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Research Article

Preservers for the p-Norm of Linear Combinations of Positive Operators

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We describe the structure of those transformations on certain sets of positive operators which preserve the p-norm of linear combinations with given nonzero real coefficients. These sets are the collection of all positive pth Schatten-class operators and the set of its normalized elements. The results of the work generalize, extend, and unify several former theorems.

1. Introduction and Statement of the Results

The characterization of preserver transformations for a quantity assigned to tuples of elements of a given structure can be regarded as one of the main types of preserver problems. The related investigations concern a huge amount of problems, a fundamental one being the description of the isometries of metric spaces. Such maps of a normed space can be regarded as those transformations which preserve the norm of a particular linear combination of vectors in the space. However, one can consider the preservation of the norm of general linear combinations or of any other given linear combination.

In the case of general combinations, there are some results concerning the general form of the corresponding preservers on function spaces and on sets of linear operators. As for the former structures, in [1], Tonev and Yates studied the so-called norm-linear maps between uniform algebras. They called a transformation between such algebras norm-linear, if it preserves the supremum norm of all linear combinations of pairs of functions. According to one of their results [1, Theorem 20], any surjective norm-linear map between uniform algebras which satisfies some other quite weak properties is a composition operator and, therefore, is an isometric unital algebra isomorphism.

As for the setting of structures of linear operators, the problem of preserving the p-norm of convex combinations of the elements in certain sets of operators has been studied in [2] ($p \geq 1$). These sets are the collection $C_p(H)^+$ of the positive pth Schatten-class operators on a complex Hilbert space H and the set $C_p(H)_1^+$ of the operators in $C_p(H)^+$ of unit p-norm. The results of the mentioned paper concern the general form of those maps on $C_p(H)^+$ or on $C_p(H)_1^+$ which preserve the p-norm of all convex combinations. However, if we assume the preservation of the norm of only a fixed linear combination, then the description of the structure of the corresponding transformations may become much more difficult. As for this problem, in [3] the author proved two results related to the general form of the isometries of $C_p(H)_1^+$. Motivated mainly by the theorems in [2, 3], in this paper, we present several statements concerning the structure of those maps on $C_p(H)^+$ or on $C_p(H)_1^+$ which preserve the norm of linear combinations with fixed real coefficients.

Before formulating our results, we introduce the notation used throughout it. The symbol H signifies a complex Hilbert space and $B(H)$ stands for the algebra of all bounded linear operators acting on H. Let $p \geq 1$ be a real number. The usual trace functional is denoted by tr. An operator $A \in B(H)$ is termed pth Schatten-class operator, if $\operatorname{tr}|A|^p < \infty$. The symbol $C_p(H)$ stands for the set of such operators and we denote by $\|\cdot\|_p$ the p-norm which is defined by

$$A \longmapsto \|A\|_p = \left(\operatorname{tr}|A|^p\right)^{1/p} \quad \left(A \in C_p(H)\right). \tag{1}$$

It is well-known that $C_p(H)$ endowed with $\|\cdot\|_p$ is a Banach space. The set of the positive operators in $C_p(H)$ is signified by $C_p(H)^+$ and $C_p(H)_1^+$ stands for the collection of the elements of $C_p(H)^+$ with p-norm 1. The members of the set $C_1(H)_1^+$ have a fundamental role in the mathematical description of quantum mechanics. They are called density operators and their collection is denoted by $S(H)$ (we will also use this notation throughout the paper). These operators represent the quantum states of the quantum system to which H corresponds. We remark that $S(H)$ is a convex subset of $B(H)$.

After these preparations, we are in a position to present the theorems of the paper. In the rest of the work α and β denote fixed nonzero real numbers. Our first result concerns the structure of those maps on $S(H)$ which preserve the 1-norm of the linear combinations with coefficients α, β. It reads as follows.

Theorem 1. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha\beta < 0$. Assume that $\phi : S(H) \to S(H)$ is a map satisfying*

$$\|\alpha\phi(A) + \beta\phi(B)\|_1 = \|\alpha A + \beta B\|_1 \quad (A, B \in S(H)). \tag{2}$$

Moreover, suppose that ϕ is bijective in the case $\dim H = \infty$. *Then there is a unitary or an antiunitary operator U on H such that*

$$\phi(A) = UAU^* \quad (A \in S(H)). \tag{3}$$

Now, we mention two former results which can be obtained as immediate corollaries of this one. The first one is a theorem on the structure of those transformations on $S(H)$ which preserve the so-called Helstrom's measure of distinguishability (see, e.g. [4]). Helstrom's measure of distinguishability of $A, B \in S(H)$ with respect to $\mu \in [0,1]$ is

$$\frac{1 + \|\mu A - (1-\mu)B\|_1}{2}. \tag{4}$$

The result [4, Theorem 7] states that, if $\dim H < \infty$, then any completely positive trace preserving linear map on $B(H)$ which preserves the latter quantity between the elements of $S(H)$ with respect to any $\lambda \in [0,1]$ is of the form (3). This statement follows very easily from Theorem 1. The second result describes the structure of the bijective isometries of $S(H)$ under the metric which comes from $\|\cdot\|_1$. This theorem was published in [5] and it asserts that those isometries are of the form (3). Clearly, the latter statement forms a particular case of Theorem 1.

In the rest of the section, $p > 1$ denotes a fixed real number. Our next theorem is related to those transformations on $S(H)$ which preserve the quantity $\|\alpha A + \beta B\|_p$ $(A, B \in S(H))$. We remark that the last quantity is well-defined, since it is well-known $C_1(H) \subset C_p(H)$.

Theorem 2. *Let $p > 1$ and $\alpha, \beta \in \mathbb{R}$ be nonzero real numbers such that $\alpha + \beta \neq 0$. Furthermore, suppose that $\phi : S(H) \to S(H)$ is a map satisfying*

$$\|\alpha\phi(A) + \beta\phi(B)\|_p = \|\alpha A + \beta B\|_p \tag{5}$$

for all $A, B \in S(H)$, and assume that ϕ is surjective in the case $\dim H = \infty$. *Then there is a unitary or an antiunitary operator U on H such that ϕ is of the form* (3).

The following discussion is related to this theorem. In quantum information science, one of the most fundamental concepts is the entropy of quantum states. In fact, there are several notions of quantum entropy, for example, the p-entropy (see, e.g. [6]). The p-entropy $S_p[A]$ of the quantum state represented by $A \in S(H)$ is defined by

$$S_p[A] = \frac{1 - \operatorname{tr} A^p}{p-1}. \tag{6}$$

It is clear that the latter quantity equals $(1 - \|A\|_p^p)/(p-1)$. Now, assume that $\alpha \in]0,1[$ and $\beta = 1-\alpha$. Then the last observation shows that the transformations $\phi : S(H) \to S(H)$ with the property that (5) holds, for any $A, B \in S(H)$, are the maps of $S(H)$ which preserve the quantity $S_p[\alpha A + (1-\alpha)B]$ $(A, B \in S(H))$. Therefore, in the previous case, Theorem 2 concerns the general form of those mappings of $S(H)$ which preserve the p-entropy of convex combinations with coefficients $\alpha, 1-\alpha$. The next result of the paper is related to the structure of those preservers of $C_p(H)_1^+$ which satisfy (5) for each $A, B \in C_p(H)_1^+$.

Theorem 3. *Let α, β and $p > 1$ be nonzero real numbers. Assume that $\phi : C_p(H)_1^+ \to C_p(H)_1^+$ is a map satisfying* (5) *for any $A, B \in C_p(H)_1^+$ and suppose that ϕ is surjective in the case* $\dim H = \infty$. *Then there is a unitary or an antiunitary operator U on H such that*

$$\phi(A) = UAU^* \quad \left(A \in C_p(H)_1^+\right). \tag{7}$$

This theorem is a common generalization of results in [2, 3]. In fact, according to [2, Theorem 1.1], the following statement holds.

(i) Assume that H is separable and that $\phi : C_p(H)_1^+ \to C_p(H)_1^+$ is a map which preserves the p-norm of all convex combinations and which is surjective if $\dim H = \infty$. Then ϕ can be written in the form (7).

It is clear that (i) forms a particular case of Theorem 3. As for the results in [3], they describe the structure of the surjective, respectively, nonsurjective isometries of $C_p(H)_1^+$ in the case $\dim H = \infty$, respectively, $\dim H < \infty$ (see Theorems 1 and 2 in [3]). They tell us that the corresponding isometries are of the form (7), a statement which is also a particular case of Theorem 3. The last three theorems of the paper form a counterpart of the previous one for preservers of $C_p(H)^+$. The following result is related to surjective transformations.

Theorem 4. *Let α, β and $p > 1$ be nonzero real numbers. Assume that $\phi : C_p(H)^+ \to C_p(H)^+$ is a surjective map satisfying* (5) *for any $A, B \in C_p(H)^+$. Then there is a unitary or an antiunitary operator U on H such that*

$$\phi(A) = UAU^* \quad \left(A \in C_p(H)^+\right). \tag{8}$$

The first finite dimensional version of this theorem reads as follows.

Theorem 5. *Let α, β and $p > 1$ be nonzero real numbers such that $\alpha + \beta \neq 0$. Moreover, suppose that $\dim H < \infty$ and that $\phi : C_p(H)^+ \to C_p(H)^+$ is a map satisfying (5), for all $A, B \in C_p(H)^+$. Then there is a unitary or an antiunitary operator U on H such that ϕ is of the form (8).*

It is mentioned in [2, Section 4] that (i) holds also for preservers on $C_p(H)^+$. We remark that this statement is an immediate consequence of the preceding two theorems. The other finite dimensional version of Theorem 4 asserts the following.

Theorem 6. *Let α, β and $p > 1$ be nonzero real numbers such that $\alpha+\beta = 0$. Furthermore, suppose that $\dim H < \infty$ and that $\phi : C_p(H)^+ \to C_p(H)^+$ is a map satisfying (5) for each $A, B \in C_p(H)^+$. Then there is a unitary or an antiunitary operator U on H and an operator $X \in C_p(H)^+$ such that*

$$\phi(A) = UAU^* + X \quad \left(A \in C_p(H)^+\right). \tag{9}$$

Concerning this theorem, observe the fact that clearly if $\dim H < \infty$, then $C_p(H) = B(H)$. Therefore, in that case for any p, the set $C_p(H)^+$ is the cone of the positive operators on H. Thus, we see that the previous theorem describes the structure of the isometries of this cone relative to the metric induced by the p-norm.

We remark that the reverse implications in our results also hold true. Namely, in the case of an arbitrary theorem of the paper, those maps which are of the form appearing in that statement have the postulated preserver property. Theorems 3–6 describe the general form of those transformations which preserve the p-norm of a linear combination of n operators with given nonzero real coefficients in the case $n = 2$. However, in the case $n = 3, 4 \cdots$ the corresponding description can be reduced to the previous case. In fact, let $n \in \mathbb{N} \setminus \{1, 2\}$ and $\mu_1, \ldots, \mu_n \in \mathbb{R} \setminus \{0\}$ be numbers; let $\mathscr{M} \subset C_p(H)$ be a class of operators and let $\phi : \mathscr{M} \to \mathscr{M}$ be a map such that

$$\left\| \sum_{i=1}^{n} \mu_i \phi(A_i) \right\|_p = \left\| \sum_{i=1}^{n} \mu_i A_i \right\|_p \quad \left(A_1, \ldots, A_n \in \mathscr{M}\right). \tag{10}$$

Then it is very easy to check that, for any $A, B \in \mathscr{M}$, by putting A in place of a suitable A_i and B in place of all other A_j's, the above relation becomes an equality of the form (5) with some scalars $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ which are independent of A and B.

2. Proofs

In this section, we will use the following notation. We denote by $P_1(H)$ the set of rank-one projections on H. For any $x, y \in H$, the operator $x \otimes y \in B(H)$ is defined by $(x \otimes y)z = \langle z, y\rangle x$ $(z \in H)$. Observe that for each unit vector $x \in H$ the operator $x \otimes x$ is a rank-one projection. In the proofs of the results of the paper, we will need the following lemmas.

Lemma 7. *Let $\alpha, \beta \in \mathbb{R}$ be nonzero numbers. Then for any $p \geq 1$ there is an injective function $f_p : [0, 1] \to \mathbb{R}$ such that*

$$\|\alpha P + \beta Q\|_p = f_p(\operatorname{tr} PQ) \quad \left(P, Q \in P_1(H)\right). \tag{11}$$

Proof. By dividing the equation in Lemma 7 by $|\alpha| + |\beta|$, one can see that regarding the assertion, we may and do assume that $\alpha \in [1/2, 1[$ and that $\beta = \pm(1-\alpha)$. Therefore we have two cases.

In what follows, it will be assumed that $\beta = \alpha - 1$. In this case, first let $P, Q \in P_1(H)$ be different projections. In order to prove the lemma, we must compute the eigenvalues of $\alpha P + (\alpha - 1)Q$. To determine the spectrum of this operator, let x and y, respectively, be a unit vector in the range of P and Q, respectively. If we restrict $\alpha P + (\alpha - 1)Q$ to its range, then the matrix of the restriction with respect to the basis $\{x, y\}$ is

$$\begin{pmatrix} \alpha & \alpha\langle y, x\rangle \\ (\alpha - 1)\langle x, y\rangle & \alpha - 1 \end{pmatrix}. \tag{12}$$

Using the formula $\operatorname{tr} PQ = |\langle x, y\rangle|^2$, we deduce that the spectrum of this matrix is

$$\left\{ \frac{2\alpha - 1 + \sqrt{1 + 4\alpha(\alpha - 1)\operatorname{tr} PQ}}{2}, \frac{2\alpha - 1 - \sqrt{1 + 4\alpha(\alpha - 1)\operatorname{tr} PQ}}{2} \right\}. \tag{13}$$

Now, let $a = \sqrt{1 + 4\alpha(\alpha - 1)\operatorname{tr} PQ}/2$. Since the nonzero eigenvalues of $\alpha P + (\alpha - 1)Q$ are the elements of the latter set, we have

$$\|\alpha P + (\alpha - 1)Q\|_p^p = \left(a + \frac{2\alpha - 1}{2}\right)^p + \left(a + \frac{1 - 2\alpha}{2}\right)^p. \tag{14}$$

Now, let the functions $\tilde{f}_p : [(2\alpha - 1)/2, \infty[\to \mathbb{R}$ and $f_p : [0, 1] \to \mathbb{R}$ be defined by

$$\tilde{f}_p(t) = \left(\left(t + \frac{2\alpha - 1}{2}\right)^p + \left(t + \frac{1 - 2\alpha}{2}\right)^p\right)^{1/p} \quad \left(t \geq \frac{2\alpha - 1}{2}\right), \tag{15}$$

$$f_p(x) = \tilde{f}_p\left(\frac{\sqrt{1 + 4\alpha(\alpha - 1)x}}{2}\right) \quad (x \in [0, 1]).$$

By the above observations, we conclude that, with the injective map f_p, one has $\|\alpha P + (\alpha - 1)Q\|_p = f_p(\operatorname{tr} PQ)$. It is clear that if P and Q were equal, then this equality would also hold.

Now assume that $\beta = 1 - \alpha$. Then, let $P, Q \in P_1(H)$ be different operators. In the same way as in the previous paragraph, we infer that the nonzero elements of the spectrum of $\alpha P + (1 - \alpha)Q$ are

$$\frac{1 + \sqrt{1 + 4\alpha(1 - \alpha)(\operatorname{tr} PQ - 1)}}{2}, \quad \frac{1 - \sqrt{1 + 4\alpha(1 - \alpha)(\operatorname{tr} PQ - 1)}}{2}. \tag{16}$$

Next, let $b = \sqrt{1+4\alpha(1-\alpha)(\operatorname{tr} PQ - 1)}/2$. We deduce that

$$\|\alpha P + (1-\alpha)Q\|_p^p = \left(\frac{1}{2}+b\right)^p + \left(\frac{1}{2}-b\right)^p. \tag{17}$$

In this paragraph, we define the maps $\widetilde{f}_p : [0,1/2] \to \mathbb{R}$ and $f_p : [0,1] \to \mathbb{R}$ by

$$\widetilde{f}_p(t) = \left(\left(\frac{1}{2}+t\right)^p + \left(\frac{1}{2}-t\right)^p\right)^{1/p} \quad \left(t \in \left[0,\frac{1}{2}\right]\right),$$
$$f_p(x) = \widetilde{f}_p\left(\frac{\sqrt{1+4\alpha(1-\alpha)(x-1)}}{2}\right) \quad (x \in [0,1]). \tag{18}$$

Then, by differentiation we get that $\widetilde{f}_p$ is strictly monotone and hence injective which yields that f_p is injective. Furthermore, we clearly have $\|\alpha P + (1-\alpha)Q\|_p = f_p(\operatorname{tr} PQ)$. It is trivial that the latter relation is valid for equal projections P, Q and now the proof of the lemma is complete. □

The following two assertions can be regarded as a kind of identification lemmas.

Lemma 8. *Let $\gamma \geq 1$ be a fixed real number. If $A, B \in S(H)$ are such that the equality*

$$\|A - \gamma P\|_1 = \|B - \gamma P\|_1 \tag{19}$$

holds, for any $P \in P_1(H)$, then $A = B$.

Proof. Pick an arbitrary density operator $T \in S(H)$. Let $c(T)$ stand for the cardinality of the set of all nonzero eigenvalues of T. For any positive integer, $n \leq c(T)$ we denote by $\lambda_n(T)$ the nth largest eigenvalue of T and by $\mathscr{M}_n(T)$ the corresponding eigensubspace. Define the function $g_T : P_1(H) \to \mathbb{R}$ by

$$g_T(P) = \|T - \gamma P\|_1 \quad (P \in P_1(H)). \tag{20}$$

We are going to show that g_T uniquely determines $\lambda_1(T)$ and $\mathscr{M}_1(T)$. To do this, we assert that

$$\min g_T(P_1(H)) = \gamma + 1 - 2\lambda_1(T) \tag{21}$$

and g_T attains its minimum exactly for those rank-one projections on H which project into $\mathscr{M}_1(T)$. For the proof, let $P \in P_1(H)$ be arbitrary and pick a unit vector $x \in \operatorname{rng} P$ (in this paper rng denotes the range of linear operators). Choose an orthonormal basis $\{e_i\}_{i\in I}$ in the orthogonal complement of $\operatorname{rng} P$. According to [7, Lemma 2.2.], for any $K \in C_1(H)$ and for each orthonormal basis $\{f_j\}_{j\in J}$ in H, we have $\sum_{j\in J} |\langle K f_j, f_j\rangle| \leq \|K\|_1$. By applying this assertion to the operator $T - \gamma P$ and to the basis $\{x\} \cup \{e_i\}_{i\in I}$, we get

$$\begin{aligned}
&\|T - \gamma P\|_1 \\
&\quad \geq |\langle Tx, x\rangle - \gamma| + \sum_{i\in I} \langle Te_i, e_i\rangle \\
&\quad = \gamma - 2\langle Tx, x\rangle + \sum_{u\in\{x\}\cup\{e_i\}_{i\in I}} \langle Tu, u\rangle \\
&\quad = \operatorname{tr} T + \gamma - 2\langle Tx, x\rangle \\
&\quad = 1 + \gamma - 2\langle Tx, x\rangle.
\end{aligned} \tag{22}$$

On the other hand, using the Cauchy-Schwarz inequality, it is easy to see that for any unit vector $v \in H$, we have $\langle Tv, v\rangle \leq \lambda_1(T)$ and equality occurs exactly when $v \in \mathscr{M}_1(T)$. By the above observations, we deduce that $g_T(P) \geq \gamma + 1 - 2\lambda_1(T)$ and equality hold, if and only if $\operatorname{rng} P \subset \mathscr{M}_1(T)$. It is clear that this proves our assertion concerning the minimum of g_T. We conclude that g_T uniquely determines $\lambda_1(T)$ and $\mathscr{M}_1(T)$.

By the previous paragraph and the conditions of the lemma, $\lambda_1(A) = \lambda_1(B)$ and $\mathscr{M}_1(A) = \mathscr{M}_1(B)$. Then it follows that the restrictions of A and B to their invariant subspace $\mathscr{M}_1(A)$ are equal. Now, we have two possibilities. In the first case $\operatorname{tr} A|_{\mathscr{M}_1(A)} = 1$. Then, it is clear that both A and B are 0 on the orthogonal complement $\widetilde{\mathscr{M}}$ of $\mathscr{M}_1(A)$ and thus it follows that $A = B$. In the second case $\operatorname{tr} A|_{\mathscr{M}_1(A)} < 1$. Then we easily infer

$$\|A|_{\widetilde{\mathscr{M}}} - \gamma Q\|_1 = \|B|_{\widetilde{\mathscr{M}}} - \gamma Q\|_1 \quad \left(Q \in P_1\left(\widetilde{\mathscr{M}}\right)\right). \tag{23}$$

It is obvious that $0 < \operatorname{tr} A|_{\widetilde{\mathscr{M}}} = \operatorname{tr} B|_{\widetilde{\mathscr{M}}} \leq 1$ and the last displayed equality implies that for every $Q \in P_1(\widetilde{\mathscr{M}})$, we have

$$\left\|\frac{1}{\operatorname{tr} A|_{\widetilde{\mathscr{M}}}} A|_{\widetilde{\mathscr{M}}} - \frac{\gamma}{\operatorname{tr} A|_{\widetilde{\mathscr{M}}}} Q\right\|_1 = \left\|\frac{1}{\operatorname{tr} A|_{\widetilde{\mathscr{M}}}} B|_{\widetilde{\mathscr{M}}} - \frac{\gamma}{\operatorname{tr} A|_{\widetilde{\mathscr{M}}}} Q\right\|_1. \tag{24}$$

Clearly, $\gamma / \operatorname{tr} A|_{\widetilde{\mathscr{M}}} \geq 1$ and

$$\frac{1}{\operatorname{tr} A|_{\widetilde{\mathscr{M}}}} A|_{\widetilde{\mathscr{M}}}, \frac{1}{\operatorname{tr} A|_{\widetilde{\mathscr{M}}}} B|_{\widetilde{\mathscr{M}}} \in S\left(\widetilde{\mathscr{M}}\right). \tag{25}$$

By applying what we have shown in the previous paragraph, by the above observations, it follows that the largest eigenvalues and the corresponding eigensubspaces of the last displayed operators are equal. This easily yields that $\lambda_2(A) = \lambda_2(B)$ and $\mathscr{M}_2(A) = \mathscr{M}_2(B)$. Continuing the above procedure, we get that $c(A) = c(B)$ and that, for any positive integer $n \leq c(A)$, the equalities $\lambda_n(A) = \lambda_n(B)$ and $\mathscr{M}_n(A) = \mathscr{M}_n(B)$ hold. Using the spectral theorem, it then follows that $A = B$ and this completes the proof of the lemma. □

We mention that in the case $\gamma = 1$, Lemma 8 has an interesting geometrical content. Before presenting it, we recall the well-known fact that the extreme points of $S(H)$ are the rank-one projections on H. In the light of the latter claim, the mentioned case of the previous lemma can be reformulated as follows. Having endowed the space $S(H)$ with the metric coming from $\|\cdot\|_1$, each operator in $S(H)$ can be uniquely recovered from its distances to the extreme points of this set. The second identification lemma of the paper reads as follows.

Lemma 9. *Let α, β and $p > 1$ be fixed nonzero real numbers. If $A, B \in C_p(H)$ are self-adjoint operators such that $\|A\|_p = \|B\|_p$ and the equality*

$$\|\alpha A + \beta P\|_p = \|\alpha B + \beta P\|_p \tag{26}$$

holds, for any $P \in P_1(H)$, then $A = B$.

Proof. Let $\gamma = \beta/\alpha$. It is clear that

$$\|A + \gamma P\|_p = \|B + \gamma P\|_p \quad (P \in P_1(H)). \tag{27}$$

Suppose that there is a projection $Q_0 \in P_1(H)$ such that at least one of the operators $A + \gamma Q_0$ and $B + \gamma Q_0$ is 0. Then it follows from the conditions of the lemma that $A = B$. In the rest of the proof, we assume that $A + \gamma Q$ and $B + \gamma Q$ are nonzero for any $Q \in P_1(H)$. We mention that the main ideas of the following argument stem from the proof of [8, Theorem]. Fix arbitrarily a rank-one projection $P \in P_1(H)$ and a self-adjoint operator S on H and pick a number $t \in \mathbb{R}$. By referring to (27), for the operator $P(t) = e^{itS}Pe^{-itS} \in P_1(H)$, we have

$$\|A + \gamma P(t)\|_p = \|B + \gamma P(t)\|_p \,(\neq 0). \tag{28}$$

We are going to show that both sides of this equality are differentiable functions with respect to the variable t; moreover, we will determine their derivatives at 0. To this end, first consider the p-norm as a map from the real Banach space $C_p(H)_s$ of the self-adjoint elements of $C_p(H)$ to $\mathbb{R}$. Using an argument similar to the proof of [9, Theorem 2.3], one can show that $\|\cdot\|_p$ is Fréchet differentiable at any nonzero point $T \in C_p(H)_s$, furthermore, for all $R \in C_p(H)_s$,

$$\left.\frac{d\|T + tR\|_p}{dt}\right|_{t=0} = \frac{\operatorname{tr} |T|^{p-1} V_T^* R}{\|T\|_p^{p-1}}, \tag{29}$$

where $V_T \in B(H)$ is the partial isometry appearing in the polar decomposition of T. Now let $0 \neq T \in C_p(H)_s$. Then it is easy to see that $V_T = \operatorname{sgn} T$ and therefore we infer that the Fréchet derivative of $\|\cdot\|_p$ at T is given by

$$N \longmapsto \frac{\operatorname{tr} N|T|^{p-1} \operatorname{sgn} T}{\|T\|_p^{p-1}} \quad \left(N \in C_p(H)_s\right). \tag{30}$$

It is obvious that the left-hand (right-hand) side of (28) is the composition of the p-norm and of the function

$$t \longmapsto A + \gamma P(t), \quad t \in \mathbb{R} \qquad (t \longmapsto B + \gamma P(t), \ t \in \mathbb{R}). \tag{31}$$

It is easy to check that the derivative of the latter maps at 0 is $i\gamma(SP - PS)$ and then it follows that we can take the differential quotients of both sides of (28) at 0 in order to get

$$\begin{aligned} &\frac{i\gamma}{\|A + \gamma P\|_p^{p-1}} \operatorname{tr}(SP - PS) f(A + \gamma P) \\ &\quad = \frac{i\gamma}{\|B + \gamma P\|_p^{p-1}} \operatorname{tr}(SP - PS) f(B + \gamma P), \end{aligned} \tag{32}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the function defined by $f(t) = |t|^{p-1} \operatorname{sgn} t$ $(t \in \mathbb{R})$. Then by the conditions of Lemma 9, we deduce that

$$\operatorname{tr}(SP - PS)\left(f(A + \gamma P) - f(B + \gamma P)\right) = 0. \tag{33}$$

Next, let $x \in H$ be an arbitrary unit vector and insert $P = x \otimes x$ into the last displayed equality. In this way—by denoting by T_x the operator $f(A + \gamma P) - f(B + \gamma P) \in B_s(H)$, we obtain

$$\operatorname{tr}(Sx \otimes x - x \otimes Sx) T_x = 0. \tag{34}$$

Using the last displayed equality, we get

$$0 = \operatorname{tr}(Sx \otimes T_x x - x \otimes T_x Sx) = \langle Sx, T_x x\rangle - \langle x, T_x Sx\rangle. \tag{35}$$

Now, let $y \in H$ be arbitrary and put $S = y \otimes y$ into this formula in order to obtain that

$$\langle x, y\rangle \langle y, T_x x\rangle = \langle y, x\rangle \langle x, T_x y\rangle = \overline{\langle x, y\rangle \langle y, T_x x\rangle}. \tag{36}$$

It means that $\langle x, y\rangle\langle y, T_x x\rangle = \langle (x \otimes T_x x)y, y\rangle$ is real which implies that $x \otimes T_x x \in B_s(H)$; that is, $x \otimes T_x x = T_x x \otimes x$. Then it follows easily that $T_x x$ is a scalar multiple of x.

Consider a normalized eigenvector $u \in H$ of A. Then it is an eigenvector of $f(A + \gamma u \otimes u)$ and—by the previous paragraph—of $T_u = f(A + \gamma u \otimes u) - f(B + \gamma u \otimes u)$. We infer that u is an eigenvector of $f(B + \gamma u \otimes u)$. One can check that f is injective and then it follows that u is an eigenvector of $B + \gamma u \otimes u$ and, hence, B. By the above observations, any eigenvector of A is an eigenvector of B and, in the same way, we deduce that the reverse statement also holds, so we conclude that A and B have the same set of eigenvectors. Since these operators are compact and self-adjoint, it follows that there is a countable set I and an orthonormal system $\{e_i\}_{i \in I} \subset H$ and one has numbers $\lambda_i, \mu_i \in \mathbb{R}$ such that

$$A = \sum_{i \in I} \lambda_i e_i \otimes e_i, \qquad B = \sum_{i \in I} \mu_i e_i \otimes e_i. \tag{37}$$

Now, let $i_0 \in I$ be an arbitrary index and denote by P_{i_0} the operator $e_{i_0} \otimes e_{i_0}$. By (27), we have

$$\operatorname{tr}\left|A + \gamma P_{i_0}\right|^p = \operatorname{tr}\left|B + \gamma P_{i_0}\right|^p. \tag{38}$$

Using the previous observations and the fact that

$$\sum_{i \in I} |\lambda_i|^p = \|A\|_p^p = \|B\|_p^p = \sum_{i \in I} |\mu_i|^p, \tag{39}$$

we easily infer that

$$\left|\lambda_{i_0} + \gamma\right|^p - \left|\lambda_{i_0}\right|^p = \left|\mu_{i_0} + \gamma\right|^p - \left|\mu_{i_0}\right|^p. \tag{40}$$

One can check that the function $x \mapsto |x + \gamma|^p - |x|^p$ $(x \in \mathbb{R})$ is strictly monotone and therefore injective. Now, it follows that $\lambda_{i_0} = \mu_{i_0}$ and, since $i_0 \in I$ was arbitrary, we deduce that $A = B$ as stated in Lemma 9. □

We remark that putting together the identification lemmas of the paper, we obtain a remarkable generalization of [3, Lemma]. The below assertion will be used several times in the proofs of our theorems.

> (∗) If $p > 1$ is a scalar and $\mathscr{M} \subset C_p(H)$ is a convex set, then any isometry of $\mathscr{M}$ is affine.

Applying an argument which is very similar to the one employed in the second paragraph of the proof of [10, Theorem 1], one can show that the above assertion holds.

Observe that, since the role of α and β in the conditions of the theorems in the paper is symmetric, from now on we may and do assume that $|\alpha| \leq |\beta|$. The verification of our first result reads as follows.

Proof of Theorem 1. Since the signs of α and β are different, due to the absolute homogeneity of the p-norm, it is clear that we may and do assume that $\beta < 0$. We assert that ϕ preserves orthogonality. As for the proof, according to [11, (2.2)], for any $S, T \in C_1(H)^+$, we have

$$ST = 0 \Longleftrightarrow \|S - T\|_1 = \|S + T\|_1. \tag{41}$$

Now let $A, B \in S(H)$ and apply this equivalence to the operators $\alpha A, -\beta B \in C_1(H)^+$; in order to get that, one has $AB = 0$ exactly when

$$\|\alpha A + \beta B\|_1 = \|\alpha A - \beta B\|_1 = \alpha - \beta. \tag{42}$$

This obviously implies that ϕ preserves orthogonality.

Now, we have two cases. First suppose that $\dim H = \infty$. Then, since ϕ is bijective and it preserves orthogonality, it follows that the set $P_1(H)$ is invariant under ϕ and $\phi|_{P_1(H)} : P_1(H) \rightarrow P_1(H)$ is bijective. This assertion can be proved using an argument which is very similar to the one employed in the second paragraph of the proof of [12, Theorem 4]. Now by the preserver property of ϕ and Lemma 7, it follows that there is an injective function $f_1 : [0,1] \rightarrow \mathbb{R}$ such that $f_1(\operatorname{tr}\phi(P)\phi(Q)) = f_1(\operatorname{tr} PQ)$ which implies that

$$\operatorname{tr}\phi(P)\phi(Q) = \operatorname{tr} PQ \quad (P, Q \in P_1(H)). \tag{43}$$

Then, the famous theorem of Wigner (see, e.g. [13, page 12]) applies and we obtain that there exists a unitary or an antiunitary operator U on H such that

$$\phi(P) = UPU^* \quad (P \in P_1(H)). \tag{44}$$

Next, assume that $\dim H < \infty$. Then, it is easy to check that. for any $A \in S(H)$, we have $A \in P_1(H)$ exactly when A belongs to a set of $\dim H$ members of $S(H)$ with pairwise products 0. It follows that ϕ leaves the set $P_1(H)$ invariant and in the same way as in the previous case, we infer that (43) holds in this case. Then, by applying the nonsurjective version of the theorem of Wigner (cf., [13, Theorem 2.1.4.]), we get that one has a unitary or an antiunitary operator U on H such that $\phi|_{P_1(H)}$ can be written in the form (44) in the present case.

To complete the proof, let $A \in S(H)$. Then by (44), for any $P \in P_1(H)$, one has

$$\|\alpha U^*\phi(A)U + \beta P\|_1 = \|\alpha\phi(A) + \beta\phi(P)\|_1 = \|\alpha A + \beta P\|_1 \tag{45}$$

from which we deduce that

$$\left\|U^*\phi(A)U - \frac{-\beta}{\alpha}P\right\|_1 = \left\|A - \frac{-\beta}{\alpha}P\right\|_1. \tag{46}$$

Clearly, $-\beta/\alpha \geq 1$, thus it follows that Lemma 8 can be applied which gives us that $U^*\phi(A)U = A$. Since this holds, for all $A \in S(H)$, we conclude that ϕ can be written in the desired form and now the proof of the theorem is complete. □

Now we are going to verify the second result of the paper.

Proof of Theorem 2. First observe that, by putting equal operators $A, B \in S(f)$ into (5), we obtain that $\|\phi(A)\|_p = \|A\|_P$. Next, we assert that ϕ preserves orthogonality. As for the proof, let $S, T \in C_p(H)^+$. Then we have

$$\begin{aligned} \|S + T\|_p^p &= \|S\|_p^p \\ &\quad + \|T\|_p^p \Longleftrightarrow ST = 0 \Longleftrightarrow \|S - T\|_p^p \\ &= \|S\|_p^p + \|T\|_p^p. \end{aligned} \tag{47}$$

The first equivalence is an immediate consequence of [7, Lemma 2.6] and of the definition of p-norms. The second one can be proved using an argument which is very similar to the one employed in the proof of [3, Equivalence 9]. Now, let $A, B \in S(H)$. Then, plug $S = |\alpha|A$ and $B = |\beta|B$ in the first or second equivalence in (47) (according to the signs of α and β) to obtain that $AB = 0$ exactly when $\|\alpha A + \beta B\|_p^p = |\alpha|^p \|A\|_p^p + |\beta|^p\|B\|_p^p$. Since ϕ leaves the p-norm invariant, it then follows easily that ϕ preserves orthogonality.

Now, we assert that ϕ is injective. In fact, using the norm preserving property of ϕ, this follows immediately from the below characterizations for the equality of operators in $S(H)$. Let $A, B \in S(H)$. In the case $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$,

$$\begin{gathered} A = B \Longleftrightarrow \|A\|_p = \|B\|_p, \\ \|\beta B + \alpha A\|_p = |\beta + \alpha| \, \|A\|_p. \end{gathered} \tag{48}$$

To prove the nontrivial implication in this equivalence, assume that the latter two equalities hold. Then,

$$\|\beta B + \alpha A\|_p = \|\beta B\|_p + \|\alpha A\|_p. \tag{49}$$

Since $C_p(H)$ is strictly convex (see, e.g., [7, Theorem 2.4]), it then follows that βB equals αA multiplied by a positive number. We deduce that A is a positive scalar multiple of B which gives us that $A = B$, completing the proof of the desired characterization. As for the case $\operatorname{sgn}\alpha \neq \operatorname{sgn}\beta$, if this condition is satisfied, then

$$\begin{gathered} A = B \Longleftrightarrow \|A\|_p = \|B\|_p, \\ \|\beta B + \alpha A\|_p = (|\beta| - |\alpha|) \, \|A\|_p. \end{gathered} \tag{50}$$

We only have to show that, in this equivalence, the nontrivial implication holds. In order to prove it, suppose that the last two equalities are valid. In this case,

$$\|\beta B - (-\alpha) A\|_p = \|\beta B\|_p - \|-\alpha A\|_p; \tag{51}$$

that is,

$$\begin{aligned}&\left\|(\beta B-(-\alpha)A)+(-\alpha A)\right\|_p\\&\quad=\left\|\beta B-(-\alpha)A\right\|_p+\|-\alpha A\|_p,\end{aligned}\tag{52}$$

which, by the strict convexity of $C_p(H)$, yields that $\beta B - (-\alpha)A = \delta(-\alpha)A$ with some number $\delta > 0$. Then, it follows that B equals A multiplied by a positive scalar; thus $A = B$ as required. Now we conclude that in the case $\dim H = \infty$, the map ϕ is bijective.

We proceed in the same way as in the corresponding part of the proof of Theorem 1 in order to obtain that there is a unitary or an antiunitary operator U on H such that for any $A \in S(H)$ and $P \in P_1(H)$

$$\begin{aligned}\phi(P)&=UPU^*,\\ \left\|\alpha U^*\phi(A)U+\beta P\right\|_p&=\left\|\alpha A+\beta P\right\|_p.\end{aligned}\tag{53}$$

Clearly $\|U^*\phi(A)U\|_p = \|A\|_p$ and hence Lemma 9 applies and we get that $U^*\phi(A)U = A$; that is, $\phi(A) = UAU^*$ $(A \in S(H))$. The proof of Theorem 2 is complete. □

Theorem 3 can be proved using the previous argument (with the exception of its first sentence). We now turn to the verification of our fourth result.

Proof of Theorem 4. In the main part of the proof which follows, we are going to verify that ϕ is a positively homogeneous norm preserving map. In the course of the verification of this assertion, we will consider two cases.

In the first case, assume that $\alpha + \beta \neq 0$. Then by inserting identical operators $A, B \in C_p(H)^+$ into (5), we see that ϕ leaves the norm of operators invariant. In particular, $\phi(0) = 0$. In what follows, we are going to show that ϕ is positively homogeneous. To this end, let $A \in C_p(H)^+$ be a nonzero operator and let τ be a positive number. Now, let us consider three subcases. In the first one, assume that $\operatorname{sgn}\alpha \neq \operatorname{sgn}\beta$ and that

$$\left\|\alpha\phi(\tau A)\right\|_p-\left\|\beta\phi(A)\right\|_p\geq 0.\tag{54}$$

Then using the norm preserving property of ϕ, we compute

$$\begin{aligned}&\left\|\alpha\phi(\tau A)\right\|_p-\left\|-\beta\phi(A)\right\|_p\\&\quad=\left|\left\|\alpha\phi(\tau A)\right\|_p-\left\|\beta\phi(A)\right\|_p\right|\\&\quad=\left|\left|\alpha\right|\left\|\tau A\right\|_p-\left|\beta\right|\left\|A\right\|_p\right|=\left|\left|\alpha\right|\tau-\left|\beta\right|\right|\cdot\left\|A\right\|_p\\&\quad=\left\|\left|\alpha\right|\tau A-\left|\beta\right|A\right\|_p\\&\quad=\left\|\alpha\tau A+\beta A\right\|_p=\left\|\alpha\phi(\tau A)+\beta\phi(A)\right\|_p\\&\quad=\left\|\alpha\phi(\tau A)-(-\beta\phi(A))\right\|_p\end{aligned}\tag{55}$$

and hence

$$\left\|\alpha\phi(\tau A)\right\|_p-\left\|-\beta\phi(A)\right\|_p=\left\|\alpha\phi(\tau A)-(-\beta\phi(A))\right\|_p.\tag{56}$$

There is a similar equality (51) in the second paragraph of the proof of Theorem 2. In the same way as in that part of the paper the last displayed relation yields that we have a number $\delta > 0$ such that $\phi(\tau A) = \delta\phi(A)$. Taking the norms of both sides of this relation, we get that $\delta = \tau$ which gives us that $\phi(\tau A) = \tau\phi(A)$. In the second subcase where $\operatorname{sgn}\alpha \neq \operatorname{sgn}\beta$ and

$$\left\|\alpha\phi(\tau A)\right\|_p-\left\|\beta\phi(A)\right\|_p<0\tag{57}$$

using an argument which is very similar to the one applied in the previous subcase, we obtain the same conclusion. As for the third subcase, let $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$. Then in a similar fashion as in the proof of the first subcase, we deduce that

$$\left\|\alpha\phi(\tau A)\right\|_p+\left\|\beta\phi(A)\right\|_p=\left\|\alpha\phi(\tau A)+\beta\phi(A)\right\|_p\tag{58}$$

which implies that

$$\alpha\phi(\tau A)=\varepsilon\beta\phi(A)\tag{59}$$

with some scalar $\varepsilon > 0$. Taking the norms of the sides of the latter equality, we obtain that $\varepsilon = \alpha\tau/\beta$ which in turn yields that $\phi(\tau A) = \tau\phi(A)$. Since ϕ sends 0 to 0, by what we have proved so far, we conclude that in the case $\alpha + \beta \neq 0$, ϕ is positively homogeneous.

Now, we turn to the second case where $\alpha + \beta = 0$. Then observe that ϕ is an isometry of $C_p(H)^+$ with respect to the metric induced by the p-norm. Referring to $(*)$, we obtain that ϕ is affine. It is easy to see that the only extreme point of $C_p(H)^+$ is 0 and, by the affinity and the surjectivity of ϕ, we infer that $\phi(0) = 0$. This assertion has two consequences. On one hand, by the affinity of ϕ, we deduce that it is positively homogeneous. On the other hand, since ϕ is an isometry, it follows that $\|\phi(A)\|_p = \|A\|_p$ $(A \in C_p(H)^+)$.

By what we have proved so far, for any $A \in C_p(H)^+$ and $\tau \geq 0$, we have $\phi(\tau A) = \tau\phi(A)$ and $\|\phi(A)\|_p = \|A\|_p$. Now we deduce that ϕ preserves the elements of $C_p(H)_1^+$ in both directions. Then, it is clear that ϕ leaves the set $C_p(H)_1^+$ invariant; moreover, the restriction of ϕ to $C_p(H)_1^+$ is a surjective map of this set. Therefore, we can apply Theorem 3 to this restriction in order to obtain that

$$\phi(A)=UAU^*\quad\left(A\in C_p(H)_1^+\right)\tag{60}$$

with some unitary or antiunitary operator U on H. By the positive homogeneity of ϕ, it follows that this equality holds for all $A \in C_p(H)^+$ and now we are done. □

Theorem 5 can be proved by applying the corresponding parts of the preceding proof. In what follows, we will verify the last result of the paper.

Proof of Theorem 6. Denote by $B(H)^+ = (C_p(H)^+)$ the set of all positive operators on H. Turning to the verification of Theorem 6, according to $(*)$, the isometry ϕ is affine. Let $X = \phi(0)$ and define the map $\psi : B(H)^+ \to B_s(H)$ by

$$\psi(A)=\phi(A)-X\quad\left(A\in B(H)^+\right).\tag{61}$$

It is clear that ψ is affine and, since it sends 0 to 0, it follows that

$$\psi(\tau A) = \tau\psi(A) \tag{62}$$

and $\|\psi(A)\|_p = \|A\|_p$ holds, for all $A \in B(H)^+$ and $\tau \geq 0$. We assert that ψ maps $B(H)^+$ into itself. To see this, let $A \in B(H)^+$. By (62), the range of ψ is closed with respect to multiplication by nonnegative scalars. Choose an arbitrary positive integer n. On one hand, by referring to the latter fact, we obtain that $n\psi(A) \in \psi(B(H)^+)$. On the other hand, the elements of the range of ψ are apparently greater than or equal to $-X$ (with respect to the usual order $\geq$ between self-adjoint operators). Hence, we deduce that $n\psi(A) \geq -X$; that is, $\psi(A) \geq -(1/n)X$, and then, by letting n tend to ∞, we infer that $\psi(A) \geq 0$. Thus $\psi(B(H)^+) \subset B(H)^+$ as required.

By what we have proved so far, we see that ψ leaves the set $C_p(H)_1^+$ invariant and therefore Theorem 3 applies and we get that

$$\psi(A) = UAU^* \quad \left(A \in C_p(H)_1^+\right) \tag{63}$$

with some unitary or antiunitary operator U on H. Then referring to (62) it follows that this equality is valid for any $A \in C_p(H)^+$ and, by transforming back to ϕ, we obtain the statement of Theorem 6. □

3. Remarks

The scalars α and β in Theorems 1, 2, and 3 are not just arbitrary nonzero real numbers; they enjoy some additional properties. However, it is a natural question to ask whether the conclusion of the latter results remains true in the case where those properties are not postulated. As for Theorem 1, its consequence is no longer valid if $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$. In fact, if the signs of α and β coincide, then any map of $S(H)$ has the preserver property postulated in Theorem 1. Theorem 2 is valid also in the case $\alpha + \beta = 0$. Observe that, in this case, the map ϕ appearing in it is an isometry of $S(H)$ under the metric induced by the p-norm. By [10, Theorem 1], in the case $\dim H < \infty$, such maps have the form (3). As for the infinite dimensional case, if $\dim H = \infty$, then, according to $(*)$, ϕ is affine and therefore it is an affine bijection of $S(H)$. It is known (see, e.g, [13, page 13]) that such bijections have the form (3).

The results of the paper concern preservers on $C_p(H)^+$ and $C_p(H)_1^+$ $(p \geq 1)$. However, there is a space among the latter ones that does not appear in the previous parts of this work, namely, $C_1(H)^+$. So, for the sake of completeness, now, we deal with the maps $\phi : C_1(H)^+ \to C_1(H)^+$ having the property that (5) holds for any $A, B \in C_1(H)^+$. Using arguments similar to the ones applied in the proofs of our theorems, it can be shown that, if $p > 1$, then such transformations are of the form $A \mapsto UAU^*$ $(A \in C_1(H)^+)$. In the case $p = 1$, we have only partial results concerning those maps $\phi : C_1(H)^+ \to C_1(H)^+$ that preserve the quantity

$$\|\alpha A + \beta B\|_1 \quad \left(A, B \in C_1(H)^+\right). \tag{64}$$

One of these is the following trivial assertion. If $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$, then a transformation preserves the former quantity exactly when it leaves the trace invariant. However, it is clear that the structure of trace preserving maps on $C_1(H)^+$ is completely irregular.

As for possible generalizations of our identification lemmas, we remark that if the requirement $\|A\|_p = \|B\|_p$ was omitted from the conditions of Lemma 9, then it would not hold true. In fact, assume that $\dim H = 2$ and denote by I the identity operator on H. Then it is very easy to check that, for the operators $(2/3)I$ and $(1/3)I$, one has $\|(2/3)I - P\|_2 = \|(1/3)I - P\|_2$, for any $P \in P_1(H)$. Turning to Lemma 8, we remark that it does not hold in the case $\gamma \leq 0$, because then the equality condition appearing in that assertion is a tautology. Moreover, our conjecture is that, in the case $\gamma \in]0, 1[$, Lemma 8 is valid.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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