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Research Article

Oscillations of Difference Equations with Several Oscillating Coefficients

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We study the oscillatory behavior of the solutions of the difference equation $\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, n \in \mathbb{N}_0 [\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, n \in \mathbb{N}]$ where $(p_i(n))$, $1 \le i \le m$ are real sequences with oscillating terms, $\tau_i(n)[\sigma_i(n)]$, $1 \le i \le m$ are general retarded (advanced) arguments, and $\Delta[\nabla]$ denotes the forward (backward) difference operator $\Delta x(n) = x(n+1) - x(n)[\nabla x(n) = x(n) - x(n-1)]$. Examples illustrating the results are also given.

1. Introduction

In the present paper, we study the oscillatory behavior of the solutions of the difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)\, x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \qquad (\mathrm{E_R})$$

where $\mathbb{N} \ni m \ge 2$, p_i, $1 \le i \le m$ are real sequences with oscillating terms, and $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$, $1 \le i \le m$ are sequences of integers such that

$$\tau_i(n) \le n-1, \quad n \in \mathbb{N}_0, \qquad \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \le i \le m \qquad (1)$$

and the (dual) advanced difference equation

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)\, x(\sigma_i(n)) = 0, \quad n \in \mathbb{N}, \qquad (\mathrm{E_A})$$

where $\mathbb{N} \ni m \ge 2$, p_i, $1 \le i \le m$ are real sequences with oscillating terms and $\{\sigma_i(n)\}_{n\in\mathbb{N}}$, $1 \le i \le m$, are sequences of integers such that

$$\sigma_i(n) \ge n+1, \quad n \in \mathbb{N}, \quad 1 \le i \le m. \qquad (2)$$

Here, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Also, as usual, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

Strong interest in $(\mathrm{E_R})$ is motivated by the fact that it represents a discrete analogue of the differential equation (see [1] and the references cited therein)

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0, \quad t \ge 0, \qquad (3)$$

where, for every $i \in \{1, \ldots, m\}$, p_i is an oscillating continuous real-valued function in the interval $[0, \infty)$, and τ_i is a continuous real-valued function on $[0, \infty)$ such that

$$\tau_i(t) \le t, \quad t \ge 0, \quad \lim_{t\to\infty} \tau_i(t) = \infty, \qquad (4)$$

while, $(\mathrm{E_A})$ represents a discrete analogue of the advanced differential equation (see [1] and the references cited therein)

$$x'(t) - \sum_{i=1}^{m} p_i(t)\, x(\sigma_i(t)) = 0, \quad t \ge 1, \qquad (5)$$

where, for every $i \in \{1, \ldots, m\}$, p_i is an oscillating continuous real-valued function in the interval $[1, \infty)$ and σ_i is a continuous real-valued function on $[1, \infty)$ such that

$$\sigma_i(t) \geq t, \quad t \geq 1. \tag{6}$$

By a *solution* of $(\mathrm{E_R})$, we mean a sequence of real numbers $\{x(n)\}_{n \geq -w}$ which satisfies $(\mathrm{E_R})$ for all $n \in \mathbb{N}_0$. Here,

$$w = -\min_{n \geq 0} \{\tau_i(n) : 1 \leq i \leq m\}. \tag{7}$$

It is clear that, for each choice of real numbers $c_{-w}, c_{-w+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $\{x(n)\}_{n \geq -w}$ of $(\mathrm{E_R})$ which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w+1) = c_{-w+1}, \ldots, x(-1) = c_{-1}$, and $x(0) = c_0$.

By a *solution* of the advanced difference equation $(\mathrm{E_A})$, we mean a sequence of real numbers $\{x(n)\}_{n \in \mathbb{N}_0}$ which satisfies $(\mathrm{E_A})$ for all $n \in \mathbb{N}$.

A solution $\{x(n)\}_{n \geq -w} [\{x(n)\}_{n \in \mathbb{N}_0}]$ of $(\mathrm{E_R})$ $[(\mathrm{E_A})]$ is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the last few decades, the oscillatory behavior of all solutions of difference equations has been extensively studied when the coefficients $p_i(n)$ are nonnegative. See, for example, [2–20] and the references cited therein. However, for the general case when $p_i(n)$ are allowed to oscillate, it is difficult to study the oscillation of $(\mathrm{E_R})$ $[(\mathrm{E_A})]$, since the difference $\Delta x(n) [\nabla x(n)]$ of any nonoscillatory solution of $(\mathrm{E_R})$ $[(\mathrm{E_A})]$ is always oscillatory. Thus, a small number of papers are dealing with this case. See, for example, [1, 21–32] and the references cited therein.

For (3) and (5) with oscillating coefficients, Fukagai and Kusano [1] established the following theorems.

Theorem 1 (see [1, Theorem 3′(i)]). *Assume* (4) *and that there is a continuous nondecreasing function* $\tau^*(t)$ *such that* $\tau_i(t) \leq \tau^*(t) \leq t$ *for* $t \geq 0$, $1 \leq i \leq m$. *Suppose moreover that there is a sequence* $\{t_n\}_{n \in \mathbb{N}}$ *such that* $\lim_{n \to \infty} t_n = \infty$, *the intervals* $\bigcup_{n \in \mathbb{N}} [(\tau^*)^n(t_n), t_n]$ *are disjoint, and*

$$p_i(t) \geq 0 \quad \forall t \in \bigcup_{n \in \mathbb{N}} \left[(\tau^*)^n(t_n), t_n\right], \; 1 \leq i \leq m. \tag{8}$$

If there is a constant c such that

$$\int_{\tau^*(t)}^{t} \sum_{i=1}^{m} p_i(s)\, ds > c > \frac{1}{e} \quad \forall t \in \bigcup_{n \in \mathbb{N}} \left[(\tau^*)^{n-1}(t_n), t_n\right], \tag{9}$$

then all solutions of (3) *oscillate.*

Theorem 2 (see [1, Theorem 3′(ii)]). *Assume* (6) *and that there is a continuous nondecreasing function* $\sigma_*(t)$ *such that* $t \leq \sigma_*(t) \leq \sigma_i(t)$ *for* $t \geq 0$, $1 \leq i \leq m$. *Suppose moreover that there is a sequence* $\{t_n\}_{n \in \mathbb{N}}$ *such that* $\lim_{n \to \infty} t_n = \infty$, *the intervals* $\bigcup_{n \in \mathbb{N}} [t_n, (\sigma_*)^n(t_n)]$ *are disjoint, and*

$$p_i(t) \geq 0 \quad \forall t \in \bigcup_{n \in \mathbb{N}} \left[t_n, (\sigma_*)^n(t_n)\right], \; 1 \leq i \leq m. \tag{10}$$

If there is a constant c such that

$$\int_{t}^{\sigma_*(t)} \sum_{i=1}^{m} p_i(s)\, ds > c > \frac{1}{e} \quad \forall t \in \bigcup_{n \in \mathbb{N}} \left[t_n, (\sigma_*)^{n-1}(t_n)\right], \tag{11}$$

then all solutions of (5) *oscillate.*

For $(\mathrm{E_R})$ and $(\mathrm{E_A})$ with oscillating coefficients, recently, Bohner et al. [21, 23] established the following theorems.

Theorem 3 (see [23, Theorem 2.4]). *Assume* (1) *and that the sequences* τ_i *are increasing for all* $i \in \{1, \ldots, m\}$. *Suppose also that for each* $i \in \{1, \ldots, m\}$ *there exists a sequence* $\{n_i(j)\}_{j \in \mathbb{N}}$ *such that* $\lim_{j \to \infty} n_i(j) = \infty$ *and*

$$p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j \in \mathbb{N}} [\tau(\tau(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{12}$$

where

$$\tau(n) = \max_{1 \leq i \leq m} \tau_i(n), \quad n \in \mathbb{N}_0. \tag{13}$$

If, moreover,

$$\limsup_{j \to \infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) > 1, \tag{14}$$

where $n(j) = \min\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* $(\mathrm{E_R})$ *oscillate.*

Theorem 4 (see [23, Theorem 3.4]). *Assume* (2) *and that the sequences* σ_i *are increasing for all* $i \in \{1, \ldots, m\}$. *Suppose also that for each* $i \in \{1, \ldots, m\}$ *there exists a sequence* $\{n_i(j)\}_{j \in \mathbb{N}}$ *such that* $\lim_{j \to \infty} n_i(j) = \infty$ *and*

$$p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j \in \mathbb{N}} [n_i(j), \sigma(\sigma(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{15}$$

where

$$\sigma(n) = \min_{1 \leq i \leq m} \sigma_i(n), \quad n \in \mathbb{N}. \tag{16}$$

If, moreover,

$$\limsup_{j \to \infty} \sum_{i=1}^{m} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) > 1, \tag{17}$$

where $n(j) = \max\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* $(\mathrm{E_A})$ *oscillate.*

Theorem 5 (see [21, Theorem 2.1]). *Assume* (1) *and that the sequences* τ_i *are increasing for all* $i \in \{1, \ldots, m\}$. *Suppose also that for each* $i \in \{1, \ldots, m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau_i(\tau_i(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{18}$$

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau_i(\tau_i(n_i(j))), n_i(j)] \cap \mathbb{N} \right\}. \tag{19}$$

If, moreover,

$$\liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j))}^{n_i(j)-1} p_i(q) > \frac{1}{e}, \tag{20}$$

then all solutions of $(\mathrm{E_R})$ *oscillate.*

Theorem 6 (see [21, Theorem 3.1]). *Assume* (2) *and that the sequences* σ_i *are increasing for all* $i \in \{1, \ldots, m\}$. *Suppose also that for each* $i \in \{1, \ldots, m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma_i(\sigma_i(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{21}$$

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma_i(\sigma_i(n_i(j)))] \cap \mathbb{N} \right\}. \tag{22}$$

If, moreover,

$$\liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n_i(j)+1}^{\sigma_i(n_i(j))} p_i(q) > \frac{1}{e}, \tag{23}$$

then all solutions of $(\mathrm{E_A})$ *oscillate.*

In the present paper, the authors study further $(\mathrm{E_R})$ $[(\mathrm{E_A})]$ and derive new sufficient oscillation conditions when neither (14) [(17)] nor (20) [(23)] is satisfied (cf. [6–8] and the references cited therein in the case of the equations $(\mathrm{E_R})$ $[(\mathrm{E_A})]$ with nonnegative coefficients p_i, $1 \leq i \leq m$). Examples illustrating the results are also given.

2. Retarded Equations

In this section, we present new sufficient conditions for the oscillation of all solutions of $(\mathrm{E_R})$ when the conditions (14) and (20) are not satisfied, under the assumption that the sequences τ_i are increasing for all $i \in \{1, \ldots, m\}$. To that end, the following lemma provides a useful tool.

Lemma 7. *Assume that* (1) *holds, the sequences* τ_i *are increasing for all* $i \in \{1, \ldots, m\}$ *and* $(x(n))_{n\geq -w}$ *is a nonoscillatory solution of* $(\mathrm{E_R})$. *Suppose also that for each* $i \in \{1, \ldots, m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$, *such that* $\lim_{j\to\infty} n_i(j) = \infty$, *and* (12) *where* τ *is defined by* (13). *Set*

$$\alpha := \liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)-1} p_i(q), \tag{24}$$

where $n(j) = \min\{n_i(j) : 1 \leq i \leq m\}$.

If $0 < \alpha < 1$, *then*

$$\liminf_{j\to\infty} \frac{x(n(j)+1)}{x(\tau(n(j)))} \geq \frac{\alpha^2}{4(1-\alpha)}. \tag{25}$$

Proof. Since the solution $\{x(n)\}_{n\geq -w}$ of $(\mathrm{E_R})$ is nonoscillatory, it is either eventually positive or eventually negative. As $\{-x(n)\}_{n\geq -w}$ is also a solution of $(\mathrm{E_R})$, we may restrict ourselves only to the case where $x(n) > 0$ eventually.

By (12), it is obvious that there exists $j_0 \in \mathbb{N}$ such that

$$p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}, \quad 1 \leq k \leq m, \tag{26}$$

$$x(\tau_k(n)) > 0 \quad \forall n \in \bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}, \quad 1 \leq k \leq m. \tag{27}$$

Also, by (24) we have

$$\sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)-1} p_i(q) \geq \alpha - \varepsilon, \tag{28}$$

where ε is an arbitrary real number with $0 < \varepsilon < \alpha$.

In view of (26) and (27), $(\mathrm{E_R})$ gives

$$x(n+1) - x(n) = -\sum_{i=1}^{m} p_i(n)\, x(\tau_i(n)) \leq 0, \tag{29}$$

for every $n \in \bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}$. This guarantees that the sequence x is decreasing on $\bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}$.

Assume that $0 < \alpha < 1$, where α is defined by (24). From inequality (28), it is clear that there exists $n^*(j_0) \geq n(j_0)$ such that

$$\sum_{i=1}^{m} \sum_{q=n(j_0)}^{n^*(j_0)-1} p_i(q) < \frac{\alpha - \varepsilon}{2}, \qquad \sum_{i=1}^{m} \sum_{q=n(j_0)}^{n^*(j_0)} p_i(q) \geq \frac{\alpha - \varepsilon}{2}. \tag{30}$$

This is because in the case where $p_i(q) < (\alpha-\varepsilon)/2$, there exists $n^*(j_0) > n(j_0)$ such that (30) is satisfied, while in the case where $p_i(q) \geq (\alpha-\varepsilon)/2$, then $n^*(j_0) = n(j_0)$, and, therefore,

$$\sum_{i=1}^{m}\sum_{q=n(j_0)}^{n^*(j_0)-1} p_i(q) = \sum_{i=1}^{m}\sum_{q=n(j_0)}^{n(j_0)-1} p_i(q) \quad (\text{by which we mean}) = 0 < \frac{\alpha-\varepsilon}{2}, \tag{31}$$

$$\sum_{i=1}^{m}\sum_{q=n(j_0)}^{n^*(j_0)} p_i(q) = \sum_{i=1}^{m}\sum_{q=n(j_0)}^{n(j_0)} p_i(q) \geq p_i(n(j_0)) \geq \frac{\alpha-\varepsilon}{2}. \tag{32}$$

That is, in both cases (30) is satisfied.

Now, we will show that $\tau(n^*(j_0)) \leq n(j_0)-1$. Indeed, in the case where $p_i(n(j_0)) \geq (\alpha-\varepsilon)/2$, since $n^*(j_0) = n(j_0)$, it is obvious that $\tau(n^*(j_0)) = \tau(n(j_0)) \leq n(j_0)-1$. In the case where $p_i(n(j_0)) < (\alpha-\varepsilon)/2$, then $n^*(j_0) > n(j_0)$. Assume, for the sake of contradiction, that $\tau(n^*(j_0)) > n(j_0)-1$. Hence, $n(j_0) \leq \tau(n^*(j_0)) \leq n^*(j_0)-1$ and then

$$\sum_{i=1}^{m}\sum_{q=\tau(n^*(j_0))}^{n^*(j_0)-1} p_i(q) \leq \sum_{i=1}^{m}\sum_{q=n(j_0)}^{n^*(j_0)-1} p_i(q) < \frac{\alpha-\varepsilon}{2}, \tag{33}$$

which contradicts (28). Thus, in both cases, we have $\tau(n^*(j_0)) \leq n(j_0)-1$. Therefore

$$\begin{aligned}\sum_{i=1}^{m}\sum_{q=\tau(n^*(j_0))}^{n(j_0)-1} p_i(q) &= \sum_{i=1}^{m}\sum_{q=\tau(n^*(j_0))}^{n^*(j_0)-1} p_i(q) - \sum_{i=1}^{m}\sum_{q=n(j_0)}^{n^*(j_0)-1} p_i(q)\\ &> (\alpha-\varepsilon) - \frac{\alpha-\varepsilon}{2} = \frac{\alpha-\varepsilon}{2}.\end{aligned} \tag{34}$$

Summing up (E_R) from $n(j_0)$ to $n^*(j_0)$, and using the fact that the function x is decreasing and the function τ (as defined by (13)) is increasing, we have

$$\begin{aligned}x(n(j_0)) &= x(n^*(j_0)+1) + \sum_{i=1}^{m}\sum_{q=n(j_0)}^{n^*(j_0)} p_i(q)\, x(\tau_i(q))\\ &\geq x(n^*(j_0)+1) + \sum_{i=1}^{m}\sum_{q=n(j_0)}^{n^*(j_0)} p_i(q)\, x(\tau(q)),\end{aligned} \tag{35}$$

or

$$x(n(j_0)) \geq x(n^*(j_0)+1) + x(\tau(n^*(j_0)))\sum_{i=1}^{m}\sum_{q=n(j_0)}^{n^*(j_0)} p_i(q), \tag{36}$$

which, in view of (30), gives

$$x(n(j_0)) \geq x(n^*(j_0)+1) + \frac{\alpha-\varepsilon}{2} x(\tau(n^*(j_0))). \tag{37}$$

Summing up (E_R) from $\tau(n^*(j_0))$ to $n(j_0)-1$, and using the same arguments, we have

$$\begin{aligned}x(\tau(n^*(j_0))) &= x(n(j_0)) + \sum_{i=1}^{m}\sum_{q=\tau(n^*(j_0))}^{n(j_0)-1} p_i(q)\, x(\tau_i(q))\\ &\geq x(n(j_0)) + \sum_{i=1}^{m}\sum_{q=\tau(n^*(j_0))}^{n(j_0)-1} p_i(q)\, x(\tau(q)),\end{aligned} \tag{38}$$

or

$$\begin{aligned}&x(\tau(n^*(j_0)))\\ &\quad\geq x(n(j_0)) + x(\tau(n(j_0)-1))\sum_{i=1}^{m}\sum_{q=\tau(n^*(j_0))}^{n(j_0)-1} p_i(q),\end{aligned} \tag{39}$$

which, in view of (34), gives

$$x(\tau(n^*(j_0))) > x(n(j_0)) + \frac{\alpha-\varepsilon}{2} x(\tau(n(j_0)-1)). \tag{40}$$

Combining inequalities (37) and (40), we obtain

$$\begin{aligned}x(n(j_0)) &> x(n^*(j_0)+1) + \frac{\alpha-\varepsilon}{2}\\ &\quad\times\left[x(n(j_0)) + \frac{\alpha-\varepsilon}{2} x(\tau(n(j_0)-1))\right],\end{aligned} \tag{41}$$

or

$$\begin{aligned}&\left(1-\frac{\alpha-\varepsilon}{2}\right)x(n(j_0))\\ &\quad> x(n^*(j_0)+1) + \left(\frac{\alpha-\varepsilon}{2}\right)^{2.} x(\tau(n(j_0)-1)).\end{aligned} \tag{42}$$

Thus

$$x(n(j_0)) > \frac{(\alpha-\varepsilon)^2}{2[2-(\alpha-\varepsilon)]} x(\tau(n(j_0)-1)). \tag{43}$$

In view of (43), inequality (42) gives

$$\begin{aligned}x(n(j_0)) &> \frac{(\alpha-\varepsilon)^2/(2[2-(\alpha-\varepsilon)])}{1-((\alpha-\varepsilon)/2)} x(\tau(n^*(j_0)))\\ &\quad+ \frac{(\alpha-\varepsilon)^2}{2[2-(\alpha-\varepsilon)]} x(\tau(n(j_0)-1)),\end{aligned} \tag{44}$$

which, in view of (40) becomes

$$\begin{aligned}x(n(j_0)) &> \frac{(\alpha-\varepsilon)^2/(2[2-(\alpha-\varepsilon)])}{1-((\alpha-\varepsilon)/2)}\\ &\quad\times\left[x(n(j_0)) + \frac{\alpha-\varepsilon}{2} x(\tau(n(j_0)-1))\right]\\ &\quad+ \frac{(\alpha-\varepsilon)^2}{2[2-(\alpha-\varepsilon)]} x(\tau(n(j_0)-1)).\end{aligned} \tag{45}$$

Thus

$$\frac{x(n(j_0))}{x(\tau(n(j_0)-1))} > \frac{(\alpha-\varepsilon)^2}{4[1-(\alpha-\varepsilon)]}, \quad (46)$$

or

$$\frac{x(n(j_0)+1)}{x(\tau(n(j_0)))} > \frac{(\alpha-\varepsilon)^2}{4[1-(\alpha-\varepsilon)]}. \quad (47)$$

Hence,

$$\liminf_{j_0\to\infty}\frac{x(n(j_0)+1)}{x(\tau(n(j_0)))} \ge \frac{(\alpha-\varepsilon)^2}{4[1-(\alpha-\varepsilon)]}, \quad (48)$$

which, for arbitrarily small values of ε, implies (25).

The proof of the lemma is complete. □

Theorem 8. *Assume that* (1) *holds, the sequences τ_i are increasing for all $i \in \{1,\ldots,m\}$ and τ is defined by* (13). *Suppose also that for each $i \in \{1,\ldots,m\}$ there exists a sequence $\{n_i(j)\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} n_i(j) = \infty$,* (12) *and define α by* (24), *where $n(j) = \min\{n_i(j) : 1 \le i \le m\}$.*

If $0 < \alpha < 1$, and

$$\limsup_{j\to\infty}\sum_{i=1}^{m}\sum_{q=\tau(n(j))}^{n(j)} p_i(q) > 1 - \frac{\alpha^2}{4(1-\alpha)}, \quad (49)$$

then all solutions of (E_R) *oscillate.*

Proof. Assume, for the sake of contradiction, that $\{x(n)\}_{n\ge -w}$ is an eventually positive solution of (E_R). Then there exists $j_0 \in \mathbb{N}$ such that

$$\begin{aligned} p_k(n) \ge 0 \quad \forall n \in \bigcap_{k=1}^{m}[\tau(\tau(n_k(j_0))), n_k(j_0)] \cap \mathbb{N}, \\ 1 \le k \le m, \\ x(\tau_k(n)) > 0 \quad \forall n \in \bigcap_{k=1}^{m}[\tau(\tau(n_k(j_0))), n_k(j_0)] \cap \mathbb{N}, \\ 1 \le k \le m. \end{aligned} \quad (50)$$

Therefore, by (E_R) we have

$$x(n+1) - x(n) = -\sum_{i=1}^{m} p_i(n)\, x(\tau_i(n)) \le 0, \quad (51)$$

for every $n \in \bigcap_{k=1}^{m}[\tau(\tau(n_k(j_0))), n_k(j_0)] \cap \mathbb{N}$. This guarantees that the sequence x is decreasing on $\bigcap_{k=1}^{m}[\tau(\tau(n_k(j_0))), n_k(j_0)] \cap \mathbb{N}$.

Summing up (E_R) from $\tau(n(j_0))$ to $n(j_0)$, and using the fact that the function x is decreasing and the function τ (as defined by (13)) is increasing, we obtain

$$\begin{aligned} x(\tau(n(j_0))) &= x(n(j_0)+1) + \sum_{i=1}^{m}\sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q)\, x(\tau_i(q)) \\ &\ge x(n(j_0)+1) + x(\tau(n(j_0))) \\ &\quad \times \sum_{i=1}^{m}\sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q). \end{aligned} \quad (52)$$

Consequently,

$$\sum_{i=1}^{m}\sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) \le 1 - \frac{x(n(j_0)+1)}{x(\tau(n(j_0)))}, \quad (53)$$

which gives

$$\limsup_{j_0\to\infty}\sum_{i=1}^{m}\sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) \le 1 - \liminf_{j_0\to\infty}\frac{x(n(j_0)+1)}{x(\tau(n(j_0)))}. \quad (54)$$

Assume that $0 < \alpha < 1$ and (49) holds. Then by Lemma 7, inequality (25) is fulfilled, and so (54) leads to

$$\limsup_{j_0\to\infty}\sum_{i=1}^{m}\sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) \le 1 - \frac{\alpha^2}{4(1-\alpha)}, \quad (55)$$

which contradicts condition (49).

The proof of the theorem is complete. □

3. Advanced Equations

Oscillation of all solutions of (E_A) is described by the theorem below. Note that the proof is an easy modification of the proof of Theorem 8 and hence is omitted.

Theorem 9. *Assume* (2) *holds, the sequences σ_i are increasing for all $i \in \{1,\ldots,m\}$ and σ is defined by* (16). *Suppose also that for each $i \in \{1,\ldots,m\}$ there exists a sequence $\{n_i(j)\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} n_i(j) = \infty$,* (15) *and*

$$\alpha := \liminf_{j\to\infty}\sum_{i=1}^{m}\sum_{q=n(j)+1}^{\sigma(n(j))} p_i(q), \quad (56)$$

where $n(j) = \max\{n_i(j) : 1 \le i \le m\}$.

If $0 < \alpha < 1$ and

$$\limsup_{j\to\infty}\sum_{i=1}^{m}\sum_{q=n(j)}^{\sigma(n(j))} p_i(q) > 1 - \frac{\alpha^2}{4(1-\alpha)}, \quad (57)$$

then all solutions of (E_A) *oscillate.*

Remark 10. When $\alpha \to 0$, then the conditions (49) and (57) reduce to the conditions (14) and (17), respectively. However the improvement is clear when $\alpha \to 1/e$. The lower bound in (49) and (57) is 0.946475699. That is, when $0 < \alpha < 1/e$, our conditions (49) and (57) essentially improve (14) and (17).

4. Examples

The significance of the results is illustrated in the following examples.

Example 1. Consider the retarded difference equation

$$\Delta x(n) + p_1(n)x(n-2) + p_2(n)x(n-3) + p_3(n)x(n-4) = 0, \quad n \in \mathbb{N}_0, \tag{58}$$

where $p_1(n)$, $p_2(n)$, and $p_3(n)$ are oscillating coefficients, as shown in Figure 1.

In view of (13), it is obvious that $\tau(n) = n-2$. Observe that for

$$n_1(j) = 20j+9, \quad j \in \mathbb{N}, \tag{59}$$

we have $p_1(n) > 0$ for every $n \in A$, where

$$A = \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_1(j))), n_1(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [20j+5, 20j+9] \cap \mathbb{N}. \tag{60}$$

For

$$n_2(j) = 20j+8, \quad j \in \mathbb{N}, \tag{61}$$

we have $p_2(n) > 0$ for every $n \in B$, where

$$B = \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_2(j))), n_2(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [20j+4, 20j+8] \cap \mathbb{N} \tag{62}$$

and, for

$$n_3(j) = 20j+9, \quad j \in \mathbb{N}, \tag{63}$$

we have $p_3(n) \geq 0$ for every $n \in C$, where

$$C = \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_3(j))), n_3(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [20j+5, 20j+9] \cap \mathbb{N}. \tag{64}$$

Therefore,

$$p_1(n) > 0, \qquad p_2(n) > 0, \qquad p_3(n) \geq 0 \quad \forall n \in A \cap B \cap C = \bigcup_{j\in\mathbb{N}} [20j+5, 20j+8] \cap \mathbb{N} \neq \emptyset. \tag{65}$$

Observe that

$$n(j) = \min\{n_i(j) : 1 \leq i \leq 3\} = 20j+8, \quad j \in \mathbb{N}. \tag{66}$$

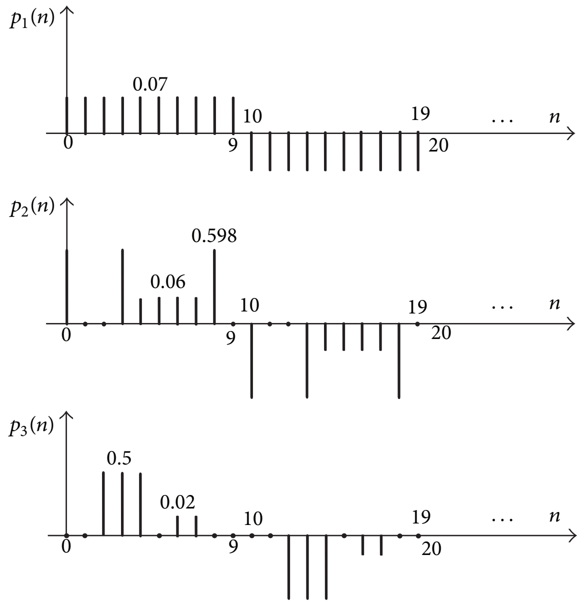

Figure 1

Now,

$$\begin{aligned}\alpha &= \liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)-1} p_i(q) \\ &= \liminf_{j\to\infty} \left[\sum_{q=20j+6}^{20j+7} p_1(q) + \sum_{q=20j+6}^{20j+7} p_2(q) + \sum_{q=20j+6}^{20j+7} p_3(q)\right] \\ &= 2\cdot\frac{7}{100} + 2\cdot\frac{6}{100} + 2\cdot\frac{2}{100} = 0.3, \\ &\limsup_{j\to\infty} \sum_{i=1}^{2} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) \\ &= \limsup_{j\to\infty} \left[\sum_{q=20j+6}^{20j+8} p_1(q) + \sum_{q=20j+6}^{20j+8} p_2(q) + \sum_{q=20j+6}^{20j+8} p_3(q)\right] \\ &= 3\cdot\frac{7}{100} + 2\cdot\frac{6}{100} + \frac{598}{1000} + 2\cdot\frac{2}{100} = 0.968.\end{aligned} \tag{67}$$

Observe that

$$0.968 > 1 - \frac{\alpha^2}{4(1-\alpha)} \simeq 0.967857142; \tag{68}$$

that is, condition (49) of Theorem 8 is satisfied and, therefore, all solutions of equation (58) oscillate.

On the other hand,

$$0.968 < 1. \tag{69}$$

Observe that $p_1(n) > 0$ for every $n \in A' = A$, $p_2(n) \geq 0$ for every $n \in B'$, where

$$B' = \bigcup_{j\in\mathbb{N}} [\tau_2(\tau_2(n_2(j))), n_2(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [20j+2, 20j+8] \cap \mathbb{N}, \tag{70}$$

and $p_3(n) \geq 0$ for every $n \in C'$, where

$$C' = \bigcup_{j\in\mathbb{N}} [\tau_3(\tau_3(n_3(j))), n_3(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [20j+1, 20j+9] \cap \mathbb{N}. \tag{71}$$

Therefore,

$$p_1(n) > 0, \quad p_2(n) > 0, \quad p_3(n) \geq 0 \quad \forall n \in A' \cap B' \cap C' = \bigcup_{j\in\mathbb{N}} [20j+5, 20j+8] \cap \mathbb{N} \neq \emptyset. \tag{72}$$

Also,

$$\begin{aligned} &\liminf_{j\to\infty} \sum_{i=1}^{2} \sum_{q=\tau_i(n_i(j))}^{n_i(j)-1} p_i(q) \\ &= \liminf_{j\to\infty} \left[\sum_{q=20j+7}^{20j+8} p_1(q) + \sum_{q=20j+5}^{20j+7} p_2(q) + \sum_{q=20j+5}^{20j+8} p_3(q) \right] \\ &= 2\cdot\frac{7}{100} + 3\cdot\frac{6}{100} + 2\cdot\frac{2}{100} = 0.36 < \frac{1}{e}. \end{aligned} \tag{73}$$

Therefore none of the conditions (14) and (20) is satisfied.

Example 2. Consider the advanced difference equation

$$\nabla x(n) - p_1(n)x(n+1) - p_2(n)x(n+2) = 0, \quad n \in \mathbb{N}, \tag{74}$$

where $p_1(n)$ and $p_2(n)$ are oscillating coefficients, as shown in Figure 2.

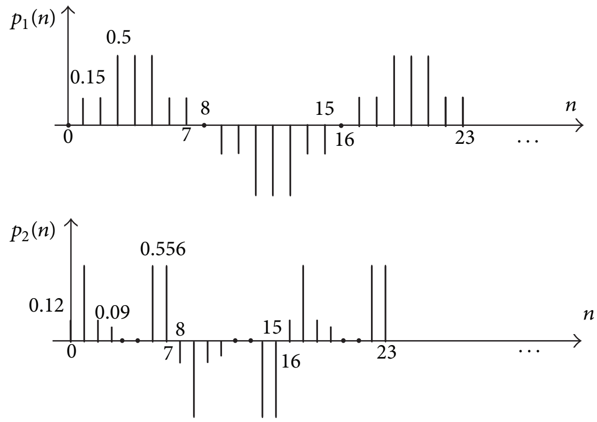

Figure 2

In view of (16), it is obvious that $\sigma(n) = n + 1$. Observe that for

$$n_1(j) = 16j, \quad j \in \mathbb{N}, \tag{75}$$

we have $p_1(n) \geq 0$ for every $n \in A$, where

$$A = \bigcup_{j\in\mathbb{N}} [n_1(j), \sigma(\sigma(n_1(j)))] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [16j, 16j+2] \cap \mathbb{N}. \tag{76}$$

Also, for

$$n_2(j) = 16j+1, \quad j \in \mathbb{N}, \tag{77}$$

we have $p_2(n) > 0$ for every $n \in B$, where

$$B = \bigcup_{j\in\mathbb{N}} [n_2(j), \sigma(\sigma(n_2(j)))] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [16j+1, 16j+3] \cap \mathbb{N}. \tag{78}$$

Therefore,

$$p_1(n) > 0, \quad p_2(n) > 0 \quad \forall n \in A \cap B = \bigcup_{j\in\mathbb{N}} [16j+1, 16j+2] \cap \mathbb{N} \neq \emptyset. \tag{79}$$

Observe that

$$n(j) = \max\{n_i(j) : 1 \leq i \leq 2\} = 16j+1, \quad j \in \mathbb{N}. \tag{80}$$

Now

$$\begin{aligned} \alpha &= \liminf_{j\to\infty} \sum_{i=1}^{2} \sum_{q=n(j)+1}^{\sigma(n(j))} p_i(q) \\ &= \liminf_{j\to\infty} \left[\sum_{q=16j+2}^{16j+2} p_1(q) + \sum_{q=16j+2}^{16j+2} p_2(q) \right] \\ &= \frac{15}{100} + \frac{12}{100} = 0.27. \end{aligned} \tag{81}$$

Also

$$\begin{aligned} &\limsup_{j\to\infty} \sum_{i=1}^{2} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) \\ &= \limsup_{j\to\infty} \left[\sum_{q=16j+1}^{16j+2} p_1(q) + \sum_{q=16j+1}^{16j+2} p_2(q) \right] \\ &= 2\cdot\frac{15}{100} + \frac{12}{100} + \frac{556}{1000} = 0.976. \end{aligned} \tag{82}$$

Observe that

$$0.976 > 1 - \frac{\alpha^2}{4(1-\alpha)} \simeq 0.975034246; \tag{83}$$

that is, condition (57) of Theorem 9 is satisfied and, therefore, all solutions of equation (74) oscillate.

On the other hand,

$$0.976 < 1. \tag{84}$$

Observe that $p_1(n) \geq 0$ for every $n \in A' = A$ and $p_2(n) \geq 0$ for every $n \in B'$, where

$$\begin{aligned} B' &= \bigcup_{j\in\mathbb{N}} \left[n_2(j), \sigma_2(\sigma_2(n_2(j)))\right] \cap \mathbb{N} \\ &= \bigcup_{j\in\mathbb{N}} \left[16j+1, 16j+5\right] \cap \mathbb{N}. \end{aligned} \tag{85}$$

Therefore,

$$\begin{aligned} p_1(n) > 0, \qquad p_2(n) > 0 \\ \forall n \in A' \cap B' \\ = \bigcup_{j\in\mathbb{N}} \left[16j+1, 16j+2\right] \neq \emptyset. \end{aligned} \tag{86}$$

Also,

$$\begin{aligned} &\liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n_i(j)+1}^{\sigma_i(n_i(j))} p_i(q) \\ &= \liminf_{j\to\infty} \left[\sum_{q=16j+1}^{16j+1} p_1(q) + \sum_{q=16j+2}^{16j+3} p_2(q)\right] \\ &= \frac{15}{100} + \frac{12}{100} + \frac{9}{100} = 0.36 < \frac{1}{e}. \end{aligned} \tag{87}$$

Therefore none of the conditions (17) and (23) is satisfied.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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