds:

$$f(\alpha x + (1 - \alpha) y) \le \max \left\{ f(x), f(y) \right\}, \tag{9}$$

or equivalently its sublevel sets

$$\left\{ x \in X : f(x) \le r \right\} \quad \text{for each } r \in \mathbb{R} \tag{10}$$

are convex. Obviously, each convex function is quasiconvex. The following definition is taken from [13].

Definition 1. Let $z \in \mathbb{R}$. The z-quasi-conjugate of f is a function $f_z^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f_z^*(x^*) = z - \inf \left\{ f(y) : \langle x^*, y \rangle \ge z \right\}. \tag{11}$$

Note that (11) implies that

$$f_z^*(x^*) \ge z - f(y) \quad \text{for each } y \in \left\{ y \in X : \langle x^*, y \rangle \ge z \right\}. \tag{12}$$

Then the z-quasi-conjugate function f_z^* provides a lower bound for the corresponding conjugate function f^* and, indeed, the conjugate function f^* is the supremum of the z-quasi-conjugates f_z^* over z. Moreover, by the definition, one finds that f_z^* is quasiconvex for each $z \in \mathbb{R}$; that is,

$$f_z^*(\alpha x_1^* + (1 - \alpha) x_2^*) \le \max \left\{ f_z^*(x_1^*), f_z^*(x_2^*) \right\}$$
$$\text{for each } x_1^*, x_2^* \in X^*, \ \alpha \in (0, 1). \tag{13}$$

For quasiconvex functions, several types of subdifferentials have been defined and observed by many researchers, for example, GP-subdifferential [13], $\mathbb{R}$-quasi-subdifferential [20], MLS-subdifferential [21], and so on. The classical Greenberg-Pierskalla subdifferential is among the simplest concepts, which is given as follows (cf. [13]).

Definition 2. The Greenberg-Pierskalla subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^* f(x) := \left\{ x^* \in X^* : f(x) + f_{\langle x^*, x \rangle}^*(x^*) = \langle x^*, x \rangle \right\}. \tag{14}$$

We also define

$$\operatorname{dom} \partial^* f := \left\{ x \in X : \partial^* f(x) \neq \emptyset \right\},$$
$$\operatorname{im} \partial^* f := \left\{ x^* \in X^* : x^* \in \partial^* f(x) \ \text{for some } x \in X \right\}. \tag{15}$$

By definition,

$$\partial^* f(x) = \left\{ x^* \in X^* : \langle x^*, y - x \rangle \ge 0 \implies f(y) \ge f(x) \right\}, \tag{16}$$

which is equivalent to the following equivalence, holds:

$$x^* \in \partial^* f(x) \Longleftrightarrow \langle x^*, y - x \rangle < 0 \quad \text{for each } y \in S_f(x),$$
(17)

where $S_f(x)$ denotes the strict sublevel sets

$$S_f(x) := \{ y \in X : f(y) < f(x) \}.$$
(18)

Moreover, the following equivalence holds:

$$0 \in \partial^* f(x) \Longleftrightarrow x \text{ is a mimimizer of } f \text{ on } X. \quad (19)$$

Recall that the subdifferential of function g at $x \in \text{dom } g$ is defined by

$$\partial g(x) := \{ x^* \in X^* : g(x) + \langle x^*, y - x \rangle$$
$$\leq g(y) \text{ for each } y \in X \}.$$
(20)

Then,

$$\partial g(x) \subseteq \partial^* g(x) \quad \text{for each } x \in \text{dom } g. \quad (21)$$

By definition, we can obtain the following lemma easily, which was proved in [13] when $X = \mathbb{R}^n$.

Lemma 3. *Let g, h be proper quasiconvex functions on X and let $x_0 \in \text{dom } g \cap \text{dom } h$. Then the following statements hold.*

(i) *If $g \leq h$, then $g_z^* \geq h_z^*$ and hence $\text{epi } g_z^* \subseteq \text{epi } h_z^*$ for each $z \in \mathbb{R}$.*

(ii) *If $g \leq h$ and $g(x_0) = h(x_0)$, then $\partial^* g(x_0) \subseteq \partial^* h(x_0)$.*

The following lemma is known in [19, Theorem 2.8.7].

Lemma 4. *Let $g, h : X \to \overline{\mathbb{R}}$ be proper convex functions. If g or h is continuous at some point of $\text{dom } g \cap \text{dom } h$, then*

$$\text{epi } (g + h)^* = \text{epi } g^* + \text{epi } h^*,$$
$$\partial (g + h)(x) = \partial g(x) + \partial h(x)$$
(22)
$$\text{for each } x \in \text{dom } g \cap \text{dom } h;$$

consequently, if $h \in X^*$, then

$$\text{epi } (g + h)^* = \text{epi } g^* + \{h\} \times [0, +\infty),$$
$$\partial (g + h)(x) = \partial g(x) + h \quad \text{for each } x \in \text{dom } g.$$
(23)

The following example shows that (22) do not necessarily hold if g is a quasiconvex function even in the case when $h \in X^*$.

Example 5. Let $X := \mathbb{R}$ and define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$
(24)

Then

$$\text{epi } g_0^* = ((0, +\infty) \times [-1, +\infty)) \cup ((-\infty, 0] \times [0, +\infty)),$$
$$\partial^* g(0) = \{ x^* \in \mathbb{R} : x^* \cdot x < 0 \text{ for each } x < 0 \} = (0, +\infty).$$
(25)

Take $p = 2$. Then for each $x^* \in \mathbb{R}$,

$$(g + p)_0^* (x^*) = \begin{cases} -1, & x^* > 0, \\ +\infty, & x^* \leq 0, \end{cases}$$
$$p_0^* (x^*) = \begin{cases} +\infty, & x^* > 0, \\ 0, & x^* \leq 0, \end{cases}$$
(26)

and hence

$$\text{epi } (g + p)_0^* = (0, +\infty) \times [-1, +\infty),$$
$$\text{epi } p_0^* = (-\infty, 0] \times [0, +\infty),$$
(27)

Therefore,

$$\text{epi } (g + p)_0^* \neq \text{epi } g_0^* + \text{epi } p_0^*. \quad (28)$$

On the other hand, take $p = -2$. Then $S_{g+p}(0) = (-\infty, 0)$ and $S_p(0) = \emptyset$. Hence,

$$\partial^* (g + p)(0) = \{ x^* \in \mathbb{R} : x^* \cdot x < 0$$
$$\text{for each } x \in S_{g+p}(0) \} = (0, +\infty),$$
$$\partial^* p(0) = \{ x^* \in \mathbb{R} : x^* \cdot x < 0 \text{ for each } x \in S_p(0) \} = \mathbb{R}.$$
(29)

Therefore,

$$\partial^* (g + p)(0) \neq \partial^* g(0) + \partial^* p(0). \quad (30)$$

3. New Regularity Conditions for Lagrange Dualities

Unless explicitly stated otherwise, let $T, C, f, \{f_t : t \in T\}$ and A be as in Section 1; namely, T is an index set, $C \subseteq X$ is a convex set, f and $f_t, t \in T$ are proper quasiconvex functions such that $f + \delta_C + \sum_{t \in T} \lambda_t f_t$ is quasiconvex for each $\lambda \in \mathbb{R}_+^{(T)}$, and $A \neq \emptyset$ is the solution set of the following system:

$$x \in C; \qquad f_t(x) \leq 0 \quad \text{for each } t \in T. \quad (31)$$

Then, A is a convex set. Throughout we also assume that $\text{dom } f \cap A \neq \emptyset$. For each $x \in X$, let $T(x)$ be the active index set of system (31); that is,

$$T(x) := \{ t \in T : f_t(x) = 0 \}. \quad (32)$$

To study the strong Lagrange duality and the total Lagrange duality, we need the following regularity conditions.

Definition 6. The family $\{\delta_C; f_t : t \in T\}$ is said to have

(a) the quasi-(WEHP) if

$$\text{epi }(f + \delta_A)_0^* \cap (\{0\} \times \mathbb{R})$$
$$= \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi }\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)_0^* \cap (\{0\} \times \mathbb{R}); \quad (33)$$

(b) the quasi-(WBCQ) at $x \in \text{dom } f \cap A$ if

$$\partial^*(f + \delta_A)(x) \cap \{0\}$$
$$= \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial^*\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)(x) \cap \{0\}; \quad (34)$$

(c) the quasi-(WBCQ) if it has the quasi-(WBCQ) at each point in $\text{dom } f \cap A$.

Remark 7. Note that $f + \delta_C + \sum_{t \in T} \lambda_t f_t \leq f + \delta_A$ holds for each $\lambda \in \mathbb{R}_+^{(T)}$. Then, by Lemma 3(i), $(f + \delta_C + \sum_{t \in T} \lambda_t f_t)_z^* \geq (f + \delta_A)_z^*$ for each $z \in \mathbb{R}$ and hence

$$\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi }\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)_z^* \subseteq \text{epi }(f + \delta_A)_z^* \quad (35)$$
$$\text{for each } z \in \mathbb{R}.$$

Moreover, for each $\lambda \in \mathbb{R}_+^{(T)}$ satisfying $\sum_{t \in T} \lambda_t f_t(x) = 0$,

$$f(x) + \delta_A(x) = f(x) = f(x) + \delta_C(x) + \sum_{t \in T} \lambda_t f_t(x) \quad (36)$$
$$\text{for each } x \in A.$$

Thus, by Lemma 3(ii), we have

$$\bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial^*\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)(x) \subseteq \partial^*(f + \delta_A)(x). \quad (37)$$

Therefore, (33) holds if and only if

$$\text{epi }(f + \delta_A)_0^* \cap (\{0\} \times \mathbb{R})$$
$$\subseteq \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi }\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)_0^* \cap (\{0\} \times \mathbb{R}); \quad (38)$$

and (34) holds if and only if the following inclusion holds:

$$\partial^*(f + \delta_A)(x) \cap \{0\}$$
$$\subseteq \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial^*\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)(x) \cap \{0\}. \quad (39)$$

The following proposition describes the relationship between the quasi-(WEHP) and the quasi-(WBCQ).

Proposition 8. *The following implication holds:*

$$\text{the quasi-(WEHP)} \implies \text{the quasi-(WBCQ)}. \quad (40)$$

Furthermore, if $0 \in \text{im } \partial^*(f + \delta_A)$, then

$$\text{the quasi-(WEHP)} \iff \text{the quasi-(WBCQ)}. \quad (41)$$

Proof. Suppose that the quasi-(WEHP) holds. To show the quasi-(WBCQ), by Remark 7, it suffices to show (39) holds. To do this, let $x_0 \in \text{dom } f \cap A$ and let $0 \in \partial^*(f + \delta_A)(x_0)$. Then by (19), $f(x_0) \leq f(x)$ for each $x \in A$. Hence, by the definition of 0-conjugate function,

$$(0, -f(x_0)) \in \text{epi }(f + \delta_A)_0^*$$
$$= \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi }\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)_0^*, \quad (42)$$

thanks to the assumed quasi-(WEHP). This implies that there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$(0, -f(x_0)) \in \text{epi }\left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)_0^*, \quad (43)$$

where $J \subseteq T$ is a finite subset and $(\lambda_t) \in \mathbb{R}_+^{(T)}$ with $J = \{t \in T : \lambda_t \neq 0\}$. By definition, (43) is equivalent to

$$\left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)(x) \geq f(x_0) \geq \left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)(x_0)$$
$$\text{for each } x \in X, \quad (44)$$

where the last inequality holds because $f_t(x_0) \leq 0$ for each $t \in J$, while, by (19), (44) holds if and only if

$$0 \in \partial^*\left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)(x_0). \quad (45)$$

Below we show that $J \subseteq T(x_0)$. Note by (43) that

$$\left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)_0^*(0) \leq -f(x_0), \quad (46)$$

while, by definition,

$$\left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)_0^*(0) \geq -\left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)(x_0). \quad (47)$$

Hence, by the above inequalities, one has that

$$\left(f + \delta_C + \sum_{t \in J} \lambda_t f_t\right)(x_0) \geq f(x_0). \quad (48)$$

Since $\lambda_t > 0$ and $f_t(x) \le 0$ for each $t \in J$, this implies that $\lambda_t f_t(x) = 0$; that is, $f_t(x) = 0$ for each $t \in J$. Thus $J \subseteq T(x)$, and hence, (39) holds.

Conversely, suppose that $0 \in \mathrm{Im}\, \partial^*(f + \delta_A)$. To show the quasi-(WEHP), by Remark 7, we only need to show (38) holds. To do this, note that $0 \in \mathrm{Im}\, \partial^*(f + \delta_A)$. Then there exists $x_0 \in \mathrm{dom}\, f \cap A$ such that

$$0 \in \partial^*(f + \delta_A)(x_0)$$

$$\subseteq \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x_0) = 0}} \partial^*\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)(x_0), \tag{49}$$

thanks to the assumed quasi-(WBCQ). Therefore, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$x^* \in \partial^*\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)(0) \tag{50}$$

with

$$\sum_{t \in T} \lambda_t f_t(x_0) = 0. \tag{51}$$

These two relations imply that

$$f(x_0) + \left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^*_0(0) = 0. \tag{52}$$

Moreover, since $0 \in \partial^*(f + \delta_A)(x_0)$, it follows that

$$f(x_0) + (f + \delta_A)^*_0(0) = 0. \tag{53}$$

Combining (53) with (52), we have that

$$(f + \delta_A)^*_0(0) = \left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^*_0(0). \tag{54}$$

Hence, for each $r \in \mathbb{R}$ satisfying $(0, r) \in \mathrm{epi}(f + \delta_A)^*_0$, we have

$$(0, r) \in \mathrm{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^*_0$$

$$\subseteq \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \mathrm{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^*_0. \tag{55}$$

Thus, (38) holds and the proof is complete. □

4. Strong and Total Lagrange Dualities for Infinite Quasiconvex Programming

Consider the following quasiconvex programming:

$$(P) \quad \text{Minimize} \quad f(x),$$

$$\text{s.t.} \quad f_t(x) \le 0, \ t \in T, \tag{56}$$

$$x \in C.$$

Its dual problem is defined by

$$(D) \quad \sup_{\lambda \in \mathbb{R}_+^{(T)}} \ \inf_{x \in C} \left\{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \right\}. \tag{57}$$

We denote by $v(P)$ and $v(D)$ the optimal objective values of (P) and (D), respectively. Clearly, $v(P) \ge v(D)$; that is, the weak Lagrange duality holds between (P) and (D). We say that the strong Lagrange duality between (P) and (D) holds if there is no duality gap (i.e., $v(P) = v(D)$) and the dual problem (D) has an optimal solution. The following theorem gives some sufficient and necessary conditions to ensure that the strong Lagrange duality holds.

Theorem 9. *The following statements are equivalent.*

(i) *For each $\alpha \in \mathbb{R}$,*

$$(0, -\alpha) \in \mathrm{epi}(f + \delta_A)^*_0$$

$$\Longleftrightarrow \left[\exists (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)} \ s.t. \ (0, -\alpha) \right. \tag{58}$$

$$\left. \in \mathrm{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^*_0 \right].$$

(ii) *The family $\{\delta_C; f_t : t \in T\}$ has the quasi-(WEHP).*

(iii) *The strong Lagrange duality holds between (P) and (D).*

Proof. It is evident that (i) ⟺ (ii). Below we show that (i) ⟺ (iii). To do this, note that, for each $\alpha \in \mathbb{R}$,

$$(0, -\alpha) \in \mathrm{epi}(f + \delta_A)^*_0$$

$$\Longleftrightarrow (f + \delta_A)^*_0(0) \le -\alpha$$

$$\Longleftrightarrow f(x) \ge \alpha \quad \text{for each } x \in A,$$

$$(0, -\alpha) \in \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \mathrm{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^*_0 \tag{59}$$

$$\Longleftrightarrow \exists (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)},$$

$$\text{s.t.} \ f(x) + \sum_{t \in T} \lambda_t f_t(x) \ge \alpha \quad \text{for each } x \in C.$$

Suppose that (i) holds. Let $r := v(P) \in \mathbb{R}$ (if $v(P) = -\infty$, then the result holds trivially). Then, by (59), $(0, -r) \in \mathrm{epi}(f + \delta_A)^*_0$ and $(0, -r) \in \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \mathrm{epi}(f + \delta_C + \sum_{t \in T} \lambda_t f_t)^*_0$ by (i). Hence, using (59) and the definition of $v(D)$, we see that $v(D) \ge r$ and the problem (D) has an optimal solution. This together with the weak Lagrange duality implies that the strong Lagrange duality holds.

Conversely, suppose that the strong Lagrange duality holds. To show (i), by Remark 7, we only need to show that (38) holds. To do this, let $(0, -r) \in \mathrm{epi}(f + \delta_A)^*_0$. Then, by (59), $v(p) \ge r$ and $v(D) \ge r$ by (iii). Hence, by (59), we see that $(0, -r) \in \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \mathrm{epi}(f + \delta_C + \sum_{t \in T} \lambda_t f_t)^*_0$. Therefore, (38) is proved and the proof is complete. □

The following two examples illustrate Theorem 9.

Example 10. Let $X = C := \mathbb{R}$ and let $T := \{1\}$. Define the function $f, f_1 : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad f_1(x) := x^2 - 1 \quad \text{for each } x \in \mathbb{R}. \tag{60}$$

Then f is quasiconvex and $A = [-1, 1]$. Note that, for each $x \in \mathbb{R}$ and each $\lambda \geq 0$,

$$(f + \delta_A)(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & -1 \leq x < 0, \\ +\infty, & x > 1 \text{ or } x < -1, \end{cases} \tag{61}$$

$$(f + \delta_C + \lambda f_1)(x) = \begin{cases} 1 + \lambda(x^2 - 1), & x \geq 0, \\ \lambda(x^2 - 1), & x < 0. \end{cases}$$

Then for each $x^* \in \mathbb{R}$, it is easy to see that

$$(f + \delta_A)_0^*(x^*) = -\inf\{(f + \delta_A)(x) : x^* x \geq 0\}$$

$$= \begin{cases} -1, & x^* > 0, \\ 0, & x^* \leq 0, \end{cases} \tag{62}$$

and for each $\lambda \geq 0$,

$$(f + \delta_C + \lambda f_1)_0^*(x^*) = -\inf\{(f + \delta_C + \lambda f_1)(x) : x^* x \geq 0\}$$

$$= \begin{cases} -1 + \lambda, & x^* > 0, \\ \lambda, & x^* \leq 0. \end{cases} \tag{63}$$

Hence,

$$\text{epi}(f + \delta_A)_0^* = \bigcup_{\lambda \geq 0} \text{epi}(f + \delta_C + \lambda f_1)_0^*$$

$$= ((-\infty, 0] \times \mathbb{R}_+) \cup ((0, +\infty) \times [-1, +\infty)). \tag{64}$$

Therefore, by Theorem 9, we see that the strong Lagrange duality holds. In fact, $v(P) = v(D) = 0$ and $\lambda = 0$ is an optimal solution to (D).

Example 11. Let $X = C := \mathbb{R}$ and let $T := \{1\}$. Define the function $f, f_1 : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad f_1(x) := -x \quad \text{for each } x \in \mathbb{R}. \tag{65}$$

Then f is quasiconvex and $A = [0, +\infty)$. Note that, for each $x \in \mathbb{R}$ and each $\lambda \geq 0$,

$$(f + \delta_A)(x) = \begin{cases} 1, & x \geq 0, \\ +\infty, & x < 0, \end{cases} \tag{66}$$

$$(f + \delta_C + \lambda f_1)(x) = \begin{cases} 1 - \lambda x, & x \geq 0, \\ -\lambda x, & x < 0. \end{cases}$$

Then for each $x^* \in \mathbb{R}$, it is easy to see that

$$(f + \delta_A)_0^*(x^*) = -\inf\{(f + \delta_A)(x) : x^* x \geq 0\} = -1, \tag{67}$$

and for each $\lambda \geq 0$,

$$(f + \delta_C + \lambda f_1)_0^*(x^*) = -\inf\{(f + \delta_C + \lambda f_1)(x) : x^* x \geq 0\}$$

$$= \begin{cases} -1, & x^* > 0, \lambda = 0, \\ +\infty, & x^* > 0, \lambda > 0, \\ 0, & x^* \leq 0. \end{cases} \tag{68}$$

Hence,

$$\text{epi}(f + \delta_A)_0^* = \mathbb{R} \times [-1, +\infty),$$

$$\bigcup_{\lambda \geq 0} \text{epi}(f + \delta_C + \lambda f_1)_0^* \tag{69}$$

$$= ((0, +\infty) \times [-1, +\infty)) \cup ((-\infty, 0] \times [0, +\infty)).$$

Therefore, the quasi-(WEHP) does not hold and, by Theorem 9, the strong Lagrange duality does not hold (in fact, $v(P) = 1$ and $v(D) = -\infty$).

Remark 12. (a) In [5, Theorem 5.1], the authors showed that the strong Lagrange duality holds between (P) and (D) if and only if the following condition holds:

(iv) $\text{epi}(f + \delta_A)^* \cap (\{0\} \times \mathbb{R})$

$$= \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^* \cap (\{0\} \times \mathbb{R}). \tag{70}$$

Thus, by Theorem 9, the statements (i), (ii), and (iii) of Theorem 9 are equivalent to (iv).

(b) Recall from [5] that the stable strong Lagrange duality holds between (P) and (D) if, for each $p^* \in X^*$, the following equality holds:

$$\inf_{x \in A}\{f(x) + \langle p, x \rangle\}$$

$$= \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C}\left\{f(x) + \langle p, x \rangle + \sum_{t \in T} \lambda_t f_t(x)\right\}. \tag{71}$$

In [5], the authors show that the stable strong Lagrange duality holds if and and only if the family $\{\delta_C; f_t : t \in T\}$ has the conical (WEHP)$_f$; that is,

$$\text{epi}(f + \delta_A)^* = \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^*. \tag{72}$$

Naturally, we wonder if the equivalence still holds if we replace the convex conjugate function by the z-quasi-conjugate function. However, the following example shows that the stable strong Lagrange duality is not equivalent to

$$\text{epi}(f + \delta_A)_0^* = \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)_0^*. \tag{73}$$

Example 13. Let $X = C := \mathbb{R}$ and $T := \{1\}$. Define $f, f_1 : \mathbb{R} \to \mathbb{R}$ as in Example 10. Then by (64), we see that (73) holds. However, it is easy to see that, for each $y \in \mathbb{R}$,

$$(f + \delta_A)^* (y) = \begin{cases} y - 1, & y \geq 1, \\ 0, & 0 \leq y < 1, \\ -y, & y < 0 \end{cases} \quad (74)$$

and, for each $\lambda > 0$,

$$(f + \delta_C + \lambda f_1)^* (y) = \frac{y^2}{4\lambda} + \lambda. \quad (75)$$

Thus,

$$\text{epi}\,(f + \delta_A)^*$$
$$= \left\{ (x, y) \in \mathbb{R}^2 : x \geq 1, y \geq x - 1 \right\} \cup \left([0, 1) \times \mathbb{R}_+ \right)$$
$$\cup \left\{ (x, y) \in \mathbb{R}^2 : x < 0, \ y \geq -x \right\},$$
$$\bigcup_{\lambda \geq 0} \text{epi}\,(f + \delta_C + \lambda f_1)^* = (\mathbb{R} \times (0, +\infty)) \cup (\{0\} \times \mathbb{R}_+).$$
$$(76)$$

This implies that

$$\text{epi}\,(f + \delta_A)^* \neq \bigcup_{\lambda \geq 0} \text{epi}\,(f + \delta_C + \lambda f_1)^*. \quad (77)$$

Hence, by [5, Theorem 5.2], the stable strong duality does not hold. Therefore, the stable strong Lagrange duality and (73) are not equivalent.

In the remainder of this section, we study the total Lagrange duality problem; that is, when does the strong duality hold between (P) and (D) (assuming that $\emptyset \neq S(P) := \{y \in A : f(x) = \min_{x \in A} f(x)\}$)? Obviously, if the strong duality holds between (P) and (D), then so does the total duality. Hence, if one of conditions in Theorem 9 holds, then the total duality holds. Below we give some sufficient and necessary conditions to ensure that the total duality holds.

Theorem 14. *The following assertions are equivalent.*

(i) *The total Lagrange duality holds between (P) and (D).*

(ii) *For each $x \in \text{dom}\, f \cap A$,*

$$0 \in \partial^* (f + \delta_A) (x)$$

$$\iff 0 \in \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial^* \left(f + \delta_C + \sum_{t \in T} \lambda_t f_t \right) (x). \quad (78)$$

(iii) *For each $x \in \text{dom}\, f \cap A$,*

$$\partial^* (f + \delta_A) (x)$$

$$= \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial^* \left(f + \delta_C + \sum_{t \in T} \lambda_t f_t \right) (x) = X^*. \quad (79)$$

(iv) *The family $\{\delta_C; f_t : t \in T\}$ has the quasi-(WBCQ).*

Proof. It is evident that (ii)$\Leftrightarrow$(iv). Note that

$$0 \in \partial^* (f + \delta_A) (x) \iff \partial^* (f + \delta_A) (x) = X^*,$$

$$0 \in \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial^* \left(f + \delta_C + \sum_{t \in T} \lambda_t f_t \right) (x)$$

$$(80)$$

$$\iff \bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial^* \left(f + \delta_C + \sum_{t \in T} \lambda_t f_t \right) (x) = X^*.$$

Hence, (ii) $\Leftrightarrow$ (iii). Suppose that (iv) holds. Let $f(x_0) = \min_{x \in A} f(x)$. Then by (19), $0 \in \partial^* (f + \delta_A)(x_0)$. This implies that $0 \in \text{Im}\,\partial^*(f + \delta_A)$. Hence, by Proposition 8, the quasi-(WEHP) holds and by Theorem 9, the strong Lagrange duality holds between (P) and (D). Therefore, (i) holds and the implication (iv) $\Rightarrow$ (i) is proved. Below we only need to show that (i) $\Rightarrow$ (iv). To do this, assume that (i) holds. Let $x_0 \in \text{dom}\, f \cap A$. To show the quasi-(WBCQ), it suffices by Remark 7 to show that (39) holds with x_0 in place of x. To do this, let $0 \in \partial^* (f + \delta_A)(x_0)$. Then by (19), $f(x_0) = \min_{x \in A} f(x)$. Since the strong duality holds between (P_f) and (D), it follows that there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$f(x_0) = \inf_{x \in C} \left\{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \right\}$$
$$\leq f(x_0) + \sum_{t \in T} \lambda_t f_t(x_0) \leq f(x_0), \quad (81)$$

where the last inequality holds because $x_0 \in A$. Hence, $\sum_{t \in T} \lambda_t f_t(x_0) = 0$ and it follows that

$$\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t \right)(x_0) \leq \left(f + \delta_C + \sum_{t \in T} \lambda_t f_t \right)(x) \quad (82)$$

$$\text{for each } x \in X,$$

which by (19) implies that $0 \in \partial^* (f + \delta_C + \sum_{t \in T} \lambda_t f_t)(x_0)$. Therefore, (39) holds and the proof is complete. $\square$

Corollary 15. *Suppose that there exists $x_0 \in \text{dom}\, f \cap A$ such that $\partial^* f(x_0) \cap (-N_A(x_0)) \neq \emptyset$. Then, for (P) and (D), the total Lagrange duality holds if and only if the strong Lagrange duality holds.*

Proof. Let $x^* \in \partial^* f(x_0) \cap (-N_A(x_0)) \neq \emptyset$. Then, by definition,

$$\forall x \in X, \quad \langle x^*, x - x_0 \rangle \geq 0 \implies f(x) \geq f(x_0),$$
$$\langle x^*, x - x_0 \rangle \geq 0 \quad \text{for each } x \in A. \quad (83)$$

Hence, x_0 is a minimizer of f on A. This implies that $0 \in \partial^*(f + \delta_A)(x_0)$. Thus, by Proposition 8, the quasi-(WEHP) and the quasi-(WBCQ) are equivalent. Therefore, by Theorems 9 and 14, the result is seen to hold. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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