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Research Article

On the Sum of Reciprocal Generalized Fibonacci Numbers

Pingzhi Yuan, Zilong He, and Junyi Zhou

School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China

Correspondence should be addressed to Pingzhi Yuan; yuanpz@scnu.edu.cn

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We consider infinite sums derived from the reciprocals of the generalized Fibonacci numbers. We obtain some new and interesting identities for the generalized Fibonacci numbers.

1. Introduction

For any integer $n \geq 0$, the famous Fibonacci numbers F_n and Pell numbers are defined by the second-order linear recurrence sequences

$$\begin{gathered} F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \ F_1 = 1, \\ P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0, \ P_1 = 1. \end{gathered} \tag{1}$$

There are many interesting results on the properties of these two sequences; see [1–9]. In 2009, Ohtsuka and Nakamura [5] studied the properties of the Fibonacci numbers and proved the following two interesting identities:

$$\begin{gathered} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}F_n, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \end{gathered} \tag{2}$$

where $\lfloor x \rfloor$ is the floor function; that is, it denotes the greatest integer less than or equal to x. Recently, Holliday and Komatsu [1] (Theorems 3 and 4) and Xu and Wang [7] proved the following interesting identities for the Pell numbers:

$$\begin{gathered} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2P_{n-1} + P_n - 1, & \text{if } n \text{ is even and } n \geq 2; \\ 2P_{n-1} + P_n, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^3} \right)^{-1} \right\rfloor = \begin{cases} P_n^2 P_{n-1} + 3P_n P_{n-1}^2 + \left\lfloor -\frac{61}{82} P_n - \frac{91}{82} P_{n-1} \right\rfloor, \\ \qquad \text{if } n \text{ is even and } n \geq 2; \\ P_n^2 P_{n-1} + 3P_n P_{n-1}^2 + \left\lfloor \frac{61}{82} P_n + \frac{91}{82} P_{n-1} \right\rfloor, \\ \qquad \text{if } n \text{ is odd and } n \geq 1, \end{cases} \end{gathered} \tag{3}$$

where providing $P_{-1} = P_1 = 1$. In [7, 8], the authors asked whether there exists a computational formula for

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^s} \right)^{-1} \right\rfloor, \tag{4}$$

where $s \geq 4$ is a positive integer.

Let p and q be integers such that $p^2 + 4q > 0$. Define the generalized Fibonacci sequence $\{U_n(p,q)\}$, briefly $\{U_n\}$, as shown: for $n \geq 2$

$$U_n = pU_{n-1} + qU_{n-2}, \tag{5}$$

where $U_0 = 0, U_1 = 1$. The Binet formula for $\{U_n\}$ is

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{6}$$

where $\alpha, \beta = (p \pm \sqrt{p^2 + 4q})/2$.

The main purpose of this paper related to the computing problem of

$$U(s,n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k^s} \right)^{-1} \right\rfloor \tag{7}$$

for $s = 3$ and $q = -1$. For easy computation, we assume that $p = a$ is a positive integer and $q = -1$ throughout the paper. We have the following.

Theorem 1. *Let $a \geq 3$ be a positive integer, and let G_n be defined by the second-order linear recurrence sequence $G_{n+2} = aG_{n+1} - G_n, G_0 = 0, G_1 = 1$. Then for all $n \geq 2$ one has*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k^3} \right)^{-1} \right\rfloor = \begin{cases} G_n^3 - G_{n-1}^3 - 3 \displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 2, & a = 3,\ n \equiv 3 \pmod 5; \\ G_n^3 - G_{n-1}^3 - 3 \displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 1, & \textit{otherwise}. \end{cases} \tag{8}$$

2. Proof of the Main Result

In this section, we will prove our main result. We consider the case that $\alpha\beta = 1$ and $s = 3$.

Proof. From the Taylor series expansion of $(1-\varepsilon)^{-3}$ as $\varepsilon \to 0$, we have

$$\begin{aligned} (1-\varepsilon)^{-3} &= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2} \varepsilon^n \\ &= 1 + 3\varepsilon + 6\varepsilon^2 + O(\varepsilon^3). \end{aligned} \tag{9}$$

Using (6), we have

$$\begin{aligned} \frac{1}{G_k^3} &= \frac{(\alpha-\beta)^3}{\alpha^{3k}} \left(1 - \frac{1}{\alpha^{2k}}\right)^{-3} \\ &= \frac{(\alpha-\beta)^3}{\alpha^{3k}} \frac{1}{(1 - 3/\alpha^{2k} + 3/\alpha^{4k} + 1/\alpha^{6k})} \\ &= \frac{(\alpha-\beta)^3}{\alpha^{3k}} \left[1 + \frac{3}{\alpha^{2k}} + \frac{6}{\alpha^{4k}} + \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{4k}(\alpha^{2k}-1)^3} \right] \\ &= (\alpha-\beta)^3 \left[\frac{1}{\alpha^{3k}} + \frac{3}{\alpha^{5k}} + \frac{6}{\alpha^{7k}} + \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3} \right]. \end{aligned} \tag{10}$$

It is easy to check that

$$\frac{10}{\alpha^{9k}} < \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3} < \frac{11}{\alpha^{9k}} \tag{11}$$

holds for $a \geq 3$ and $k \geq 2$.

Thus

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{1}{G_k^3} &= (\alpha-\beta)^3 \\ &\quad \times \left[\frac{1}{\alpha^{3n}} \cdot \frac{\alpha^3}{\alpha^3 - 1} + \frac{3}{\alpha^{5n}} \cdot \frac{\alpha^5}{\alpha^5 - 1} + \frac{6}{\alpha^{7n}} \cdot \frac{\alpha^7}{\alpha^7 - 1} \right. \\ &\quad \left. + \sum_{k=n}^{\infty} \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3} \right] \\ &= \frac{(\alpha-\beta)^3 \alpha^3}{\alpha^{3n}(\alpha^3 - 1)} \\ &\quad \times \left[1 + \frac{3}{\alpha^{2n}} \frac{\alpha^2(\alpha^3-1)}{\alpha^5 - 1} + \frac{6\alpha^4}{\alpha^{4n}} \frac{(\alpha^3-1)}{\alpha^7 - 1} + R_n \right], \end{aligned} \tag{12}$$

where

$$R_n = \frac{(\alpha^3 - 1)\alpha^{3n}}{\alpha^3} \sum_{k=n}^{\infty} \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3}. \tag{13}$$

Since $\sum_{k=n}^{\infty}(1/\alpha^{9k}) = \alpha^9/\alpha^{9n}(\alpha^9 - 1)$, we have

$$\frac{10\alpha^6}{\alpha^{6n}(\alpha^6 + \alpha^3 + 1)} < R_n < \frac{11\alpha^6}{\alpha^{6n}(\alpha^6 + \alpha^3 + 1)} \tag{14}$$

holds for $a \geq 3$ and $k \geq 2$.

Taking reciprocal, we get

$$\begin{aligned}
&\left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1}\\
&=\frac{\left(\alpha^3-1\right)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2\left(\alpha^3-1\right)}{\alpha^5-1}\right.\right.\\
&\qquad\left.\left.+\frac{6\alpha^4}{\alpha^{4n}}\frac{\left(\alpha^3-1\right)}{\alpha^7-1}+R_n\right)^{-1}\right)\\
&<\frac{\left(\alpha^3-1\right)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2\left(\alpha^3-1\right)}{\alpha^5-1}+\frac{6\alpha^4}{\alpha^{4n}}\frac{\left(\alpha^3-1\right)}{\alpha^7-1}\right.\right.\\
&\qquad\left.\left.+\frac{10\alpha^6}{\alpha^{6n}\left(\alpha^6+\alpha^3+1\right)}\right)^{-1}\right)\\
&<\frac{\left(\alpha^3-1\right)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}-\frac{3\left(\alpha^3-1\right)^2\alpha^n}{(\alpha-\beta)^3\alpha\left(\alpha^5-1\right)}+\delta_1,
\end{aligned}\tag{15}$$

where

$$\begin{aligned}
\delta_1=&-\frac{6\alpha\left(\alpha^3-1\right)^2}{\alpha^n(\alpha-\beta)^3\left(\alpha^7-1\right)}\\
&-\frac{10\alpha^3\left(\alpha^3-1\right)}{\alpha^{3n}(\alpha-\beta)^3\left(\alpha^6+\alpha^3+1\right)}+\frac{9\alpha\left(\alpha^3-1\right)^3}{\alpha^n(\alpha-\beta)^3\left(\alpha^5-1\right)^2}\\
&+\frac{36\alpha^3\left(\alpha^3-1\right)^3}{\alpha^{3n}(\alpha-\beta)^3\left(\alpha^5-1\right)\left(\alpha^7-1\right)}\\
&+\frac{36\alpha^5\left(\alpha^3-1\right)^3}{\alpha^{5n}(\alpha-\beta)^3\left(\alpha^7-1\right)^2}\\
&+\frac{60\alpha^5\left(\alpha^3-1\right)^2}{\alpha^{5n}(\alpha-\beta)^3\left(\alpha^5-1\right)\left(\alpha^6+\alpha^3+1\right)}\\
&+\frac{120\alpha^7\left(\alpha^3-1\right)^2}{\alpha^{7n}(\alpha-\beta)^3\left(\alpha^7-1\right)\left(\alpha^6+\alpha^3+1\right)}\\
&+\frac{100\alpha^9\left(\alpha^3-1\right)}{\alpha^{9n}(\alpha-\beta)^3\left(\alpha^6+\alpha^3+1\right)^2}
\end{aligned}\tag{16}$$

since

$$\frac{1}{1+\varepsilon}=1-\varepsilon+\varepsilon^2-\frac{\varepsilon^3}{1+\varepsilon}.\tag{17}$$

An easy calculation shows that $\delta_1 \le 4/\alpha^{n+3}$ holds for $a \ge 3$ and $k \ge 2$. Therefore,

$$\begin{aligned}
\left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1}&<\frac{\left(\alpha^3-1\right)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}-\frac{3\left(\alpha^3-1\right)^2\alpha^n}{(\alpha-\beta)^3\alpha\left(\alpha^5-1\right)}+\delta_1\\
&\le\frac{\alpha^{3n}-\alpha^{3n-3}}{(\alpha-\beta)^3}-\frac{3\left(\alpha^3-1\right)^2\alpha^n}{(\alpha-\beta)^3\alpha\left(\alpha^5-1\right)}+\frac{4}{\alpha^{n+3}}\\
&=G_n^3-G_{n-1}^3-\frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha\left(\alpha^5-1\right)}\\
&\quad+\frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n}-\frac{\alpha^3-1}{(\alpha-\beta)^3\alpha^{3n}}+\frac{4}{\alpha^{n+3}}\\
&<G_n^3-G_{n-1}^3-\frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha\left(\alpha^5-1\right)}\\
&\quad+\frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n}+\frac{4}{\alpha^{n+3}}\\
&=G_n^3-G_{n-1}^3-\frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha\left(\alpha^5-1\right)}+\lambda_1,
\end{aligned}\tag{18}$$

where

$$\lambda_1=\frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n}+\frac{4}{\alpha^{n+3}}<0.1681\tag{19}$$

for $a \ge 3$ and $n \ge 2$.

Similarly, we have

$$\begin{aligned}
&\left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1}\\
&=\frac{\left(\alpha^3-1\right)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2\left(\alpha^3-1\right)}{\alpha^5-1}\right.\right.\\
&\qquad\left.\left.+\frac{6\alpha^4}{\alpha^{4n}}\frac{\left(\alpha^3-1\right)}{\alpha^7-1}+R_n\right)^{-1}\right)\\
&>\frac{\left(\alpha^3-1\right)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2\left(\alpha^3-1\right)}{\alpha^5-1}+\frac{6\alpha^4}{\alpha^{4n}}\frac{\left(\alpha^3-1\right)}{\alpha^7-1}\right.\right.\\
&\qquad\left.\left.+\frac{11\alpha^6}{\alpha^{6n}\left(\alpha^6+\alpha^3+1\right)}\right)^{-1}\right)\\
&>G_n^3-G_{n-1}^3-\frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha\left(\alpha^5-1\right)}+\lambda_2.
\end{aligned}\tag{20}$$

Since

$$\frac{1}{1+\varepsilon}=1-\varepsilon+\varepsilon^2-\varepsilon^3+\frac{\varepsilon^4}{1+\varepsilon},\tag{21}$$

and $\varepsilon = (3/\alpha^{2n})(\alpha^2(\alpha^3-1)/(\alpha^5-1)) + (6\alpha^4/\alpha^{4n})((\alpha^3-1)/(\alpha^7-1)) + 11\alpha^6/\alpha^{6n}(\alpha^6+\alpha^3+1) < 0.3$ for $a \geq 3$ and $n \geq 2$, we have $\varepsilon^2 - \varepsilon^3 > 0.7\varepsilon^2$, whence we can take

$$\begin{aligned}\lambda_2 = {} & \frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n} - \frac{\alpha^3-1}{(\alpha-\beta)^3\alpha^{3n}} - \frac{6\alpha(\alpha^3-1)^2}{\alpha^n(\alpha-\beta)^3(\alpha^7-1)} \\ & - \frac{11\alpha^3(\alpha^3-1)}{\alpha^{3n}(\alpha-\beta)^3(\alpha^6+\alpha^3+1)} \\ & + \frac{6.3\alpha(\alpha^3-1)^3}{\alpha^n(\alpha-\beta)^3(\alpha^5-1)^2} > 0\end{aligned} \tag{22}$$

for $a \geq 3$ and $n \geq 2$.

Consequently, we have shown that

$$\begin{aligned}& G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \lambda_2 \\ & < \left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1} < G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \lambda_1,\end{aligned} \tag{23}$$

where $0 < \lambda_2 < \lambda_1 < 0.1681$ for $a \geq 3$ and $n \geq 2$, and $\lambda_1 < 0.0053$ for $a \geq 4$ and $n \geq 3$.

Now the calculations show that

$$\frac{\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} = \begin{cases} G_{n-3} + G_{n-8} + \cdots + G_7 + G_2 - \dfrac{\alpha^2+\alpha^3}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 0 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_8 + G_3 - \dfrac{\alpha^2+\alpha^3}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 1 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_9 + G_4 - \dfrac{\alpha+\alpha^4}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 2 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_{10} + G_5 - \dfrac{1+\alpha^5}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 3 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_6 + G_1 - \dfrac{\alpha+\alpha^4}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 4 \pmod 5. \end{cases} \tag{24}$$

The calculations also show that $3(\alpha^2+\alpha^3)/(\alpha-\beta)(\alpha^5-1) > \lambda_1$ for $a \geq 3$ and $n \geq 2$; $3(\alpha+\alpha^4)/(\alpha-\beta)(\alpha^5-1) > \lambda_1$ for $a \geq 3$ and $n \geq 2$; and $3(1+\alpha^5)/(\alpha-\beta)(\alpha^5-1) > \lambda_1 + 1$ for $a = 3$ and $n \geq 3$; $0.87 < 3(1+\alpha^5)/(\alpha-\beta)(\alpha^5-1) < 1$ for $a > 3$ and $n \geq 3$. Combining the calculations and (23), we obtain

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1} \right\rfloor = \begin{cases} G_n^3 - G_{n-1}^3 - 3\displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 2, & a = 3,\ n \equiv 3 \pmod 5; \\ G_n^3 - G_{n-1}^3 - 3\displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 1, & \text{otherwise}. \end{cases} \tag{25}$$

Therefore we have proved Theorem 1. □

Remark 2. We can also compute the cases $s > 3$ or $q = 1$; however, the computations are much more complicated. So we stop here.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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