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Research Article

Properties of Solutions to Stochastic Set Differential Equations under Non-Lipschitzian Coefficients

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A class of stochastic set differential equations (SSDEs) with non-Lipschitzian coefficients is investigated. We first give the preliminaries on the stochastic set differential equations. Then the nonexplosion of solutions to the SSDEs is discussed. Moreover, the existence and uniqueness of the solutions to SSDEs are proven. Finally, the continuous dependence of the solutions to SSDEs is studied.

1. Introduction

Set-valued differential equations which were started in 1969 by de Blasi and Lervolino [1] have been employed in investigations of dynamic systems. The evidence of set differential equations for such areas as control theory, differential inclusions, and fuzzy differential equations can be found in [2–12] and references therein. The set differential equations also are explored in [13–15]. One of the main advantages of investigating deterministic set differential equations is that they can be used as a tool for studying properties of solutions of differential inclusions. On the other hand, the set-valued random processes are first introduced by van Cutsem [16]. Since then the subject has attracted the interest of many mathematicians and further contributions are made from both the theoretical and applied viewpoints (see, e.g., [17–26]). In [27–31], the set-valued random differential equations are studied. The strong solution of Itô type set-valued stochastic differential equation is analyzed in [32].

As far as we know, there exists a wide literature where attempts have been made to investigate stochastic differential inclusions (see, e.g., [33–40] and references therein). And recently, in [27], a kind of the SSDEs disturbed by Wiener processes is investigated, where under the Lipschitzian condition the existence and uniqueness of solutions to the SSDEs are proven. Under the non-Lipschitzian condition, the existence and uniqueness of solutions to the stochastic set differential equations are proven in [41, 42]. Moreover, in our present paper, under the non-Lipschitzian condition the nonexplosion and continuous dependence of solutions to the SSDEs are studied. The mathematical tool employed in the paper is the Bihari inequality and the notion of the support function. The work presented here generalizes results obtained both for deterministic and for random set differential equations. Also, it should be noted that the work related to this paper is the discussions of fuzzy-valued processes and stochastic differential equations (see, e.g., [43–51]).

The paper is organized as follows. Section 2 gives an appropriate framework on a set-valued analysis within which the notion of a set-valued stochastic integral is given. In Section 3, moreover, the continuous dependence of the solutions for SSDEs on initial conditions and nonexplosion are discussed. Finally, the conclusions are made in Section 4.

2. Preliminaries

Let $\mathscr{K}(\mathbf{R}^d)$ be the family of all nonempty compact and convex subsets of $\mathbf{R}^d$. In $\mathscr{K}(\mathbf{R}^d)$, we define the Hausdorff metric d_H of two sets $A, B \in \mathscr{K}(\mathbf{R}^d)$ as follows:

$$d_H = \max\left(\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right). \quad (1)$$

Throughout this paper, let $(\Omega, \mathscr{A}, P)$ be complete probability space. $\mathscr{A} \times \mathscr{B}_+$ is a product σ-field of $\Omega \times \mathbf{R}_+$. $\mathscr{M}(\Omega, \mathscr{A}; \mathscr{K}(\mathbf{R}^d))$ denotes the family of $\mathscr{A}$-measurable multifunctions with values in $\mathbf{R}^d$. A multifunction $F \in \mathscr{M}(\Omega, \mathscr{A}; \mathscr{K}(\mathbf{R}^d))$ is said to be L^p-integrably bounded, $p \ge 1$, if there exists $h \in L^p(\Omega, \mathscr{A}, P; \mathbf{R}_+)$ such that $\|\|F\|\| \le h$ a.s., where

$$\|\|A\|\| := d_H(A, \{0\}) = \sup_{a \in A} \|a\| \quad \text{for } A \in \mathscr{K}\left(\mathbf{R}^d\right). \tag{2}$$

Let us denote

$$\mathscr{L}^p\left(\Omega, \mathscr{A}, P; \mathscr{K}\left(\mathbf{R}^d\right)\right) := \left\{F \in \mathscr{M}\left(\Omega, \mathscr{A}; \mathscr{K}\left(\mathbf{R}^d\right)\right) : \|\|F\|\| \in L^p\left(\Omega, \mathscr{A}, P; \mathbf{R}_+\right)\right\}. \tag{3}$$

Denote $I := [0, \infty)$. Let $(\Omega, \mathscr{A}, \{\mathscr{A}_t\}_{t \in I}, P)$ be a complete, filtered probability space where the sub-σ-field family $(\mathscr{A}_t, t \in I)$ of $\mathscr{A}$ satisfies the usual conditions. We call $X : I \times \Omega \to \mathscr{K}(\mathbf{R}^d)$ a set-valued stochastic process, if for every $t \in I$ a mapping $X(t, \cdot) = X(t) : \Omega \to \mathscr{K}(\mathbf{R}^d)$ is a set-valued random variable. If $X : I \times \Omega \to \mathscr{K}(\mathbf{R}^d)$ is $\{\mathscr{A}_t\}_{t \in I}$-adapted and measurable, then it will be called nonanticipating. Equivalently, the set-valued process X is nonanticipating if and only if X is measurable with respect to the σ-algebra $\mathscr{N}$, which is defined as follows:

$$\mathscr{N} := \left\{A \in \mathscr{B}(I) \otimes \mathscr{A} : A^t \in \mathscr{A}_t \text{ for every } t \in I\right\}, \tag{4}$$

where $A^t = \{\omega : (t, \omega) \in A\}$ for $t \in I$.

Let $p \ge 1$ and $L^p(I \times \Omega, \mathscr{N}; \mathbf{R}^d)$ denote the set of all nonanticipating $\mathbf{R}^d$-valued stochastic processes $\{h(t)\}_{t \in I}$ such that $E(\int_0^T \|h(s)\|^p ds) < \infty$. A set-valued stochastic process X is called L^p-integrably bounded, if there exists a real-valued stochastic process $\hbar \in L^p(I \times \Omega, \mathscr{N}; \mathbf{R}_+)$ such that

$$\|\|X(t, \omega)\|\| \le \hbar(t, \omega) \quad \text{for a.a. } (t, \omega) \in I \times \Omega. \tag{5}$$

We define the operation on $\mathscr{K}(\mathbf{R}^d)$ as follows. For two sets $A, B \in \mathscr{K}(\mathbf{R}^d)$, if there exists such a $C \in \mathscr{K}(\mathbf{R}^d)$ that $A = B + C$, then C is the Hukuhara difference of A and B denoted by $C = A \ominus B$. We note that $A + B = \{0\}$ implies that $A = -B$. However, $A \ominus A \ne \{0\}$. Indeed, take $A = [0, 1]$.

It is well known that

$$\begin{gathered} d_H(A + C, B + C) = d_H(A, B), \\ d_H(\lambda A, \lambda B) = |\lambda| d_H(A, B), \end{gathered} \tag{6}$$

for all $A, B, C \in \mathscr{K}(\mathbf{R}^d)$, $\lambda \in \mathbf{R}$.

Let $\sigma(A, r) \triangleq \sup\{\langle x, r\rangle : x \in A\}; r \in \mathbf{B}$ is the support function of A, where $\mathbf{B}$ is a unit sphere centered at origin. The support function $\sigma(\cdot, \cdot)$ satisfies the following properties.

(i) $\sigma(A, \cdot)$ is bounded on $\mathbf{B}$; that is, $|\sigma(A, r)| \le \|\|A\|\|, \forall r \in \mathbf{B}$.

(ii) $\sigma(A, \cdot)$ is Lipschitz continuous in r

$$\left|\sigma(A, r) - \sigma(A, r^*)\right| \le \|\|A\|\| \cdot \left|r - r^*\right|, \quad \forall r, r^* \in \mathbf{B}. \tag{7}$$

(iii) For all $A, B \in \mathscr{K}(\mathbf{R}^d)$,

$$H(A, B) = \sup_{r \in \mathbf{B}} |\sigma(A, r) - \sigma(B, r)|. \tag{8}$$

(iv) For all $A, B \in \mathscr{K}(\mathbf{R}^d)$, $\sigma(A + B, r) = \sigma(A, r) + \sigma(B, r) \forall r \in \mathbf{B}$.

(v) For all $A \in \mathscr{K}(\mathbf{R}^d)$, $\sigma(\lambda A, r) = \lambda\sigma(A, r) \forall r \in \mathbf{B}$, $\lambda \ge 0$.

(vi) For all $A, B \in \mathscr{K}(\mathbf{R}^d)$, if $\sigma(A, r) = \sigma(A, r) + \sigma(B, r)$, $\forall r \in \mathbf{B}$; then we have $A = B$.

By $\mathscr{L}^p(I \times \Omega, \mathscr{N}; \mathscr{K}(\mathbf{R}^d))$ we denote the set of nonanticipating and L^p-integrably bounded set-valued stochastic processes. Let $X \in \mathscr{L}^1(I \times \Omega, \mathscr{N}; \mathscr{K}(\mathbf{R}^d))$. For such X and a fixed $t \in I$, by the Fubini Theorem, we can define Aumann's integral $\int_0^t X(s, \omega) ds$, for $\omega \in \Omega$. Obviously, for every $t \in I$ and $\omega \in \Omega$ the Aumann integral $\int_0^t X(s, \omega) ds$ belongs to $\mathscr{K}(\mathbf{R}^d)$ (see, e.g., [7, 19]).

We say that a set-valued stochastic process X is d_H-continuous, if almost all its trajectories, that is, the mappings $X(\cdot, \omega) : I \to \mathscr{K}(\mathbf{R}^d)$, are d_H-continuous functions. It is easy to know that if $X \in \mathscr{L}^p(I \times \Omega, \mathscr{N}; \mathscr{K}(\mathbf{R}^d))$, then the set-valued stochastic process $\int_0^t X(s) ds$ is d_H-continuous (see, e.g., Corollary 1 in [27]).

In what follows, we state the generalized Bihari inequality (cf., Mao [52]) which plays an important role in the following section.

Lemma 1 (generalized Bihari inequality). *Let u be Borel measurable, bounded, nonnegative, and left limit function on $[0, T]$ and $c > 0$. Let $K : \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous nondecreasing function such that $K(t) > 0$ for all $t > 0$.*

(i) If $\mu(t)$ is a continuous nonnegative nondecreasing function on $[0, T]$, then the inequality

$$u(t) \le c + \int_0^t K(u(s-)) d\mu(s), \quad \forall t \in [0, T] \tag{9}$$

implies that

$$u(t) \le G^{-1}(G(c) + \mu(t)), \tag{10}$$

for all $t \in [0, T]$ such that

$$G(c) + \mu(t) \in \operatorname{Dom}\left(G^{-1}\right), \tag{11}$$

where

$$G(q) = \int_a^q \frac{1}{K(v)} dv, \quad q > 0, \tag{12}$$

where G^{-1} is the inverse function of G and $a \in [0, T]$.

(ii) If $\nu(t)$ is a continuous nonpositive nonincreasing function on $[0, T]$, then the inequality

$$u(t) \ge c + \int_0^t K(u(s-)) d\nu(s), \quad \forall t \in [0, T], \tag{13}$$

implies that

$$u(t) \geq G^{-1}(G(c) + \nu(t)), \tag{14}$$

for all $t \in [0, T]$ *such that*

$$G(c) + \nu(t) \in \operatorname{Dom}(G^{-1}). \tag{15}$$

3. Properties of Solutions to SSDEs

In this section, we consider the following stochastic set differential equation (in the integral form):

$$X(t) = x_0 + \int_0^t f(X(s))\,ds + \int_0^t g(X(s))\,dW(s), \tag{16}$$

where f takes values in $C(\mathscr{K}(\mathbf{R}^d), \mathscr{K}(\mathbf{R}^d))$, g in $C(\mathscr{K}(\mathbf{R}^d), \mathbf{R}^d \times \mathbf{R}^m)$, $W = (W(t), t \geq 0)$ is an m-dimensional Brownian motion, and $x_0 \in \mathscr{K}(\mathbf{R}^d)$ is a set-valued random variable. Here, $C(\mathscr{K}(\mathbf{R}^d), \mathscr{K}(\mathbf{R}^d))$ (resp., $C(\mathscr{K}(\mathbf{R}^d), \mathbf{R}^d \times \mathbf{R}^m)$ stands for the family of the continuous functions from the space $\mathscr{K}(\mathbf{R}^d)$ to the space $\mathscr{K}(\mathbf{R}^d)$ (resp., $\mathscr{K}(\mathbf{R}^d)$ to $\mathbf{R}^d \times \mathbf{R}^m$). In (16), the integral $\int_0^t f(X(s))ds$ is Aumann's one, and the integral $\int_0^t g(X(s))dW(s)$ is a general Itô type stochastic one, whose definition can refer to [53].

Due to the continuity of f, we can know that $\sigma(f(\cdot), r)$ takes values in $C(\mathscr{K}(\mathbf{R}^d), \mathbf{R})$. In order to discuss the solutions to (16), by the concept and properties of the support function we consider the following single valued stochastic differential equation, for $\forall r \in \mathbf{B}$:

$$\begin{aligned} &d\sigma(X(t), r) = \sigma(f(X(t)), r)\,dt + r^\top g(X(t))\,dW(t), \\ &\sigma(X(0), r) = \sigma(x_0, r). \end{aligned} \tag{17}$$

We first claim that (16) is equivalent to (17). Indeed, from the properties of the support function and (16) we deduce that, for $r \in \mathbf{B}$,

$$\begin{aligned} \sigma(X(t), r) &= \sigma(x_0, r) + \sigma\left(\int_0^t f(X(t))\,dt, r\right) \\ &\quad + \int_0^t r^\top g(X(t))\,dW(t) \\ &= \sigma(x_0, r) + \int_0^t \sigma(f(X(t)), r)\,dt \\ &\quad + \int_0^t r^\top g(X(t))\,dW(t), \end{aligned} \tag{18}$$

which shows that (17) holds. Conversely, (16) can be derived from (17).

It is known that (17) has a solution up to a lifetime $\zeta(r)$ which depends on $r \in \mathbf{B}$. Set

$$\zeta = \inf_{r \in \mathbf{B}} \zeta(r). \tag{19}$$

We call ζ the lifetime of solution to (16). Obviously, $|||X(\zeta)||| = +\infty$ a.s. Here, by the concepts of explosion time and lifetime time from Pages 158 and 191 in [54], the lifetime of solution to (16) or (17) is the same as the explosion time of the solution of (16) or (17).

If (17) has the pathwise uniqueness, then we show the existence and uniqueness of the solution to the SSDE (16). So the study of pathwise uniqueness is of great interest. It is a classical result that, under the Lipschitz coefficients, the pathwise uniqueness holds and the solution of (17) can be constructed by using Picard iteration; moreover, the solution depends on the initial values continuously. However, under non-Lipschitzian condition, Fei [42] presents the existence and uniqueness of solutions to (16); hence, the existence and uniqueness of solutions to (17) also are proven.

In what follows, we discuss the nonexplosion of the solutions to (16) under non-Lipschitzian condition. Our idea is to derive an inequality so that the generalized Bihari inequality (Lemma 1) can be applied.

Theorem 2. *Let* $\rho : \mathbf{R}_+ \to [1, +\infty)$ *be a continuous function such that*

(i) $s\rho(s)$ *is nondecreasing and concave;*

(ii) $\int_0^\infty ds/(s\rho(s) + 1) = +\infty$.

Assume that, for some constant $C > 0$,

$$\begin{aligned} &|||f(x)||| \leq C\left(|||x|||\,\rho\left(|||x|||^2\right) + 1\right), \\ &\|g(x)\|^2 \leq C\left(|||x|||^2 \rho\left(|||x|||^2\right) + 1\right), \quad x \in \mathscr{K}\left(\mathbf{R}^d\right). \end{aligned} \tag{20}$$

If $E|||x_0||| < +\infty$, *then the lifetime of the solution* $X(t, x_0)$ *to* (16) *is infinite:* $\zeta = +\infty$ *a.s. Moreover,*

$$\lim_{E|||x_0||| \to +\infty} E\,|||X(t, x_0)||| = +\infty \quad \forall t \geq 0. \tag{21}$$

Proof. Let $\xi^r(t) = |\sigma(X(t), r)|^2, \forall r \in \mathbf{B}$, where $X(t)$ is a solution to (16). Hence, we have

$$\begin{aligned} d\xi^r(t) &= \left[2\sigma(X(t), r)\,\sigma(f(X(t)), r) + \left\|g(X(t))^\top r\right\|^2\right] dt \\ &\quad + 2\sigma(X(t), r)\, r^\top g(X(t))\,dW(t). \end{aligned} \tag{22}$$

Since $r \in \mathbf{B}$, we get $\|g(X(s))^\top r\|^2 \leq \|g(X(s))\|^2$. Set $\xi(t) = |||X(t)|||^2$; it follows that from (20) and property (i) of the support function $\sigma(\cdot)$

$$\begin{aligned} E\xi^r(t) = \xi^r(0) + 2E\int_0^t &\left(\sigma(X(s), r)\,\sigma(f(X(s)), r)\right. \\ &\left. + \left\|r^\top g(X(s))\right\|^2\right) ds \end{aligned}$$

$$\le E\|\|x_0\|\|^2 + 2C\int_0^t E\Big(2\xi(s)\rho(\xi(s)) + \sqrt{\xi(s)} + 1\Big)\,ds$$
$$\le E\|\|x_0\|\|^2 + 2C\int_0^t \Big(2E\xi(s)\rho(E\xi(s)) + \sqrt{E\xi(s)} + 1\Big)\,ds, \tag{23}$$

where we have utilized the concavity of the functions $v\rho(v)$ and $\sqrt{v}$.

Denote $\eta(t) = E\xi(t)$. Noticing $\rho(\eta) \ge 1$, we get

$$\frac{\eta\rho(\eta) + \sqrt{\eta}}{\eta\rho(\eta) + 1} \le 1 + \frac{\sqrt{\eta}}{\eta\rho(\eta) + 1} \le 1 + \frac{\sqrt{\eta}}{\eta + 1} \le 1 + \frac{1}{2} < 2, \tag{24}$$

where $\varphi(\eta) = \sqrt{\eta}/(1+\eta)$ takes the maximum at 1 on the interval $(0,\infty)$.

It is easy to see that

$$\sup_{\eta\ge 0}\frac{\eta\rho(\eta) + \sqrt{\eta}}{\eta\rho(\eta) + 1} \le 2, \tag{25}$$

which deduce that

$$\frac{2\eta\rho(\eta) + \sqrt{\eta}}{\eta\rho(\eta) + 1} \le \frac{2\left(\eta\rho(\eta) + \sqrt{\eta}\right)}{\eta\rho(\eta) + 1} + \frac{1}{\eta\rho(\eta) + 1} \le 5. \tag{26}$$

Thus we have

$$2\eta\rho(\eta) + \sqrt{\eta} + 1 \le 6\left(\eta\rho(\eta) + 1\right). \tag{27}$$

In virtue of (23), we have

$$\eta(t) \le E\|\|x_0\|\|^2 + 12C\int_0^t \left(\eta(s)\rho(\eta(s)) + 1\right)ds. \tag{28}$$

Set

$$G(u) = \int_1^u \frac{ds}{s\rho(s) + 1}, \quad u > 0. \tag{29}$$

By condition (ii) it is easy to show that $G(u)$ is strictly increasing, $G(u) \to +\infty$ as $u \to +\infty$ and $G^{-1}(u) \to +\infty$ as $u \to +\infty$.

From (28), the generalized Bihari inequality (Lemma 1 (i)), and $E\|\|x_0\|\| < +\infty$ we obtain

$$\eta(t) \le G^{-1}\left(G\left(E\|\|x_0\|\|^2\right) + 12Ct\right) < +\infty, \quad \forall t \ge 0, \tag{30}$$

which proves that $\zeta = +\infty$.

On the other hand, from (20) and property (i) of the support function $\sigma(\cdot)$ we have

$$E\xi^r(t) = \xi^r(0) + 2E\int_0^t \Big(\sigma(X(s),r)\,\sigma(f(X(s)),r) + \left\|r^{\top}g(X(s))\right\|^2\Big)\,ds$$
$$\ge E\|\|x_0\|\|^2 - 2C\int_0^t E\Big(\xi(s)\rho(\xi(s)) + \sqrt{\xi(s)}\Big)\,ds$$
$$\ge E\|\|x_0\|\|^2 - 2C\int_0^t \Big(E\xi(s)\rho(E\xi(s)) + \sqrt{E\xi(s)}\Big)\,ds. \tag{31}$$

Due to inequality (24), we have

$$\eta(t) \ge E\|\|x_0\|\|^2 - 4C\int_0^t \left(\eta(s)\rho(\eta(s)) + 1\right)ds. \tag{32}$$

By Lemma 1(ii), we get

$$E\|\|X(t,x_0)\|\|^2 = \eta(t) \ge G^{-1}\left(G\left(E\|\|x_0\|\|^2\right) - 4Ct\right), \quad \forall t \ge 0, \tag{33}$$

which shows that $\lim_{E\|\|x_0\|\| \to +\infty} E\|\|X(t,x_0)\|\| = +\infty$ by the property of the function $G(\cdot)$ in (29). Thus, the proof of the theorem is complete. □

Theorem 3. *Let $\psi : (0,1) \to [1,+\infty)$ be a continuous function such that*

(i) *$s\psi(s)$ is nondecreasing and concave;*

(ii) $\int_0^{1/2} ds/s\psi(s) = +\infty$.

Assume that for some constant $C > 0$, $d_H(x,y) < 1$, $x, y \in \mathscr{K}(\mathbf{R}^d)$, $t \in I$,

$$\begin{aligned} d_H\left(f(t,x), f(t,y)\right) &\le C d_H(x,y)\,\psi\left(d_H^2(x,y)\right), \\ \|g(t,x) - g(t,y))\|^2 &\le C d_H^2(x,y)\,\psi\left(d_H^2(x,y)\right). \end{aligned} \tag{34}$$

Then SSDE (16) *has a unique solution.*

Proof. By constructing the sequence $\{X_n\}$ of set-valued random variables as that in [42], the existence of the solutions to (16) is similarly proven. Next, we prove the uniqueness of the solutions to (16).

Let $(X(t))_{t\ge 0}$ and $(Y(t))_{t\ge 0}$ be two solutions of (16). Set $\eta^r(t) = \sigma(X(t),r) - \sigma(Y(t),r)$, $\xi^r(t) = |\eta^r(t)|^2$, $\forall r \in \mathbf{B}$. Hence, by Itô formula we have

$$\begin{aligned} d\xi^r(t) = \Big[&2\eta^r(t)\left(\sigma\left(f(t,X(t)),r\right) - \sigma\left(f(t,Y(t)),r\right)\right) \\ &+ \left\|\left(g(t,X(t)) - g(t,Y(t))\right)^{\top} r\right\|^2\Big]\,dt \\ &+ 2\eta^r(t)\,r^{\top}\left(g(t,X(t)) - g(t,Y(t))\right)dW(t). \end{aligned} \tag{35}$$

Since $r \in \mathbf{B}$, we get $\|(g(t,X(s)) - g(t,Y(s)))^\top r\|^2 \le \|g(t,X(s)) - g(t,Y(s))\|^2$. Set $\xi(t) = d_H^2(X(t),Y(t))$.

From condition (34) and the property of the support function, we get

$$\begin{aligned} &\left|\eta^r(t)\right| \le d_H(X(t),Y(t)), \\ &\left|\sigma(f(t,X(t)),r) - \sigma(f(t,Y(t)),r)\right| \\ &\quad \le d_H(f(t,X(t)), f(t,Y(t))) \\ &\quad \le C d_H(X(t),Y(t))\,\psi\left(d_H^2(X(t),Y(t))\right), \end{aligned} \tag{36}$$

which deduces

$$\begin{aligned} &E\xi^r(t) \\ &\quad = 2E\int_0^t \Big(\eta^r(s)\left(\sigma(f(t,X(s)),r) - \sigma(f(t,Y(s)),r)\right) \\ &\qquad\qquad + \left\|(g(t,X(s)) - g(t,Y(s)))^\top r\right\|^2\Big)\,ds \\ &\quad \le 2C\int_0^t E\left(2d_H^2(X(s),Y(s))\,\psi\right. \\ &\qquad\qquad \left.\times\left(d_H^2(X(s),Y(s))\right)\right)ds \\ &\quad \le 4C\int_0^t E\xi(s)\,\psi(E\xi(s))\,ds, \end{aligned} \tag{37}$$

where we have utilized the concavity of the functions $v\psi(v)$.

Denote $\vartheta(t) = E\xi(t)$. In virtue of (37), we have

$$\vartheta(t) \le 4C\int_0^t \vartheta(s)\,\psi(\vartheta(s))\,ds. \tag{38}$$

Let

$$G(u) = \int_{1/2}^{u} \frac{ds}{s\psi(s)}, \quad u > 0. \tag{39}$$

We easily show that $G(u)$ is strictly increasing, $G(u) \to -\infty$ as $u \to 0$ and $G^{-1}(u) \to 0$ as $u \to -\infty$.

By Lemma 1 (i), we obtain

$$\vartheta(t) \le G^{-1}(G(0+) + 4Ct) = 0, \quad \forall t \ge 0, \tag{40}$$

which shows that $X(t) = Y(t)$. Thus we complete the proof. □

Note that function

$$\psi(s) = \begin{cases} \log\dfrac{1}{s}, & 0 < s \le \dfrac{1}{e}, \\ 1, & s > \dfrac{1}{e}, \end{cases} \tag{41}$$

is a typical example satisfying conditions (i) and (ii).

Next, we will study the dependence of the solutions to the SSDE (16) on initial data. For the mapping $x_0 \to X(t,x_0)$, we call $X_t(x_0) = X(t,x_0)$ mean square continuous on x_0, uniformly with respect to $t \in I$ if $Ed_H(X(t,y_0),X(t,x_0)) \to 0$ as $d_H - \lim y_0 = x_0$ on any compact subset I of t, where the limit x_0 of y_0 is in sense of the metric d_H.

Theorem 4. *Assume that the conditions in Theorem 3 hold. Then the mapping $x_0 \to X_t(x_0)$ is mean square continuous, uniformly with respect to t in any compact subset, where $X_t(x_0) = X(t,x_0)$ is the solution to SSDE (16).*

Proof. Take $\epsilon \in (0,1)$. Consider a small parameter $0 < \delta < \epsilon$. Assume $x_0, y_0 \in \mathscr{K}(\mathbf{R}^d)$ such that $d_H(x_0,y_0) < \delta$. For $\forall r \in \mathbf{B}$, let

$$\begin{aligned} \eta^r(t) &= \sigma(X_t(x_0),r) - \sigma(X_t(y_0),r), \\ \xi^r(t) &= \left|\eta^r(t)\right|^2, \\ \xi(t) &= d_H^2(X_t(x_0),X_t(y_0)). \end{aligned} \tag{42}$$

By (17) and Itô formula, we have

$$\begin{aligned} &d\xi^r(t) \\ &\quad = \Big[2\eta^{\alpha,r}(t)\left(\sigma(f(t,X(t,x_0)),\alpha,r)\right. \\ &\qquad\qquad \left.-\sigma(f(t,X(t,y_0)),r)\right) \\ &\qquad + \left\|(g(t,X(t,x_0)) - g(t,X(t,y_0)))^\top r\right\|^2\Big]\,dt \\ &\qquad + 2\eta^{\alpha,r}(t)\,r^\top\left(g(t,X(t,x_0))\right. \\ &\qquad\qquad \left.-g(t,X(t,y_0))\right)dW(t). \end{aligned} \tag{43}$$

From condition (34) and the property of the support function, we get

$$\begin{aligned} &\left|\sigma(f(t,X(t,x_0)),r) - \sigma(f(t,X(t,y_0)),r)\right| \\ &\le d_H(f(t,X(t,x_0)), f(t,X(t,y_0))) \\ &\le C d_H(X(t,x_0),X(t,y_0))\,\psi\left(d_H^2(X(t,x_0),X(t,y_0))\right), \end{aligned} \tag{44}$$

which deduces

$$\begin{aligned} &E\xi^r(t) \\ &\quad = E\xi^r(0) \\ &\qquad + 2E\int_0^t \Big(\eta^{\alpha,r}(s)\Big(\sigma(f(t,X(s,x_0)),r) \\ &\qquad\qquad -\sigma(f(t,X(s,y_0)),r)\Big) \\ &\qquad\qquad + \left\|(g(t,X(s,x_0)) - g(t,X(s,y_0)))^\top r\right\|^2\Big)\,ds \\ &\quad \le Ed_H^2(x_0,y_0) \\ &\qquad + 2C\int_0^t E\left(2d_H^2(X(s,x_0),X(s,y_0))\,\psi\right. \\ &\qquad\qquad \left.\times\left(d_H^2(X(s,x_0),X(s,y_0))\right)\right)ds \\ &\quad \le Ed_H^2(x_0,y_0) + 4C\int_0^t E\xi(s)\,\psi(E\xi(s))\,ds, \end{aligned} \tag{45}$$

where we have utilized the concavity of the functions $v\psi(v)$.

Setting $\vartheta(t) = Ed_H^2(X_t(x_0), X_t(y_0))$, from (45) we have

$$\vartheta(t) \le Ed_H^2(x_0, y_0) + 4C\int_0^t \vartheta(s)\psi(\vartheta(s))\,ds. \tag{46}$$

Define

$$G(u) = \int_a^u \frac{ds}{s\psi(s)}, \quad u > 0, \text{ for some } a \in (0,1). \tag{47}$$

We easily have that $G(u)$ is strictly increasing, $G(u) \to -\infty$ as $u \to 0$ and $G^{-1}(u) \to 0$ as $u \to -\infty$.

By Lemma 1 (i), we have

$$\vartheta(t) \le G^{-1}\left(G\left(Ed_H^2(x_0, y_0)\right) + 4Ct\right). \tag{48}$$

For arbitrary $\epsilon > 0$ and given t, it is easy to deduce that there exists $\delta > 0$ with $d_H(x_0, y_0) < \delta$, which shows $Ed_H(x_0, y_0) < \delta$, such that

$$\vartheta(t) \le G^{-1}\left(G\left(Ed_H^2(x_0, y_0)\right) + 4Ct\right) < \epsilon^2. \tag{49}$$

Since $G(u)$ is increasing, we have that

$$\begin{aligned}\sup_{0\le s\le t}\vartheta(s) &\le \sup_{0\le s\le t} G^{-1}\left(G\left(Ed_H^2(x_0, y_0)\right) + 4Cs\right)\\ &\le G^{-1}\left(G\left(Ed_H^2(x_0, y_0)\right) + 4Ct\right) < \epsilon^2,\end{aligned} \tag{50}$$

which shows that

$$\begin{aligned}&\sup_{0\le s\le t} Ed_H\left(X_s(x_0), X_s(y_0)\right)\\ &\qquad\le \sup_{0\le s\le t}\sqrt{Ed_H^2\left(X_s(x_0), X_s(y_0)\right)} < \epsilon.\end{aligned} \tag{51}$$

Thus, we show $X_t(x_0)$ is mean square continuous on x_0, with respect to t in any compact subset. Therefore, we complete the proof. □

Definition 5. The two solutions $X_t(x_0)$ and $Y_t(y_0)$ of SSDE (16) with initial value x_0 and y_0, respectively, if for any t, $P(d_H(X_t(x_0), X_t(y_0)) > 0) > 0$, can be called *nonconfluence*.

The following theorem gives the sufficient condition.

Theorem 6. *Suppose that the conditions in Theorem 3 hold. For $\forall x_0, y_0 \in \mathscr{K}(\mathbf{R}^d)$, $x_0 \neq y_0$, we have that $X_t(x_0)$ and $X_t(y_0)$ are nonconfluence.*

Proof. Let

$$\begin{aligned}\eta^r(t) &= \sigma\left(X_t(x_0), r\right) - \sigma\left(X_t(y_0), r\right),\\ \xi^r(t) &= \left|\eta^r(t)\right|^2,\\ \xi(t) &= d_H^2\left(X_t(x_0), X_t(y_0)\right),\\ \vartheta(t) &= E\xi(t).\end{aligned} \tag{52}$$

From condition (34) and the property of the support function, for $\forall r \in \mathbf{B}$, we get

$$\begin{aligned}&\left|\sigma\left(f\left(t, X(t,x_0)\right), r\right) - \sigma\left(f\left(t, X(t,y_0)\right), r\right)\right|\\ &\quad\le d_H\left(f\left(t, X(t,x_0)\right), f\left(t, X(t,y_0)\right)\right)\\ &\quad\le Cd_H\left(X(t,x_0), X(t,y_0)\right)\psi\left(d_H^2\left(X(t,x_0), X(t,y_0)\right)\right).\end{aligned} \tag{53}$$

Thus, similar to the discussion in the proof of Theorem 4, we obtain

$$\begin{aligned}&E\xi^r(t)\\ &\quad= E\xi^r(0)\\ &\qquad+ 2E\int_0^t\Big(\eta^r(s)\Big(\sigma\left(f\left(t, X(s,x_0)\right), r\right)\\ &\qquad\qquad -\sigma\left(f\left(t, X(s,y_0)\right), r\right)\Big)\\ &\qquad\qquad +\left\|\left(g\left(t, X(s,x_0)\right) - g\left(t, X(s,y_0)\right)\right)^{\top} r\right\|^2\Big)\,ds\\ &\quad\ge Ed_H^2(x_0, y_0)\\ &\qquad - 2C\int_0^t E\Big(2d_H^2\left(X(s,x_0), X(s,y_0)\right)\psi\\ &\qquad\qquad \times\left(d_H^2\left(X(s,x_0), X(s,y_0)\right)\right)\Big)\,ds\\ &\quad\ge Ed_H^2(x_0, y_0) - 4C\int_0^t E\xi(s)\psi(E\xi(s))\,ds,\end{aligned} \tag{54}$$

which shows that

$$\vartheta(t) \ge Ed_H^2(x_0, y_0) - 4C\int_0^t \vartheta(s)\psi(\vartheta(s))\,ds. \tag{55}$$

Take $G(u)$ as in the proof of Theorem 4.

By Lemma 1 (ii), we have

$$\vartheta(t) \ge G^{-1}\left(G\left(Ed_H^2(x_0, y_0)\right) - 4Ct\right), \quad \forall t \ge 0. \tag{56}$$

Since $Ed_H^2(x_0, y_0) \neq 0$, from the property of $G(u)$, we obtain

$$\vartheta(t) > 0, \quad \forall t \ge 0, \tag{57}$$

which means that $P(d_H(X_t(x_0), X_t(y_0)) > 0) > 0$. Thus, the proof is complete. □

Finally, through using (34), Theorem 6, and the standard arguments, we obtain the following theorem.

Theorem 7. *Let condition (34) hold. Then for any $t > 0$, $x_0 \to X_t(x_0)$ defines a flow of homeomorphisms of $\mathscr{K}(\mathbf{R}^d)$.*

4. Conclusions

In many real dynamic systems, we are often faced with random experiments whose outcomes might be multivalued. Moreover, the stochastic set differential equations may be employed in characterizing a large class of physically important dynamic systems which can be applied in such areas as control, economics, and finance. In this paper, we study the behavior of solutions to SSDEs disturbed by a Wiener process with the non-Lipschitzian coefficients. First, the nonexplosion theorem of the Itô type SSDEs is proven. Then the existence and uniqueness theorem of solutions to SSDEs is given. Moreover, the continuous dependence of solutions to the SSDEs is investigated. Main mathematical tool is the notion of the support function and the generalized Bihari inequality. Besides, the present case can be extended to the SSDEs driven by a multidimensional semimartingale in future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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