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Research Article

A Weak Limit Theorem for Galton-Watson Processes in Varying Environments

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We extend Donsker's theorem and the central limit theorem of classical Galton-Watson process to the Galton-Watson processes in varying environment.

1. Introduction

There has been a lot of interesting works on Markov chains in random environments, which is mainly concentrated in branching processes in random environments and random walks in random environments (see [1]).

The study of branching processes in random environments dates back to late 60s or early 70s in the last century (see [2–5]). Our paper deals with a Galton-Watson branching process in the varying environment (GWVE) which is a special case of branching processes in random environments. The main concern is the weak convergence for a GWVE, which is an extension of Donsker's theorem (see [6, 7]).

In the following context, $\{X_{ni}, n \geq 0, i \geq 1\}$ is a double sequence of independent and nonnegative integer valued random variables, where for fixed n, $\{X_{ni}, i \geq 1\}$ have the same distribution $\{p_{ni}, i = 0, 1, 2, \ldots\}$ with mean $\mu_n > 0$ and variance $\sigma_n^2 > 0$.

Definition 1. Assume $Z_0 \equiv 1$ and for any $n \geq 1$, define

$$Z_{n+1} = \begin{cases} \displaystyle\sum_{j=1}^{Z_n} X_{nj}, & \text{if } Z_n \neq 0; \\ 0, & \text{if } Z_n = 0, \end{cases} \tag{1}$$

then $\{Z_n, n \geq 0\}$ is said to be a GWVE.

Define $m_n = E(Z_n)$; it is well known that $m_n = \mu_0 \cdot \mu_1, \ldots, \mu_{n-1}$ and there exists a nonnegative random variable V such that $Z_n/m_n \xrightarrow{\text{a.e.}} V$, as $n \to +\infty$ (see [8]).

For any fixed r, let $\xi_{nj} := X_{n,r}^{(j)}$ be the size of the rth generation of GWVE starting with the jth particle at time n; then $\{\xi_{nj}, j \geq 1\}$ are i.i.d. with mean $m_{n,r}$ and variance $\sigma_{n,r}^2$ (see (4) and (5)). For each n, define

$$Y_n(t, \omega) = \frac{1}{\sigma_n \sqrt{Z_n}} \left(\sum_{j=1}^{[Z_n t]} \xi_{nj} - m_{n,r} [Z_n t] \right), \quad t \in [0, 1], \tag{2}$$

where $[x]$ is the largest integer that is less than x. Our main result is a weak limit theorem for GWVE, which is an extension of Donsker's theorem.

Theorem 2. *Suppose that $m_n \to \infty$ and $P(V = 0) = 0$; then $Y_n \xrightarrow{d} B$, where B is the standard Brown motion on* $[0, 1]$.

Let D be the space of functions defined on $[0, 1]$ and having discontinuities of at most the first kind. For any $\alpha \in R$, define $A_\alpha = \{x \in D : x(1) \leq \alpha\}$; it turns out that $W(\partial A_\alpha) = 0$, where W is the Wiener measure on D. Note that $Z_{n+r} = \sum_{j=1}^{Z_n} \xi_{nj}$; by Theorem 2 one has the following.

Corollary 3 (CLT). *Suppose that $m_n \to \infty$ and $P(V = 0) = 0$; then for any fixed r,*

$$\frac{1}{\sqrt{Z_n \sigma_{n,r}^2}} \left(Z_{n+r} - m_{n,r} Z_n\right) \xrightarrow{d} N(0,1), \quad (3)$$

where $N(0,1)$ is the standard normal random variable.

So, Theorem 2 is an extension of the central limit theorem for classical Galton-Watson process (see [9, 10]).

2. Auxiliary Results

Let us begin with a result of ξ_{nj}.

Proposition 4. *$\{\xi_{nj}, j \geq 1\}$ are independent and identically distributed with*

$$m_{n,r} = E\left(\xi_{nj}\right) = \mu_n \mu_{n+1} \cdots \mu_{n+r-1}, \quad (4)$$

$$\sigma_{n,r}^2 = \operatorname{Var}\left(\xi_{nj}\right) = \left(m_{n,r}\right)^2 \sum_{j=n}^{n+r-1} \frac{\sigma_j^2}{\mu_j^2 m_{n,j-n}}. \quad (5)$$

Proof. According to the definition of definition of GWVE, $\{\xi_{nj}, j \geq 1\}$ are independent and identically distributed.

Denote the generating functions of X_{n1} and $\xi_{n,1}$ by $\phi_n(s)$ and $g_{n,r}(s)$, respectively; then it can be proved that

$$g_{n,r}(s) = \phi_n\left(\phi_{n+1}\left(\cdots \phi_{n+r-1}(s)\cdots\right)\right). \quad (6)$$

Therefore,

$$m_{n,r} = E\left(\xi_{n1}\right) = g_{n,r}'(1) = \prod_{j=n}^{n+r-1} \phi_j'(1) = \prod_{j=n}^{n+r-1} \mu_j. \quad (7)$$

So (4) is proved. In addition, the first and second derivatives of $g_{n,r}(s)$ are as follows:

$$\begin{aligned} g_{n,r}'(s) &= g_{n,r-1}'\left(\phi_{n+r-1}(s)\right) \phi_{n+r-1}'(s), \\ g_{n,r}''(s) &= g_{n,r-1}''\left(\phi_{n+r-1}(s)\right)\left(\phi_{n+r-1}'(s)\right)^2 \\ &\quad + g_{n,r-1}'\left(\phi_{n+r-1}(s)\right) \phi_{n+r-1}''(s). \end{aligned} \quad (8)$$

By (8) one has

$$\begin{aligned} &\operatorname{Var}\left(X_{n,r}^{(1)}\right) \\ &\quad = \left(\operatorname{Var}\left(X_{n,r-1}^{(1)}\right) - m_{n,r-1} + m_{n,r-1}^2\right) \mu_{n+r-1}^2 \\ &\qquad + m_{n,r-1}\left(\sigma_{n+r-1}^2 - \mu_{n+r-1} + \mu_{n+r-1}^2\right) + m_{n,r} - m_{n,r}^2 \\ &\quad = \operatorname{Var}\left(X_{n,r-1}^{(1)}\right) \mu_{n+r-1}^2 + \sigma_{n+r-1}^2 m_{n,r-1}. \end{aligned} \quad (9)$$

Thus,

$$\frac{\operatorname{Var}\left(X_{n,r}^{(1)}\right)}{m_{n,r}^2} = \frac{\operatorname{Var}\left(X_{n,r-1}^{(1)}\right)}{m_{n,r-1}^2} + \frac{\sigma_{n+r-1}^2}{\mu_{n+r-1}^2 m_{n,r-1}}. \quad (10)$$

Since $m_{n,1} = \mu_n$, $\sigma_{n,1}^2 = \sigma_n^2$, $\xi_{n1} = X_{n,r}^{(1)}$, we complete the proof of (5) by (10). □

For any n, define

$$\begin{aligned} \eta_{nj} &= \frac{\xi_{nj} - m_{n,r}}{\sigma_{n,r}}, \\ X_n(t,\omega) &= \frac{1}{\sqrt{m_n}} \sum_{j=1}^{[m_n t]} \eta_{nj}, \quad t \in [0,1]. \end{aligned} \quad (11)$$

The proof of Theorem 2 depends on the following proposition.

Proposition 5. *$X_n \xrightarrow{d} B$, where B is standard Brown motion on $[0,1]$.*

Proof. It lose no generality if we assume that $\{m_n\}$ are integers. The proof is divided into two steps. We first show that the finite-dimensional distributions of the X_n are convergent to those of B. Consider first a single time point s. We must prove that $X_n(s,\cdot) \xrightarrow{d} W_s$.

Since $\{\eta_{nj}, j \geq 1\}$ have the same distribution, we can set

$$\varphi_n(t) = E\left(\exp\left(it\eta_{nj}\right)\right). \quad (12)$$

Note $E(\eta_{nj}) \equiv 0$ and $\operatorname{Var}(\eta_{nj}) \equiv 1$, according to (3.8) of [11] P101; one obtains

$$\begin{aligned} \varphi_n(s) &= \varphi_n(0) + \varphi_n'(0) s + \frac{\varphi_n''(0)}{2!} s^2 + o\left(s^2\right) \\ &= 1 - \frac{s^2}{2} + o\left(\frac{s^2}{2}\right), \quad (s \longrightarrow 0). \end{aligned} \quad (13)$$

For any fixed t and k large enough,

$$\varphi_n\left(\frac{t}{\sqrt{k}}\right) = 1 - \frac{t^2}{2k} + o\left(\frac{1}{2k}\right). \quad (14)$$

Since $m_n \to \infty$, for n large enough, we have

$$\begin{aligned} E\exp\left(itX_n(s)\right) &= E\left(\exp\left(it\frac{1}{\sqrt{m_n}} \sum_{j=1}^{[m_n s]} \eta_{nj}\right)\right) \\ &= \left[1 - \frac{t^2}{2m_n} + o\left(\frac{1}{2m_n}\right)\right]^{[m_n s]} \\ &\longrightarrow \exp\left\{-\frac{st^2}{2}\right\} \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (15)$$

This means that the characteristic function of $X_n(s)$ is convergent to that of B_s; by Lévy continuous theorem we complete the proof of single point case.

Consider now two time points s and t with $s < t$; we are to prove

$$\left(X_n(s), X_n(t)\right) \xrightarrow{d} \left(B_s, B_t\right). \quad (16)$$

Note that

$$\left(X_n(s), X_n(t)\right) = \left(X_n(s), X_n(t) - X_n(s)\right) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (17)$$

By Corollary 1 to Theorem 5.1 in [12], it is only needed to prove

$$(X_n(s), X_n(t) - X_n(s)) \xrightarrow{d} (B_s, B_t - B_s). \tag{18}$$

Since the components on the left are independent by the independence of the $\{\xi_{ni}, i \geq 1\}$. Equation (16) follows from the case of one time point and Theorem 3.2 of [12].

A set of three or more time points can be treated in the same way, and hence the finite-dimensional distributions converge properly.

In the next step, we will show that $\{X_n\}$ is tight. According to Theorem 15.6 of [12], it is enough to establish the inequality

$$\begin{aligned}\Delta_n &:= E\left\{|X_n(t) - X_n(t_1)|^2 |X_n(t_2) - X_n(t)|^2\right\} \\ &\leq 4(t_2 - t_1)^2, \quad \forall\, 0 \leq t_1 \leq t \leq t_2 \leq 1.\end{aligned} \tag{19}$$

Since $\{\eta_{nj}, j \geq 1\}$ are i.i.d. with $E(\eta_{nj}) \equiv 0$ and $\mathrm{Var}(\eta_{nj}) \equiv 1$; by the definition of X_n, we have

$$\begin{aligned}\Delta_n &= E\left\{\left|\frac{1}{\sqrt{m_n}} \sum_{j=[m_n t_1+1]}^{[m_n t]} \eta_{nj}\right|^2 \cdot \left|\frac{1}{\sqrt{m_n}} \sum_{j=[m_n t+1]}^{[m_n t_2]} \eta_{nj}\right|^2\right\} \\ &= \frac{([m_n t] - [m_n t_1]) \cdot ([m_n t_2] - [m_n t])}{m_n^2} \\ &\leq \frac{([m_n t_2] - [m_n t_1])^2}{m_n^2}.\end{aligned} \tag{20}$$

If $t_2 - t_1 \geq 1/m_n$, then there exist $k_1 < k_2$ such that

$$t_1 \in \left[\frac{k_1}{m_n}, \frac{k_1+1}{m_n}\right), \qquad t_2 \in \left[\frac{k_2}{m_n}, \frac{k_2+1}{m_n}\right). \tag{21}$$

Hence,

$$\begin{aligned}\frac{[m_n t_2] - [m_n t_1]}{m_n} &= \frac{k_2 - k_1}{m_n} = \frac{k_2 - (k_1+1)}{m_n} + \frac{1}{m_n} \\ &\leq (t_2 - t_1) + (t_2 - t_1).\end{aligned} \tag{22}$$

So (19) is true when $t_2 - t_1 \geq 1/m_n$. Next, if $t_2 - t_1 < 1/m_n$, then either t_1 and t lie in the same subinterval $[k/m_n, (k+1)/m_n)$ or else t and t_2 do. In either of these cases $\Delta_n = 0$ by (20). This establishes (19) in general and proves the proposition. □

3. The Proof of Theorem 2

We are now ready to prove Theorem 2.

Proof. Note that for each n,

$$\begin{aligned}Y_n(t, \omega) &= \frac{1}{\sigma_{n,r}\sqrt{Z_n}} \left(\sum_{j=1}^{[Z_n t]} \xi_{nj} - \mu_n [Z_n t]\right) \\ &= \frac{1}{\sqrt{Z_n}} \sum_{j=1}^{[Z_n t]} \eta_{nj}, \quad t \in [0,1].\end{aligned} \tag{23}$$

We assume at first that V is bounded, so that there exists a constant k such that $0 < V \leq k$ with probability 1. We may adjust the m_n so that they are integer and so that $k < 1$.

If we define

$$|\Phi_n(t) - tV| = \begin{cases} t\dfrac{Z_n}{m_n}, & \text{if } \dfrac{Z_n}{m_n} \leq 1; \\ tV & \text{otherwise.}\end{cases} \tag{24}$$

Since

$$|\Phi_n(t) - tV| \leq \left|\frac{Z_n}{m_n} - V\right| \xrightarrow{\text{a.e.}} 0, \tag{25}$$

Φ_n converges in probability in the sense of the Skorohod topology to the elements $\Phi(t) = Vt$ of D_0, where D_0 consists of those elements φ of D that are nondecreasing and satisfy $0 \leq \varphi(t) \leq 1$ for all t. Define

$$X_n'(t, \omega) = \begin{cases} \dfrac{1}{\sqrt{m_n}} \displaystyle\sum_{l_n \leq i \leq m_n t} \eta_{ni}(\omega), & \text{if } t \geq \dfrac{l_n}{m_n}; \\ 0, & \text{otherwise,}\end{cases} \tag{26}$$

where $\{l_n, n \geq 0\}$ is a sequence of nonnegative integers going to infinity slowly enough that $l_n/\sqrt{m_n} \to 0$ as $n \to +\infty$. Define $\delta_n = \sup_t |X_n(t) - X_n'(t)|$; then

$$\delta_n \leq \frac{1}{\delta\sqrt{m_n}} \sum_{i=1}^{l_n} |\eta_{ni}|. \tag{27}$$

By Minkowski's inequality and the fact that $l_n/\sqrt{m_n} \to 0$, one has

$$E^{1/2}\{\delta_n^2\} \leq \frac{1}{\delta\sqrt{m_n}} \sum_{i=1}^{l_n} E^{1/2}\{\eta_{ni}^2\} = \frac{l_n}{\delta\sqrt{m_n}} \longrightarrow 0. \tag{28}$$

So that by Chebyshev's inequality $\delta_n \xrightarrow{P} 0$. By Proposition 4, $X_n \xrightarrow{d} B$. Since $d(X_n, X_n') \leq \delta_n$, where d is the metric in D which generates the Skorohod topology, it follows by Theorem 4.1 of [12] that $X_n' \xrightarrow{d} B$. So, if A is a W-continuity set in D, we have

$$P\{X_n' \in A\} \longrightarrow W(A). \tag{29}$$

Let $\mathscr{B}_0$ be the field of cylinders sets; that is, $\mathscr{B}_0$ consists of the form

$$\{\omega; (\xi_{ij}(w), 0 \leq i \leq m, 1 \leq j \leq k) \in H\} \tag{30}$$

with $H \in \mathscr{B}(R^{mk})$, the Borel σ-field of R^{mk}.

If $E \in \mathscr{B}_0$, since $l_n \to \infty$ and $\{X_{ni}, n \geq 0, i \geq 1\}$ are independent, then for large n,

$$P(\{X_n' \in A\} \cap E) = P\{X_n' \in A\} P(E). \tag{31}$$

It follows by (29) that

$$P(\{X_n' \in A\} \cap E) \longrightarrow W(A) P(E). \tag{32}$$

Since $(\Phi_n, Z_n/m_n) \xrightarrow{P} (\Phi, V)$ in the sense of the product topology on $D_0 \times R$ and every X'_n is $\sigma(\mathscr{B}_0)$ measurable, it follows by Theorem 4.5 of [12] that

$$\left(X'_n, \Phi_n, \frac{Z_n}{m_n}\right) \xrightarrow{d} (B, \Phi_0, V_0) \tag{33}$$

is relative to the product topology in $D \times D_0 \times R^1$, where V_0 is independent of B and has the same distribution as V, $\Phi_0(t) = V_0 t$. By the fact that $\delta_n \xrightarrow{P} 0$,

$$\left(X_n, \Phi_n, \frac{Z_n}{m_n}\right) \xrightarrow{d} (B, \Phi_0, V_0). \tag{34}$$

Now the mapping that carries the point (x, ϕ, α) to $\alpha^{-1/2}(x \circ \phi)$ is continuous at that point $x \in C$, $\phi \in C \cap D_0$ and $\alpha > 0$. By Corollary 1 to Theorem 5.1 of [12],

$$\left(\frac{Z_n}{m_n}\right)^{-1/2} (X_n \circ \Phi_n) \xrightarrow{d} (V_0)^{-1/2} (B \circ \Phi_0). \tag{35}$$

Since V_0 and B are independent, $(V_0)^{-1/2}(B \circ \Phi_0)$ has the same distribution as B. Moreover $(Z_n/m_n)^{-1/2}(X_n \circ \Phi_n)$ coincides with Y_n if $Z_n/m_n < 1$, the probability of which goes to 1 since $k < 1$. Thus $Y_n \xrightarrow{d} B$ if V is bounded.

Suppose V is not bounded. For $u > 0$, define

$$V_u = \begin{cases} V, & \text{if } V \le u; \\ u, & \text{if } V > u, \end{cases}$$
$$Z_{un} = \begin{cases} Z_n, & \text{if } V \le u; \\ m_n u, & \text{if } V > u. \end{cases} \tag{36}$$

Then for each u, $Z_{un}/m_n \xrightarrow{P} V_u$ and by the case already treated if

$$Y_{un}(t, \omega) = \frac{1}{\sigma_{n,r}\sqrt{Z_{un}}} \left(\sum_{j=1}^{[Z_{un}t]} \xi_{nj} - \mu_n [Z_{un}t] \right) \tag{37}$$

then $Y_{un} \xrightarrow{d} B$. Since $P(Y_{un} \ne Y_n) \le P(V > u)$, $Y_n \xrightarrow{d} B$ follows Theorem 4.2 of [12]. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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