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Research Article

Some New Generating Functions for q-Hahn Polynomials

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We obtain some new generating functions for q-Hahn polynomials and give their proofs based on the homogeneous q-difference operator.

1. Introduction

Throughout this paper we suppose that $q \in \mathbb{C}$, $|q| < 1$, and the q-shifted factorials are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right),$$
$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right), \quad n \geq 1. \tag{1}$$

Clearly,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2}$$

We also adopt the following compact notation for the multiple q-shifted factorials:

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty. \tag{3}$$

The basic hypergeometric series or q-series ${}_r\phi_s$ are defined by

$${}_r\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_s; \end{matrix}\quad q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \tag{4}$$

Euler identity is as follows:

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}. \tag{5}$$

The q-binomial theorem is as follows:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{6}$$

The usual q-differential operator or q-derivative operator D_q is defined by (see [1, Page 177, (2.1)])

$$D_q\{f(a)\} = \frac{f(a)-f(aq)}{a},$$
$$D_q^n\{f(a)\} = D_q\left\{D_q^{n-1}\{f(a)\}\right\}. \tag{7}$$

In [1], Chen and Liu introduced the q-exponential $T(bD_q)$ operator as follows (see [1, Page 17, (2.5)]):

$$T\left(bD_q\right) = \sum_{n=0}^{\infty}\frac{\left(bD_q\right)^n}{(q;q)_n}, \tag{8}$$

and they get the q-operator identity of $T(bD_q)$ (see [1, Page 178, Theorems 2.2 and 2.3]) as follows:

$$T\left(bD_q\right)\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{1}{(at,bt;q)_\infty}\quad |bt|<1,$$
$$T\left(bD_q\right)\left\{\frac{1}{(as,at;q)_\infty}\right\}=\frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}\quad |bt|<1. \tag{9}$$

Recently Chen et al. [2] introduced the following homogeneous q-difference D_{xy}

$$D_{xy}\{f(x,y)\}=\frac{f\left(x,q^{-1}y\right)-f(qx,y)}{x-q^{-1}y} \tag{10}$$

and the homogeneous q-difference operator $E(D_{xy})$:

$$E\left(D_{xy}\right)=\sum_{k=0}^{\infty}\frac{D_{xy}^k}{(q;q)_k}. \tag{11}$$

They obtained some properties of D_{xy} as follows:

$$\begin{aligned} D_{xy}\{P_n(x,y)\}&=(1-q^n)P_{n-1}(x,y),\\ D_{xy}\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}&=t\frac{(yt;q)_\infty}{(xt;q)_\infty},\\ D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}&=t^k\frac{(yt;q)_\infty}{(xt;q)_\infty},\\ E\left(D_{xy}\right)\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}&=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}. \end{aligned} \tag{12}$$

The classical Rogers-Szegö polynomial is defined by means of the generating function:

$$\sum_{n=0}^{\infty}h_n(x\mid q)\frac{t^n}{(q;q)_n}=\frac{1}{(t,xt;q)_\infty},\quad |t|<1; \tag{13}$$

obviously, we have

$$T\left(D_q\right)\{x^n\}=h_n(x\mid q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k. \tag{14}$$

The homogeneous Rogers-Szegö polynomial is defined by

$$h_n(x,y\mid q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(x,y), \tag{15}$$

where $P_n(x,y)=(x-y)(x-yq)\cdots(x-yq^{n-1})$. Clearly, $h_n(x,y\mid q)=\Phi_n^{(y/x)}(x)$ are the Cauchy polynomials with the following generating function:

$$\sum_{k=0}^{\infty}P_k(x,y)\frac{z^k}{(q;q)_k}=\frac{(yz;q)_\infty}{(xz;q)_\infty},\quad |xz|<1. \tag{16}$$

From the above properties, we have

$$E\left(D_{xy}\right)\{P_n(x,y)\}=h_n(x,y\mid q), \tag{17}$$

$$\sum_{n=0}^{\infty}h_n(x,y\mid q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}. \tag{18}$$

Lemma 1 (see [3, Lemma 2.3]). *For* $|t|,|xt|<1$,

$$\begin{aligned} &E\left(D_{xy}\right)\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_n(x,y)}{(yt;q)_n}\right\}\\ &\quad=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(y,xt;q)_k}{(yt;q)_k}x^{n-k}. \end{aligned} \tag{19}$$

q-Hahn polynomial is defined by [4]

$$\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\frac{t^n}{(q;q)_n}=\frac{(axt;q)_\infty}{(t,xt;q)_\infty}. \tag{20}$$

We have

$$\Phi_n^{(a)}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_k x^k. \tag{21}$$

Clearly, $\Phi_n^{(0)}(x)=h_n(x\mid q)$.

Recently, Chen et al. [3] gave some new proofs of the following results based on the method of homogeneous q-difference operator $E(D_{xy})$.

Theorem 2. *Consider the following:*

$$\begin{aligned} &\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\Phi_n^{(b)}(y)\frac{t^n}{(q;q)_n}\\ &\quad=\frac{(xat,ybt;q)_\infty}{(t,xt,yt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}t,a,b\\ xat,ybt;\end{matrix}\;q,xyt\right). \end{aligned} \tag{22}$$

Theorem 3. *Consider the following:*

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Phi_{m+n}^{(a)}(x)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\ &\quad=\frac{(xas;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}xa,xs\\ xas;\end{matrix}\;q,t\right). \end{aligned} \tag{23}$$

For more references on the q-difference operators, see [1, 5–16].

In the present paper, we obtain some new generating functions for q-Hahn polynomials and give their proofs based on the homogeneous q-difference operator.

2. Some New Generating Functions for q-Hahn Polynomial

In the present section we obtain the following new generating functions of q-Hahn polynomial.

Theorem 4. *For* $|z| < 1$,

$$\sum_{k=0}^{\infty}\Phi_{n+k}^{(a)}(x)\frac{z^k}{(q;q)_k}=\frac{(axz;q)_\infty}{(z,xz;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a,z;q)_k}{(axz;q)_k}x^k. \tag{24}$$

Proof. Let $x\mapsto y$ and $a\mapsto b$ in (21), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\Phi_n^{(b)}(y)\frac{z^n}{(q;q)_n}&=\sum_{n=0}^{\infty}\Phi_n^{(a)}(x)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(b;q)_k y^k\frac{z^n}{(q;q)_n}\\&=\sum_{k,n=0}^{\infty}\Phi_{n+k}^{(a)}(x)\frac{(b;q)_k z^n}{(q;q)_n(q;q)_k}(yz)^k.\end{aligned}\tag{25}$$

By the q-binomial theorem (6) and noting that $(b;q)_{n+k}=(bq^k;q)_n(b;q)_k$, we have

$$\begin{aligned}&\frac{(xaz,ybz;q)_\infty}{(z,xz,yz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,z;q)_k}{(axz,byz,q;q)_k}(xyz)^k\\&=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,z;q)_k}{(axz,q;q)_k}(xyz)^k\frac{(byzq^k;q)_\infty}{(yz;q)_\infty}\\&=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,z;q)_k}{(axz,q;q)_k}(xyz)^k\sum_{n=0}^{\infty}\frac{(bq^k;q)_n}{(q;q)_n}(yz)^n\\&=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{n,k=0}^{\infty}\frac{(a,z;q)_k}{(axz,q;q)_k}\frac{(b;q)_{n+k}}{(q;q)_n}x^k(yz)^{n+k}.\end{aligned}\tag{26}$$

By (17), (25), and (26), we obtain

$$\begin{aligned}\sum_{k,n=0}^{\infty}\Phi_{n+k}^{(a)}(x)\frac{(b;q)_k z^n}{(q;q)_n(q;q)_k}(yz)^k\\=\frac{(xaz;q)_\infty}{(z,xz;q)_\infty}\sum_{n,k=0}^{\infty}\frac{(a,z;q)_k}{(axz,q;q)_k}\frac{(b;q)_{n+k}}{(q;q)_n}x^k(yz)^{n+k}.\end{aligned}\tag{27}$$

Comparing the coefficients of $y^k/(q;q)_k$ on both sides of (27), we obtain the formula (24) immediately. This proof is complete. □

Theorem 5. *For* $|t| < 1$,

$$\begin{aligned}&\sum_{n=0}^{\infty}\Phi_{m+n}^{(a)}(x)\Phi_n^{(b)}(y)\frac{t^n}{(q;q)_n}\\&=\frac{(xat;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(b;q)_k(xyt)^k}{(q;q)_k}\sum_{j=0}^{m+k}\begin{bmatrix}m+k\\j\end{bmatrix}\frac{(xa,xt;q)_j}{(xat;q)_j}x^{m-j}.\end{aligned}\tag{28}$$

Proof. By (17) and (19), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}h_{m+n}(x,y\mid q)h_n(u,v\mid q)\frac{t^n}{(q;q)_n}\\&=\sum_{n=0}^{\infty}E(D_{xy})\{P_{m+n}(x,y)\}h_n(u,v\mid q)\frac{t^n}{(q;q)_n}\\&=E(D_{xy})\left\{\sum_{n=0}^{\infty}P_{m+n}(x,y)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(u,v)\frac{t^n}{(q;q)_n}\right\}\\&=E(D_{xy})\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{P_{m+n+k}(x,y)P_k(u,v)t^{n+k}}{(q;q)_k(q;q)_n}\right\}\\&=E(D_{xy})\left\{\sum_{k=0}^{\infty}\frac{P_{m+k}(x,y)P_k(u,v)t^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{P_n(x,yq^{m+k})t^n}{(q;q)_n}\right\}\\&=E(D_{xy})\left\{\sum_{k=0}^{\infty}\frac{P_{m+k}(x,y)P_k(u,v)t^k}{(q;q)_k}\frac{(yt;q)_\infty}{(xt;q)_\infty(yt;q)_{m+k}}\right\}\\&=\sum_{k=0}^{\infty}\frac{P_k(u,v)t^k}{(q;q)_k}E(D_{xy})\left\{\frac{(yt;q)_\infty P_{m+k}(x,y)}{(xt;q)_\infty(yt;q)_{m+k}}\right\}\\&=\sum_{k=0}^{\infty}\frac{P_k(u,v)t^k}{(q;q)_k}\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{j=0}^{m+k}\begin{bmatrix}m+k\\j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{m+k-j}\\&=\frac{(yt;q)_\infty}{(t,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(v/u;q)_k(utx)^k}{(q;q)_k}\\&\quad\times\sum_{j=0}^{m+k}\begin{bmatrix}m+k\\j\end{bmatrix}\frac{(y,xt;q)_j}{(yt;q)_j}x^{m-j}.\end{aligned}\tag{29}$$

Setting $y/x=a$, $v/u=b$, $u=y$ in the last sum, we obtain the formula (28) of Theorem 5. This proof is complete. □

Theorem 6. *For* $|l| < 1$, $|s| < 1$, $|t| < 1$,

$$\begin{aligned}&\sum_{m,n,k=0}^{\infty}\Phi_{m+k}^{(a)}(x)\Phi_{n+k}^{(b)}(y)\frac{l^m s^n t^k}{(q;q)_m(q;q)_n(q;q)_k}\\&=\frac{(xal,ybs;q)_\infty}{(l,xl,s,ys;q)_\infty}\sum_{k=0}^{\infty}\frac{t^k}{(q;q)_k}\\&\quad\times\sum_{i,j=0}^{\infty}\begin{bmatrix}k\\i\end{bmatrix}\begin{bmatrix}k\\j\end{bmatrix}\frac{(xa,xl;q)_i}{(xal;q)_i}\frac{(yb,ys;q)_j}{(ybs;q)_j}x^{k-i}y^{k-j}.\end{aligned}\tag{30}$$

Proof. By (17) and (19), we have

$$\begin{aligned}&\sum_{m,n,k=0}^{\infty}h_{m+k}(x,y\mid q)h_{n+k}(u,v\mid q)\\&\quad\times\frac{l^m s^n t^k}{(q;q)_m(q;q)_n(q;q)_k}\end{aligned}$$

$$= \sum_{m,n,k=0}^{\infty} E\left(D_{xy}\right)\{P_{m+k}(x,y)\} E\left(D_{uv}\right)\{P_{n+k}(u,v)\} \times \frac{l^m s^n t^k}{(q;q)_m (q;q)_n (q;q)_k}$$

$$= E\left(D_{xy}\right) E\left(D_{uv}\right)\left\{\sum_{m,n,k=0}^{\infty} \frac{P_{m+k}(x,y) P_{n+k}(u,v) l^m s^n t^k}{(q;q)_m (q;q)_n (q;q)_k}\right\}$$

$$= E\left(D_{xy}\right) E\left(D_{uv}\right)\left\{\sum_{k=0}^{\infty} \frac{P_k(x,y) P_k(u,v) t^k}{(q;q)_k} \times \sum_{m=0}^{\infty} \frac{P_m\left(x, yq^k\right) l^m}{(q;q)_m} \sum_{n=0}^{\infty} \frac{P_n\left(u, vq^k\right) s^n}{(q;q)_n}\right\}$$

$$= E\left(D_{xy}\right) E\left(D_{uv}\right)\left\{\sum_{k=0}^{\infty} \frac{P_k(x,y) P_k(u,v) t^k}{(q;q)_k} \times \frac{\left(ylq^k;q\right)_\infty}{(xl;q)_\infty} \frac{\left(vsq^k;q\right)_\infty}{(us;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty} E\left(D_{xy}\right)\left\{\frac{(yl;q)_\infty P_k(x,y)}{(xl;q)_\infty (yl;q)_k}\right\} \times E\left(D_{uv}\right)\left\{\frac{(vs;q)_\infty P_k(u,v)}{(us;q)_\infty (vs;q)_k}\right\} \frac{t^k}{(q;q)_k}$$

$$= \sum_{k=0}^{\infty}\left\{\frac{(yl;q)_\infty}{(l,xl;q)_\infty} \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \frac{(y,xl;q)_i}{(yl;q)_i} x^{k-i}\right\} \times \left\{\frac{(vs;q)_\infty}{(s,us;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(v,us;q)_j}{(vs;q)_j} u^{k-j}\right\} \frac{t^k}{(q;q)_k}$$

$$= \frac{(yl,vs;q)_\infty}{(l,s,xl,us;q)_\infty} \times \sum_{k=0}^{\infty} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(y,xl;q)_i (v,us;q)_j x^{k-i} u^{k-j} t^k}{(yl;q)_i (vs;q)_j (q;q)_k}. \tag{31}$$

Setting $y/x = a$, $v/u = b$, $u = y$ in the last sum, we obtain the formula (30) of Theorem 6. This proof is complete. □

Theorem 7. *For* $|t| < 1$,

$$\sum_{k=0}^{\infty} \Phi_{m+k}^{(a)}(x) \Phi_{n+k}^{(b)}(y) \frac{t^k}{(q;q)_k} = \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{\infty} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (by;q)_j \times x^{k-i} y^{k-j} \Phi_m^{(a)}\left(xq^i\right) \Phi_n^{(b)}\left(yq^j\right). \tag{32}$$

Proof. Applying (2) and the Euler identity (5) and noting (21), then the right-hand side is equal to (30) as follows:

$$\frac{(xal,ybs;q)_\infty}{(l,xl,s,ys;q)_\infty} \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \times \sum_{i,j=0}^{\infty} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(xa,xl;q)_i}{(xal;q)_i} \frac{(yb,ys;q)_j}{(ybs;q)_j} x^{k-i} y^{k-j}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \times \frac{\left(xalq^i, ybsq^j;q\right)_\infty}{\left(l,s,xlq^i,ysq^j;q\right)_\infty}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \times \sum_{u,v,m,n=0}^{\infty} \frac{(a;q)_m (b;q)_n \left(xlq^i\right)^m \left(ysq^j\right)^n l^u s^v}{(q;q)_m (q;q)_n (q;q)_u (q;q)_v}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \times \sum_{u,v,m,n=0}^{\infty} \frac{(a;q)_m (b;q)_n \left(xlq^i\right)^m \left(ysq^j\right)^n}{(q;q)_m (q;q)_n (q;q)_u (q;q)_v} l^{m+u} s^{n+v}. \tag{33}$$

By (30) and (33), we have

$$\sum_{m,n,k=0}^{\infty} \Phi_{m+k}^{(a)}(x) \Phi_{n+k}^{(b)}(y) \frac{l^m s^n t^k}{(q;q)_m (q;q)_n (q;q)_k} = \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{i,j=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{k-i} y^{k-j} \times \sum_{u,v,m,n=0}^{\infty} \frac{(a;q)_m (b;q)_n \left(xlq^i\right)^m \left(ysq^j\right)^n}{(q;q)_m (q;q)_n (q;q)_u (q;q)_v} l^{m+u} s^{n+v}. \tag{34}$$

Comparing the coefficients of $l^m s^n/(q;q)_m (q;q)_n$ on both sides of (34), we obtain the formula (32) immediately. □

Theorem 8. *For* $|t| < 1$,

$$\sum_{n=0}^{\infty} \Phi_{m+n}^{(a)}(x) \Phi_n^{(b)}(y) \frac{t^n}{(q;q)_n} = \frac{(xyat,xybt;q)_\infty}{(xyt,xt,yt;q)_\infty} \times \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \frac{(yt;q)_s}{(xyat;q)_s} \times {}_3\phi_2\left(\begin{matrix} xyt, xa, yb \\ xyatq^s, xybt; \end{matrix}\ q, tq^s\right). \tag{35}$$

Proof. Set $n = 0$ and then let $k \mapsto n$ in (32) and note that $\Phi_0^{(b)}(x) = 1$; by (21) and (22), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \Phi_{m+n}^{(a)}(x)\, \Phi_n^{(b)}(y)\, \frac{t^n}{(q;q)_n} \\
&\quad = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{i,j=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} (xa;q)_i \\
&\qquad\qquad \times (yb;q)_j x^{n-i} y^{n-j} \Phi_m^{(a)}\left(xq^i\right) \\
&\quad = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{i,j=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} (xa;q)_i (yb;q)_j x^{n-i} y^{n-j} \\
&\qquad \times \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s \left(xq^i\right)^s \\
&\quad = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s y^n \\
&\qquad \times \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (ax;q)_i x^{s+n} \left(\frac{q^s}{x}\right)^i \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (by;q)_j \left(\frac{1}{y}\right)^j \\
&\quad = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^{s+n} y^n \Phi_n^{(xa)}\left(\frac{q^s}{x}\right) \Phi_n^{(yb)}\left(\frac{1}{y}\right) \\
&\quad = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \sum_{n=0}^{\infty} \Phi_n^{(xa)}\left(\frac{q^s}{x}\right) \Phi_n^{(yb)}\left(\frac{1}{y}\right) \frac{(xyt)^n}{(q;q)_n} \\
&\quad = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \frac{\left(xytaq^s, xybt;q\right)_\infty}{\left(xyt, ytq^s, xt;q\right)_\infty} \\
&\qquad \times {}_3\phi_2\left(\begin{matrix} xyt, xa, yb \\ xyatq^s, xybt; \end{matrix}\; q, tq^s\right) \\
&\quad = \frac{(xyat, xybt;q)_\infty}{(xyt, xt, yt;q)_\infty} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} (a;q)_s x^s \frac{(yt;q)_s}{(xyat;q)_s} \\
&\qquad\qquad \times {}_3\phi_2\left(\begin{matrix} xyt, xa, yb \\ xyatq^s, xybt; \end{matrix}\; q, tq^s\right).
\end{aligned} \tag{36}
$$

This proof is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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