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Research Article

On k-Distance Pell Numbers in 3-Edge-Coloured Graphs

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We define in this paper new distance generalizations of the Pell numbers and the companion Pell numbers. We give a graph interpretation of these numbers with respect to a special 3-edge colouring of the graph.

1. Introduction

The Fibonacci sequence is defined by the following recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = F_1 = 1$. Among sequences of the Fibonacci type there is the Pell sequence defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with the initial conditions $P_0 = 0$ and $P_1 = 1$. The companion Pell sequence is closely related to the Pell sequence and is given by formula $Q_0 = Q_1 = 2$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. The Pell sequences play an important role in the number theory and they have many interesting interpretations. We recall some of them.

(i) The number of lattice paths from the point $O(0,0)$ to the line $x = n$ consisting of $a = [1,1]$, $b = [1,-1]$, and $c = [2,0]$ steps is equal to P_{n+1}; see [1].

(ii) The number of compositions (i.e., ordered partitions) of a number n into two sorts of of 1's and one sort of 2's is equal to P_{n+1}, see [1].

(iii) The number of n-step non-self-intersecting paths starting at the point $O(0,0)$ with steps of types $p = [1,0]$, $q = [-1,0]$, or $r = [0,1]$ is equal to $(1/2)Q_{n+1}$, see [1].

Interesting generalizations of the numbers of the Fibonacci type (also generalizations of the Pell sequences) are studied, for instance, by Kilic in [2–4]. Among others Kilic in [4] introduced the generalization of the Pell numbers. He defined the generalized Pell (p,i)-numbers for any given p, where $p \geq 1$ and for $n > p+1$ and $0 \leq i \leq p$ in the following way $P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1)$ with initial conditions and $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$, $P_p^{(i)}(i+1) = \cdots = P_p^{(i)}(p+1) = 1$. If $i = 0$ then $P_p^{(i)}(1) = \cdots = P_p^{(i)}(p+1) = 1$.

In this paper we describe new kinds of generalized Pell sequence and the companion Pell sequence. Our generalization is closely related to the recurrence given in [4] by Kilic. By other initial conditions we obtain other generalized Pell sequences. We give their graph interpretations which are closely related to a concept of edge colouring in graphs. Graph interpretations of the Fibonacci numbers and the like are study intensively; see, for example, [5–9].

2. k-Distance Pell Sequences $P_k(n)$ and $Q_k(n)$

Let $k \geq 2$, $n \geq 0$ be integers. The k-distance Pell sequence $P_k(n)$ is defined by the kth order linear recurrence relation:

$$P_k(n) = 2P_k(n-1) + P_k(n-k) \quad \text{for } n \geq k, \tag{1}$$

with the following initial conditions:

$$P_k(n) = \begin{cases} 0 & \text{if } n = 0, \\ 2^{n-1} & \text{if } n = 1, 2, 3, \ldots, k-1. \end{cases} \tag{2}$$

If $k = 2$, then this definition reduces to the classical Pell numbers; that is, $P_2(n) = P_n$.

Table 1 includes a few first words of the $P_k(n)$ for special values of k.

Firstly we give an interpretation of the k-distance Pell sequence, which generalizes result given in (i).

Table 1: The k-distance Pell sequence $P_k(n)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741	13860	33461
$P_3(n)$	0	1	2	4	9	20	44	97	214	472	1041	2296	5064	11169
$P_4(n)$	0	1	2	4	8	17	36	76	160	337	710	1496	3152	6641
$P_5(n)$	0	1	2	4	8	16	33	68	140	288	592	1217	2502	5144
$P_6(n)$	0	1	2	4	8	16	32	65	132	268	514	1104	2240	4545

Table 2: The k-distance companion Pell sequence $Q_k(n)$.

n	0	1	2	3	4	5	6	7	8	9	10	11
Q_n	2	2	6	14	34	82	198	478	1154	2786	6726	16238
$Q_3(n)$	3	2	4	11	24	52	115	254	560	1235	2724	6008
$Q_4(n)$	4	2	4	8	20	42	88	184	388	818	1724	3632
$Q_5(n)$	3	2	4	8	16	37	76	156	320	656	1349	2774
$Q_6(n)$	6	2	4	8	16	32	70	142	288	584	1184	2400

Theorem 1. *Let $k \geq 2$ and $n \geq 1$ be integers. Then the number of lattice paths from the point $O(0,0)$ to the line $x = n$ consisting of $a = [1,1]$, $b = [1,-1]$, and $c = [k,0]$ steps is equal to $P_k(n+1)$.*

Proof. Let $s(k,n)$ be the number of paths from the point $O(0,0)$ to the line $x = n$. It is easy to notice that $s(k,n) = P_k(n+1)$ for $n = 1,2,\ldots,k$.

Let $n \geq k+1$ and let $s_a(k,n)$, $s_b(k,n)$, and $s_c(k,n)$ denote the number of such paths from the point $O(0,0)$ to the line $x = n$, for which the last step is of the form $a = [1,1]$, $b = [1,-1]$, and $c = [k,0]$, respectively.

It can be easily seen that $s_a(k,n) = s(k,n-1)$, $s_b(k,n) = s(k,n-1)$, and $s_c(k,n) = s(k,n-k)$. Since $s(k,n) = s_a(k,n) + s_b(k,n) + s_c(k,n)$ then we have $s(k,n) = 2s(k,n-1) + s(k,n-k)$ and consequently $s(k,n) = P_k(n+1)$, for all $n \geq 1$. □

In the same way we can prove the generalization of the result (ii).

Theorem 2. *Let $k \geq 2$ and $n \geq 1$ be integers. Then the number of all compositions of the number n into two sorts of 1's and one sort of k's is equal to $P_k(n+1)$.*

By analogy to the Pell sequence we introduce a generalization of the companion Pell sequence which generalizes the classical companion Pell sequence in the distance sense.

Let $k \geq 2$, $n \geq 0$ be integers. The k-distance companion Pell sequence $Q_k(n)$ is defined by the kth order linear recurrence relation:

$$Q_k(n) = 2Q_k(n-1) + Q_k(n-k) \quad \text{for } n \geq k, \tag{3}$$

with the initial conditions

$$Q_k(n) = \begin{cases} k & \text{if } n = 0, \\ 2^n & \text{if } n = 1,2,3,\ldots,k-1. \end{cases} \tag{4}$$

If $k = 2$ then $Q_2(n)$ gives the classical companion Pell numbers Q_n; that is, $Q_2(n) = Q_n$.

Table 2 includes a few first words of the $Q_k(n)$ for special values of k.

The following theorem gives the basic relation between $P_k(n)$ and $Q_k(n)$.

Theorem 3. *Let $k \geq 2$ and $n \geq k-1$ be integers. Then*

$$Q_k(n) = 2P_k(n) + kP_k(n-k+1). \tag{5}$$

Proof (by induction on n). For $n = k-1$ the result follows immediately by the definitions of $P_k(n)$ and $Q_k(n)$. Let $n \geq k$. Assume that formula (5) is true for $t = k, k+1, \ldots, n$. We will prove that $Q_k(n+1) = 2P_k(n+1) + kP_k(n-k+2)$.

By the induction hypothesis and the definitions of $P_k(n)$ and $Q_k(n)$, we have that

$$\begin{aligned} 2P_k(n+1) &+ kP_k(n-k+2) \\ &= 2[2P_k(n) + P_k(n-k+1)] \\ &\quad + k[2P_k(n-k+1) + P_k(n-2k+2)] \\ &= 2[2P_k(n) + kP_k(n-k+1)] \\ &\quad + 2P_k(n-k+1) + kP_k(n-2k+2) \\ &= 2Q_k(n) + Q_k(n-k+1) = Q_k(n+1), \end{aligned} \tag{6}$$

which ends the proof. □

For $k = 2$ Theorem 3 gives the well-known relation between the classical Pell numbers and the companion Pell numbers; namely, $Q_n = 2(P_n + P_{n-1})$.

Theorem 4. *Let $k \geq 2$, $m \geq 1$, and $n \geq 0$ be integers. Then*

$$P_k(n) + 2\sum_{i=1}^{m} P_k(n+ki-1) = P_k(n+km), \tag{7}$$

$$Q_k(n) + 2\sum_{i=1}^{m} Q_k(n+ki-1) = Q_k(n+km). \tag{8}$$

Proof of (7) *(by induction on m).* If $m = 1$ then the equation is obvious. Let $m \geq 2$. Assume that formula (7) is true for an arbitrary m. We will prove that $P_k(n) + 2\sum_{i=1}^{m+1} P_k(n+ik-1) = P_k(n+(m+1)k)$.

By the induction hypothesis and the definition of $P_k(n)$ we have

$$\begin{aligned} P_k(n) &+ 2\sum_{i=1}^{m+1} P_k(n+ik-1) \\ &= P_k(n) + 2\sum_{i=1}^{m} P_k(n+ik-1) \\ &\quad + 2P_k(n+(m+1)k-1) \\ &= P_k(n+mk) + 2P_k(n+(m+1)k-1) \\ &= P_k(n+(m+1)k), \end{aligned} \tag{9}$$

which ends the proof of (7).

In the same way we can prove the equality (8), so we omit the proof. □

If $n = 0$ or $n = 1$, respectively, then from Theorem 4 the following follows.

Corollary 5. *Let $k \geq 2$ and $m \geq 1$ be integers. Then*

$$\begin{aligned} 2\sum_{i=1}^{m} P_k(ki-1) &= P_k(mk), \\ 2\sum_{i=1}^{m} P_k(ki) &= P_k(mk+1) - 1, \\ 2\sum_{i=1}^{m} Q_k(ki-1) &= Q_k(mk) - k, \\ 2\sum_{i=1}^{m} Q_k(ki) &= Q_k(mk+1) - 2. \end{aligned} \tag{10}$$

For $k = 2$ we obtain the well-known identities for the classical Pell numbers and the companion Pell numbers; namely,

$$\begin{aligned} 2\sum_{i=1}^{m} P_{2i-1} &= P_{2m}, \\ 2\sum_{i=1}^{m} P_{2i} &= P_{2m+1} - 1, \\ 2\sum_{i=1}^{m} Q_{2i-1} &= Q_{2m} - 2, \\ 2\sum_{i=1}^{m} Q_{2i} &= Q_{2m+1} - 2. \end{aligned} \tag{11}$$

Theorem 6. *Let $k \geq 2$ and $m = 0, 1, 2, \ldots, k$. Then*

$$P_k(k+m) = 2^{k+m-1} + m2^{m-1}, \tag{12}$$

$$Q_k(k+m) = 2^{k+m+1} + k2^m + m2^m. \tag{13}$$

Proof of (12) *(by induction on m).* If $m = 0$ then the result is obvious. Assume that $P_k(k+m) = 2^{k+m-1} + m2^{m-1}$ for an arbitrary m, such that $1 \leq m \leq k-1$. We will prove that $P_k(k+m+1) = 2^{k+m} + (m+1)2^m$.

By the induction hypothesis and the definition of $P_k(n)$, we have

$$\begin{aligned} P_k(k+m+1) &= 2P_k(k+m) + P_k(m+1) \\ &= 2\left[2^{k+m-1} + m2^{m-1}\right] + 2^m \\ &= 2^{k+m} + (m+1)2^m, \end{aligned} \tag{14}$$

which ends the proof.

In the same way we can prove the equality (13). □

If $m = 0$, $m = 1$, or $m = k-1$, respectively, then from Theorem 6 we obtain the following corollary.

Corollary 7. *Let $k \geq 2$. Then*

$$\begin{aligned} P_k(k) &= 2^{k-1}, \\ P_k(k+1) &= 2^k + 1, \\ P_k(2k-1) &= 2^{k-2}\left(2^k + k - 1\right), \\ Q_k(k) &= 2^{k+1} + k, \\ Q_k(k+1) &= 2^{k+2} + 2k + 2, \\ Q_k(2k-1) &= 2^{2k} + 2^{k-1}(2k-1). \end{aligned} \tag{15}$$

Theorem 8. *Let $k \geq 2$ and $n \geq 2k-1$ be integers. Then*

$$P_k(n) = \left(2^k + 1\right)P_k(n-k) + \sum_{i=1}^{k-1} 2^i P_k(n-k-i), \tag{16}$$

$$Q_k(n) = \left(2^k + 1\right)Q_k(n-k) + \sum_{i=1}^{k-1} 2^i Q_k(n-k-i). \tag{17}$$

Proof of (16) *(by induction on n).* For $n = 2k-1$ we have the equation

$$P_k(2k-1) = \left(2^k + 1\right)P_k(k-1) + \sum_{i=1}^{k-1} 2^i P_k(k-1-i). \tag{18}$$

By the initial conditions of the sequence $P_k(n)$ and Theorem 6 we can see that this equation is an identity (because it is equivalent to third identity from Corollary 7).

Let $n \geq 2k$. Assume that formula (16) is true for an arbitrary t where $2k \leq t \leq n$. We will prove that

$P_k(n+1) = (2^k+1)P_k(n+1-k) + \sum_{i=1}^{k-1} 2^i P_k(n+1-k-i)$. By the induction hypothesis and the definition of $P_k(n)$, we have

$$\begin{aligned}
&\left(2^k+1\right) P_k(n+1-k) + \sum_{i=1}^{k-1} 2^i P_k(n+1-k-i) \\
&= \left(2^k+1\right)\left[2P_k(n-k) + P_k(n+1-2k)\right] \\
&\quad + \sum_{i=1}^{k-1} 2^i \left[2P_k(n-k-i) + P_k(n+1-2k-i)\right] \\
&= 2\left[\left(2^k+1\right) P_k(n-k) + \sum_{i=1}^{k-1} 2^i P_k(n-k-i)\right] \\
&\quad + \left(2^k+1\right) P_k(n-2k+1) \\
&\quad + \sum_{i=1}^{k-1} 2^i P_k(n+1-2k-i) \\
&= 2P_k(n) + P_k(n+1-k) = P_k(n+1),
\end{aligned} \tag{19}$$

which ends the proof of (16). In the same way we can prove (17) which completes the proof of the theorem. □

If $k = 2$ then from Theorem 8 we obtain the following formula for the classical Pell numbers and companion Pell numbers:

$$\begin{aligned}
P_n &= 5P_{n-2} + 2P_{n-3}, \\
Q_n &= 5Q_{n-2} + 2Q_{n-3}, \quad n \geq 3.
\end{aligned} \tag{20}$$

3. (A, B, kC)-Coloured Graphs

For concepts not defined here see [10]. The numbers of the Fibonacci type have many applications in distinct areas of mathematics. There is a large interest of modern science in the applications of the numbers of the Fibonacci type. These numbers are studied intensively in a wide sense also in graphs and combinatorials problem. In graphs Prodinger and Tichy initiated studying the Fibonacci numbers and the like. In [11] they showed the relations between the number of independent sets in $\mathscr{P}_n$ and $\mathscr{C}_n$ with the Fibonacci numbers and the Lucas numbers, where $\mathscr{P}_n$ and $\mathscr{C}_n$ denote an n-vertex path and an n-vertex cycle, respectively. This short paper gave an impetus for counting problems related to the numbers of the Fibonacci type. Many of these problems and results are closely related with the Merrifield-Simmons index $\sigma(G)$ and the Hosoya index $Z(G)$ in graphs; see [6, 12]. The Pell numbers also have a graph interpretation. It is well-known that $Z(\mathscr{P}_n \circ K_1) = P_{n+1}$, where $G \circ H$ denotes the corona of two graphs.

In this section we give a graph interpretation of the k-distance Pell numbers with respect to special edge colouring of a graph.

Let G be a 3-edge coloured graph with the set of colours $\{A, B, C\}$. Let $M \in \{A, B, C\}$. We say that a path is M-monochromatic if all its edges are coloured alike by colour M. By $l(M)$ we denote the length of the M-monochromatic path. For $xy \in E(G)$ notation $M(xy)$ means that the edge xy has the colour M.

Let $k \geq 1$ be an integer. In the graph G we define a (A, B, kC)-edge colouring, such that $l(A) \geq 0$, $l(B) \geq 0$, and $l(C) = qk$, where $q \geq 0$ is an integer.

Theorem 9. *Let $k \geq 2$ and $n \geq 2$ be integers. The number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_n$ is equal to $P_k(n)$.*

Proof. Let $V(\mathscr{P}_n) = \{x_1, x_2, \ldots, x_n\}$ be the set of vertices of a graph $\mathscr{P}_n$ with the numbering in the natural fashion. Then $E(\mathscr{P}_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. Let $\sigma(k, n)$ be the number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_n$. By inspection we obtain that $\sigma(k, n) = P_k(n)$ for $n = 2, \ldots, k+1$.

Let $n \geq k+2$ and let $\sigma_A(k, n)$, $\sigma_B(k, n)$, and $\sigma_C(k, n)$ denote the number of (A, B, kC)-edge colouring of the graph $\mathscr{P}_n$, with $A(x_{n-1}x_n)$, $B(x_{n-1}x_n)$, and $C(x_{n-1}x_n)$, respectively. It is obvious that $\sigma_A(k, n) = \sigma_B(k, n)$.

Clearly $\sigma_A(k, n)$ and $\sigma_B(k, n)$ are equal to the number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_{n-1}$ and $\sigma_C(k, n)$ is equal to the number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_{n-k}$. In the other words $\sigma_A(k, n) = \sigma(k, n-1)$, $\sigma_B(k, n) = \sigma(k, n-1)$, and $\sigma_C(k, n) = \sigma(k, n-k)$. Since

$$\sigma(k, n) = \sigma_A(k, n) + \sigma_B(k, n) + \sigma_C(k, n), \tag{21}$$

then we have $\sigma(k, n) = 2\sigma(k, n-1) + \sigma(k, n-k)$. By the initial conditions we have that $\sigma(k, n) = P_k(n)$, for all $n \geq 2$, which ends the proof. □

Corollary 10. *Let $n \geq 2$ be integer. The number of all $(A, B, 2C)$-edge colouring of the graph $\mathscr{P}_n$ is equal to P_n.*

Using the concept of (A, B, kC)-edge colouring of the graph $\mathscr{P}_n$ we can obtain the direct formula for the numbers $P_k(n)$ and $Q_k(n)$.

Let $k \geq 1$, $n \geq 2$, and $0 \leq t \leq [(n-1)/k]$ be integers and let $p_k(n, t)$ be the number of (A, B, kC)-edge colouring of graph $\mathscr{P}_n$, such that C-monochromatic path appears in this colouring exactly t times. In the other words tk edges of $\mathscr{P}_n$ have colour C.

Theorem 11. *Let $n \geq 2$ and $0 \leq t \leq n-1$ be integers. Then $p_1(n, t) = \binom{n-1}{t} 2^{n-1-t}$.*

Proof. Since $\mathscr{P}_n$ has $n-1$ edges, then for $t = 0, 1, 2, \ldots$ we obtain that

$$\begin{aligned}
p_1(n, 0) &= 2^{n-1}, \\
p_1(n, 1) &= \binom{n-1}{1} 2^{n-2}, \\
p_1(n, 2) &= \binom{n-1}{2} 2^{n-3}, \ldots,
\end{aligned} \tag{22}$$

and so $p_1(n, t) = \binom{n-1}{t} 2^{n-1-t}$. □

Theorem 12. *Let $k \geq 1$, $n \geq 2$, and $0 \leq t \leq [(n-1)/k]$ be integers. Then for all $s = 0, 1, 2, \ldots, k-1$ holds $p_k(n, t) = p_{k-s}(n - ts, t)$.*

Proof. Consider the graph $\mathscr{P}_n$ with the (A, B, kC)-edge colouring. By contracting the C-monochromatic paths of length k to the C-monochromatic paths of length $k-s$ we can see that the number of (A, B, kC)-edge colouring of graph $\mathscr{P}_n$, such that C-monochromatic path appears in this colouring exactly t times is equal to the number of $(A, B, (k-s)C)$-edge colouring of graph $\mathscr{P}_{n-ts}$, such that C-monochromatic path appears in this colouring exactly t times and the Theorem follows. □

If $s = k-1$ or $s = k-2$ then Theorem 12 gives the following result.

Corollary 13. *Let $k \geq 1$, $n \geq 2$, and $0 \leq t \leq [(n-1)/k]$ be integers. Then*

$$p_k(n,t) = p_1(n - t(k-1), t), \tag{23}$$

$$p_k(n,t) = p_2(n - t(k-2), t). \tag{24}$$

By (23) and Theorem 11 we obtain the direct formula for $p_k(n,t)$.

Corollary 14. *Let $k \geq 1$, $n \geq 2$, $0 \leq t \leq [(n-1)/k]$ be integers. Then*

$$p_k(n,t) = \binom{n-1-(k-1)t}{t} 2^{n-1-kt}. \tag{25}$$

Moreover Theorem 9 immediately gives that

$$\sum_{t\geq 0} p_k(n,t) = \sigma(k,n) = P_k(n). \tag{26}$$

Using (26) and Corollary 14 we can give the direct formula for the number $P_k(n)$.

Theorem 15. *Let $k \geq 2$ and $n \geq 1$ be integers. Then*

$$P_k(n) = \sum_{i=0}^{[(n-1)/k]} \binom{n-1-(k-1)i}{i} 2^{n-1-ki}. \tag{27}$$

If $k = 2$ then we obtain the direct formula for the classical Pell numbers of the form

$$P_n = \sum_{i=0}^{[(n-1)/2]} \binom{n-1-i}{i} 2^{n-1-2i} = \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} 2^i. \tag{28}$$

From Theorems 3 and 15 follows the direct formula for the sequence $Q_k(n)$.

Theorem 16. *Let $k \geq 2$ and $n \geq 1$ be integers. Then*

$$\begin{aligned} Q_k(n) = 2 &\sum_{i=0}^{[(n-1)/k]} \binom{n-1-(k-1)i}{i} 2^{n-1-ki} \\ &+ k \sum_{i=0}^{[(n-k)/k]} \binom{n-k-(k-1)i}{i} 2^{n-k-ki}. \end{aligned} \tag{29}$$

If $k = 2$ then using the above theorem and after some calculations we obtain that

$$\begin{aligned} Q_n = 2 \Bigg(&\sum_{i=0}^{[(n-1)/2]} \binom{n-1-i}{i} 2^{n-1-2i} \\ &+ \sum_{i=0}^{[(n-2)/2]} \binom{n-2-i}{i} 2^{n-2-2i} \Bigg). \end{aligned} \tag{30}$$

Now we give the recurrence relation for the number $p_k(n,k)$.

Theorem 17. *Let $k \geq 1$, $n \geq k+2$, and $0 \leq t \leq [(n-1)/k]$ be integers. Then*

$$p_k(n,t) = 2p_k(n-1,t) + p_k(n-k,t-1). \tag{31}$$

Proof. Let $V(\mathscr{P}_n) = \{x_1, x_2, \ldots, x_n\}$ be the set of vertices of a graph $\mathscr{P}_n$ with the numbering in the natural fashion. Then $E(\mathscr{P}_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$.

Let $p_k^A(n,t)$, $p_k^B(n,t)$, and $p_k^C(n,t)$ denote the number of (A, B, kC)-edge colouring of the graph $\mathscr{P}_n$, such that C-monochromatic path appears in this colouring exactly t times with $A(x_{n-1}x_n)$, $B(x_{n-1}x_n)$, and $C(x_{n-1}x_n)$, respectively.

It can be easily seen that both $p_k^A(n,t)$ and $p_k^B(n,t)$ are equal to the number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_{n-1}$, such that C-monochromatic path appears in this colouring exactly t times and $p_k^C(n,t)$ is equal to the number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_{n-k}$, such that C-monochromatic path appears in this colouring exactly $t-1$ times. In the other words $p_k^A(n,t) = p_k(n-1,t)$, $p_k^B(n,t) = p_k(n-1,t)$, and $p_k^C(k,n) = p_k(n-k,t-1)$. Since

$$p_k(n,t) = p_k^A(n,t) + p_k^B(n,t) + p_k^C(n,t), \tag{32}$$

so the result immediately follows. □

By (24) and by Theorem 17 we obtain the following.

Corollary 18. *Let $k \geq 1$, $n \geq k+2$, and $0 \leq t \leq [(n-1)/k]$ be integers. Then*

$$\begin{aligned} p_k(n,t) = &\,2p_2(n - t(k-2) - 1, t) \\ &+ p_2(n - t(k-2) - k, t-1). \end{aligned} \tag{33}$$

Using (26) and Corollary 18 we have the following.

Corollary 19. *Let $k \geq 2$, $n \geq k+2$, and $0 \leq t \leq [(n-1)/k]$ be integers. Then*

$$\begin{aligned} P_k(n) = \sum_{t=0}^{[(n-1)/k]} &\big(2p_2(n - t(k-2) - 1, t) \\ &+ p_2(n - t(k-2) - k, t-1)\big). \end{aligned} \tag{34}$$

Now we consider the (A, B, kC)-edge colouring of the cycle $\mathscr{C}_n$ with the numbering of edges in the natural fashion. For explanation for cycle $\mathscr{C}_4$ with $E(\mathscr{C}_4) = \{e_1, e_2, e_3, e_4\}$ there are two distinct $(A, B, 2C)$-edge colouring using only colour C. The first $(A, B, 2C)$-edge colouring gives a partition of the set $E(\mathscr{C}_4) = \{e_1, e_2\} \cup \{e_3, e_4\}$ and the second $(A, B, 2C)$-edge colouring gives a partition $E(\mathscr{C}_4) = \{e_2, e_3\} \cup \{e_4, e_1\}$.

Let $\rho(k, n)$ be the number of all $(A, B, 2C)$-edge colouring of the cycle $\mathscr{C}_n$.

Theorem 20. *Let $k \geq 2$ and $n \geq 3$ be integers. The number of all (A, B, kC)-edge colouring of the cycle $\mathscr{C}_n$ is equal to $Q_k(n)$.*

Proof. Let $V(\mathscr{C}_n) = \{x_1, x_2, \ldots, x_n\}$ be the set of vertices of a graph $\mathscr{C}_n$ with the numbering in the natural fashion. Then $E(\mathscr{C}_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$.

It is easy to check that $\rho(k, n) = Q_k(n)$ for $n = 3, \ldots, k$.

Let $n \geq k+1$ and let $\rho_A(k, n)$, $\rho_B(k, n)$, and $\rho_C(k, n)$ denote the number of (A, B, kC)-edge colouring of the cycle $\mathscr{C}_n$ with $A(x_nx_1)$, $B(x_nx_1)$, and $C(x_nx_1)$, respectively.

It can be easily seen that $\rho_A(k, n)$ and $\rho_B(k, n)$ are equal to the number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_n$ and $\rho_C(k, n)$ is equal to the number of all (A, B, kC)-edge colouring of the graph $\mathscr{P}_{n-k+1}$ multiplied by k. By Theorem 9 we have $\rho_A(k, n) = P_k(n)$, $\rho_B(k, n) = P_k(n)$, and $\rho_C(k, n) = kP_k(n-k+1)$. Since

$$\rho(k, n) = \rho_A(k, n) + \rho_B(k, n) + \rho_C(k, n), \tag{35}$$

then we have $\rho(k, n) = 2P_k(n) + kP_k(n-k+1)$ and so by Theorem 3 we have $\rho(k, n) = Q_k(n)$ for all $n \geq 3$. □

Corollary 21. *Let $n \geq 3$ be an integer. Then the number of all $(A, B, 2C)$-edge colouring of the cycle $\mathscr{C}_n$ is equal to Q_n, $n \geq 3$.*

4. Concluding Remarks

The interpretation of the generalized Pell numbers with respect to (A, B, kC)-colouring of a 3-edge coloured graph gives a motivation for studying this type of colouring in graphs. For an arbitrary $k \geq 2$ this problem seems to be difficult and more interesting results can be obtained for special value of k (e.g., if we study $(A, B, 2C)$-colouring in graphs). In the class of trees some interesting results can be obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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