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Research Article

Multiplicity of Solutions for Neumann Problems for Semilinear Elliptic Equations

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Using the minimax methods in critical point theory, we study the multiplicity of solutions for a class of Neumann problems in the case near resonance. The results improve and generalize some of the corresponding existing results.

1. Introduction

The aim of this paper is to study the following semilinear Neumann problem:

$$\begin{gathered} -\Delta u+\beta(x)u=\lambda u+f(x,u), \quad x\in\Omega, \\ \frac{\partial u}{\partial n}=0, \quad x\in\partial\Omega. \end{gathered} \tag{1}$$

Here $\Omega \subset R^N$ ($N \geq 3$) is a bounded domain with a C^1 boundary $\partial\Omega$ and $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. Also $\beta \in L^s(\Omega)$ with $s > N/2$ and it may change sign. The reaction $f(x,u)$ is a Carathéodory function and satisfies the following assumptions:

(f_0) for every $M > 0$, there exists a function $L_M \in L^2(\Omega)$ such that $|f(x,t)| \leq L_M(x)$ for all $|t| \leq M$ and a.e. $x \in \Omega$;

(f_∞) $\lim_{|t|\longrightarrow\infty}(f(x,t)/t) = 0$ uniformly for $x \in \Omega$.

Recently, there have been many papers concerned with the Neumann problems; see [1–5] and the references therein. Or, more specifically, in Li [1] and Qian [2], the left hand side differential operator is $-\Delta u + \beta u$, with $\beta \in R$, $\beta > 0$. In Motreanu et al. [3], Tang and Wu [4], and Motreanu et al. [5], the differential operator is $-\Delta u$ (i.e., $\beta = 0$). Semilinear Neumann problems with unbounded and indefinite potential, especially, were studied by Gasiński and Papageorgiou [6]. They obtained two multiplicity theorems. In addition, the same problems were studied by Papageorgiou and Rădulescu [7]. They dealt with equations in which the reaction $f(x,u)$ exhibits an asymmetric behavior at $+\infty$ and at $-\infty$ (jumping nonlinearity) and they proved multiplicity theorems providing sign information for all the solutions.

On the other hand, for the perturbed problem, Mawhin and Schmitt [8] first considered the two-point boundary value problem

$$-u''-\lambda u=f(x,u)+h(x), \quad u(0)=u(\pi)=0. \tag{2}$$

Under the assumption that f is bounded and satisfies a sign condition, if the parameter λ is sufficiently close to λ_1 from left, problem (2) has at least three solutions; if $\lambda_1 \leq \lambda < \lambda_2$, problem (2) has at least one solution, where λ_1, λ_2 are the first and second eigenvalues of the corresponding linear problem. Ma et al. [9] considered the boundary value problem $\Delta u + \lambda u + f(x,u) = h(x)$ defined on a bounded open set $\Omega \subset R^N$, no matter whether the boundary conditions are Dirichlet or Neumann condition; as the parameter λ approaches λ_1 from left, there exist three solutions. Moreover, existence of three solutions was obtained for the quasilinear problem in bounded domains as the parameter λ approaches λ_1 from left. In [10, 11], these results were extended to the perturbed p-Laplacian equation in R^N. In [12], Ou and Tang extended above some results to some elliptic systems with the Dirichlet boundary conditions. de Paiva and Massa in [13], especially, studied the semilinear elliptic boundary value problem in any spatial dimension and using variational techniques; they showed that a suitable perturbation will turn the almost resonant situation (λ near to λ_k, i.e., near resonance with a nonprincipal eigenvalue) in a situation where the solutions are at least two. In [14], those results were extended to the degenerate elliptic equations in the bounded domain.

Motivated by the above idea, we have the goal in this paper of extending these results in [6, 12–14] to some elliptic equations with the Neumann boundary conditions. Here, it is worth pointing out that (f_∞) is weaker than (f_1) in [13] (or (A) in [14]). More to the point, there are functions satisfying the assumptions of our main results in Section 2 and not satisfying the assumptions in [13, 14]. For example, let $f(x,t) = t/\ln(1+|t|)$. Then f satisfies the assumptions for our Theorem 5 in Section 2 and does not satisfy (f_1) in [13] (or (A) in [14]).

The rest of our paper is organized as follows. In Section 2 we give some preliminary lemmas and our main results. Section 3 gives the detailed proofs of our main results based on several estimates, whose proof will be presented in Sections 4 and 5.

2. Preliminaries and Main Results

Let the Sobolev space $X = H^1(\Omega)$. Denote

$$\|u\|_{1,2} = \left(\int_\Omega \left(|\nabla u|^2 + u^2\right) dx\right)^{1/2} \tag{3}$$

to be the norm of u in X and $\|u\|_p$ the norm of u in $L^p(\Omega)$. The space X is a Hilbert space. For a discussion about the space setting, we refer to [15] and the references therein.

Again, we recall the properties of the eigenvalue problem as follows (see [6]):

$$\begin{aligned} -\Delta u + \beta(x) u &= \lambda u, \quad x \in \Omega, \\ \frac{\partial u}{\partial n} &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{4}$$

The eigenvalue problems in (4) have a sequence $\{\lambda_k\}_{k\geq 1}$ of eigenvalues, such that

$$\begin{aligned} -\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots &\longrightarrow +\infty \\ \text{as } k &\longrightarrow +\infty. \end{aligned} \tag{5}$$

There is also a corresponding sequence $\{\phi_k\}_{k\geq 1} \subset H^1(\Omega)$ of eigenfunctions which form an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$. Moreover, we know that $\phi_k \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ and all $k \geq 1$.

We denote by E_k the eigenspace corresponding to an eigenvalue λ_k, and we can decompose $H^1(\Omega) = \bigoplus_{k=1}^\infty E_k$. We set

$$\sigma(u) = \int_\Omega \left(|\nabla u|^2 + \beta(x) u^2\right) dx. \tag{6}$$

The eigenvalues admit the following variational characterizations in terms of the Rayleigh quotient $\sigma(u)/\|u\|_2^2$ for all $u \in H^1(\Omega)$:

$$\lambda_1 = \inf\left\{\frac{\sigma(u)}{\|u\|_2^2} : u \in H^1(\Omega) \setminus \{0\}\right\}, \tag{7}$$

$$\begin{aligned} \lambda_k &= \inf\left\{\frac{\sigma(u)}{\|u\|_2^2} : u \in \bigoplus_{i=k}^{\infty} E_i,\ u \neq 0\right\} \\ &= \sup\left\{\frac{\sigma(u)}{\|u\|_2^2} : u \in \bigoplus_{i=1}^{k} E_i,\ u \neq 0\right\}. \end{aligned} \tag{8}$$

In (7), the infimum is realized on E_1. Also, in (8), both the infimum and the supremum are realized on E_k. All the eigenspaces have the so-called unique continuation property. The first eigenvalue λ_1 is simple and it is clear from (7) that the corresponding eigenfunctions do not change sign. Namely, we can suppose that ϕ_1 is strictly positive on Ω. We mention that all the other eigenvalues have nodal eigenfunctions. For more properties to the eigenvalue problem (4), see [6, 7].

By the presence of function β, weak solutions of (1) must be found in a suitable space. To this purpose, letting $\theta > \max\{-\lambda_1, 0\}$, we introduce a new inner product on $H^1(\Omega)$ by

$$\langle u, v\rangle = \int_\Omega \left(\nabla u \nabla v + \beta(x) uv + \theta uv\right) dx \tag{9}$$

for $u, v \in H^1(\Omega)$ and the associated norm

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 + \beta(x) u^2 + \theta u^2\right) dx\right)^{1/2}, \quad u \in H^1(\Omega). \tag{10}$$

Lemma 1. *Let $\beta \in L^s(\Omega)$ with $s > N/2$ and it may change sign, if $\theta > \max\{-\lambda_1, 0\}$; then $\|\cdot\|$ is equivalent to the usual Sobolev norm $\|\cdot\|_{1,2}$.*

Proof. By virtue of Hölder's inequality, we have

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^2 + \beta(x) u^2 + \theta u^2\right) dx \\ &\quad\leq \int_\Omega \left(|\nabla u|^2 + \theta u^2\right) dx + \|\beta\|_s \|u\|_{2s'}^2 \\ &\quad\leq \max\{1, \theta\} \|u\|_{1,2}^2 + \|\beta\|_s \|u\|_{2s'}^2 \\ &\quad\leq \max\{1, \theta\} \|u\|_{1,2}^2 + K^2 \|\beta\|_s \|u\|_{1,2}^2 \\ &\quad\leq \xi_2 \|u\|_{1,2}^2, \end{aligned} \tag{11}$$

where K is the constant of Sobolev imbedding from $H^1(\Omega) \hookrightarrow L^{2s'}(\Omega)$, $s' = s/(s-1)$, $\xi_2 = \max\{1, \theta, K^2\|\beta\|_s\}$.

On the other hand, if $\theta > \max\{-\lambda_1, 0\}$, then there exists $\xi_1 > 0$ such that

$$\int_\Omega \left(|\nabla u|^2 + \beta(x) u^2 + \theta u^2\right) dx \geq \xi_1 \|u\|_{1,2}^2. \tag{12}$$

If not, it is clear from (7) that

$$\chi(u) = \int_\Omega \left(|\nabla u|^2 + \beta(x) u^2 + \theta u^2\right) dx \geq 0. \tag{13}$$

Exploiting the 2-homogeneity of χ we can find a sequence $\{u_n\}_{n\geq 1} \subset H^1(\Omega)$, such that

$$\|u_n\|_{1,2} = 1, \quad \forall n \geq 1, \quad \chi(u_n) \longrightarrow 0^+. \tag{14}$$

Passing to a subsequence if necessary, we may assume that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\Omega), \tag{15}$$

$$u_n \longrightarrow u \quad \text{strongly in } L^{2s'}(\Omega). \tag{16}$$

The sequential weak lower semicontinuity of σ and (15) imply that

$$\sigma(u) \leq -\int_\Omega \theta u^2 dx \leq \lambda_1 \|u\|_2^2. \tag{17}$$

So $\sigma(u) = \lambda_1 \|u\|_2^2$ and thus $u = \gamma\phi_1$, with some $\gamma \in R$.

If $\gamma = 0$, then $u_n \longrightarrow 0$ in $H^1(\Omega)$, which contradicts the fact that $\|u_n\|_{1,2} = 1$ for all $n \geq 1$. If $\gamma \neq 0$, then, from (17), we have $\sigma(\phi_1) < \lambda_1 \|\phi_1\|_2^2$, which contradicts (7).

Combining (11) and (12), we have

$$\xi_1 \|u\|_{1,2}^2 \leq \int_\Omega \left(|\nabla u|^2 + \beta(x) u^2 + \theta u^2\right) dx \leq \xi_2 \|u\|_{1,2}^2. \tag{18}$$

Namely, $\|\cdot\|$ is equivalent to the usual Sobolev norm $\|\cdot\|_{1,2}$. □

From now on we take $(H^1(\Omega), \langle\cdot\rangle, \|\cdot\|)$ as our working space, $\theta > \max\{-\lambda_1, 0\}$, $\|\phi_1\| = 1$. In view of Lemma 1 and the Rellich-Kondrachov Compactness theorem (see [15, Theorem 1]), we directly get the following lemma.

Lemma 2. *Let $\beta \in L^s(\Omega)$ with $s > N/2$ and it may change sign. Then the space $H^1(\Omega)$ is compactly embedded in $L^p(\Omega)$ for $1 \leq p < 2^*$ and continuously embedded in $L^{2^*}(\Omega)$; hence there exists $S > 0$ such that*

$$\|u\|_p \leq S\|u\|, \quad \forall u \in H^1(\Omega). \tag{19}$$

In addition, we also need the following lemmas.

Lemma 3 (from Lemma 4.6 of [13]). *Let X be a Hilbert space with orthonormal direct sum splitting $X = V \oplus Z \oplus W$. Moreover, let $\dim(V \oplus Z) < \infty$. For $\rho > R > 0$, set*

$$\begin{gathered} A = \{u \in W : \|u\| \geq R\} \cup \{u \in Z \oplus W : \|u\| = R\}, \\ B = \{u \in V \oplus Z : \|u\| = \rho\}. \end{gathered} \tag{20}$$

Then A and B link.

Lemma 4 (from Theorem 8.1 of [16]). *Let $X = X_1 \oplus X_2$ be a Hilbert space where X_1 has finite dimension and $J \in C^1(X, R)$ satisfying the (P.S.) condition and such that, for given $\rho_1, \rho_2 > 0$,*

$$\sup_{u\in\rho_1 S_1} J(u) < a = \inf_{u\in\rho_2 B_2} J(u) \leq b = \sup_{u\in\rho_1 B_1} J(u) < \inf_{u\in\rho_2 S_2} J(u), \tag{21}$$

where B_i and S_i represent the unit ball and the unit sphere in X_i : $i = 1, 2$. Then there exists a critical point u_0 such that $J(u_0) \in [a, b]$.

Next, let $F(x,t) = \int_0^t f(x,s)ds$; in order to state our main results, we introduce the following assumptions on the nonlinear term:

(f_1) $\lim_{|t|\longrightarrow\infty}(f(x,t)t/|t|) = +\infty$ uniformly with respect to $x \in \Omega$;

(f_2) $\lim_{|t|\longrightarrow\infty} F(x,t) = +\infty$ uniformly with respect to $x \in \Omega$;

(f_3) $\lim_{|t|\longrightarrow\infty}(f(x,t)t/|t|) = -\infty$ uniformly with respect to $x \in \Omega$;

(f_4) $\lim_{|t|\longrightarrow\infty} F(x,t) = -\infty$ uniformly with respect to $x \in \Omega$.

Our main results are given by the following theorems.

Theorem 5. *Let λ_k $(k \geq 2)$ be an eigenvalue of multiplicity m and $\beta \in L^s(\Omega)$ with $s > N/2$ and it may change sign. Suppose that the conditions (f_0) and (f_∞) hold and one of the sets of hypotheses (f_1) and (f_2). Then there exists $\delta_0 > 0$ such that for $\lambda \in (\lambda_k - \delta_0, \lambda_k)$ problem (1) has at least two solutions.*

Theorem 6. *Let λ_k $(k \geq 2)$ be an eigenvalue of multiplicity m and $\beta \in L^s(\Omega)$ with $s > N/2$ and it may change sign. Suppose that the conditions (f_0) and (f_∞) hold and one of the sets of hypotheses (f_3) and (f_4). Then there exists $\delta_1 > 0$ such that for $\lambda \in (\lambda_k, \lambda_k + \delta_1)$ problem (1) has at least two solutions.*

Theorem 7. *Let $\beta \in L^s(\Omega)$ with $s > N/2$ and it may change sign. Suppose that the conditions (f_0) and (f_∞) hold, and the nonlinearity f satisfies (f_2). Then, for $\lambda < \lambda_1$ sufficiently close to λ_1, problem (1) has at least three solutions.*

3. Proof of Theorems

The associated functional of problem (1) is

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \beta(x) u^2\right) dx - \frac{\lambda}{2}\int_\Omega u^2 dx \\ &\quad - \int_\Omega F(x,u)\,dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\lambda+\theta}{2}\int_\Omega u^2 dx - \int_\Omega F(x,u)\,dx \end{aligned} \tag{22}$$

for $u \in H^1(\Omega)$. Under the conditions (f_0) and (f_∞), it is easy to verify that, for every $\lambda \in R$, $J \in C^1(H^1(\Omega), R)$ and

$$\begin{aligned}\left\langle J'(u), v\right\rangle = &\int_\Omega (\nabla u \nabla v + \beta(x) uv)\, dx - \lambda \int_\Omega uv\, dx \\ &- \int_\Omega f(x,u) v\, dx,\end{aligned} \tag{23}$$

for $u, v \in H^1(\Omega)$. Moreover, critical points of J are exactly weak solutions of problem (1).

It follows from (f_0) and (f_∞) that for every $\varepsilon > 0$ there exist $\widetilde{M}_\varepsilon > 0$ and $L_{\widetilde{M}_\varepsilon} \in L^2(\Omega)$ such that

$$|f(x,t)| \le \varepsilon |t| + \left|L_{\widetilde{M}_\varepsilon}(x)\right|, \tag{24}$$

for all $t \in R$ and a.e. $x \in \Omega$, which implies that

$$|F(x,t)| \le \frac{\varepsilon}{2}|t|^2 + \left|L_{\widetilde{M}_\varepsilon}(x)\right| |t|, \tag{25}$$

for all $t \in R$ and a.e. $x \in \Omega$. From this and Hölder's inequality, we have

$$\left|\int_\Omega F(x,u)\, dx\right| \le \frac{\varepsilon}{2} S^2 \|u\|^2 + \widetilde{L}_\varepsilon \|u\|, \tag{26}$$

where $\widetilde{L}_\varepsilon = S\|L_{\widetilde{M}_\varepsilon}\|_2$ and S is the best embedding constant.

In addition, we will use several times the estimates below.

From (7) and (8), we have

$$\|u\|^2 \ge (\theta + \lambda_1) \int_\Omega u^2 dx, \quad \forall u \in H^1(\Omega), \tag{27}$$

$$\|u\|^2 \le (\theta + \lambda_k) \int_\Omega u^2 dx, \quad \forall u \in \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_k\}, \tag{28}$$

$$\|u\|^2 \ge (\theta + \lambda_{k+1}) \int_\Omega u^2 dx, \quad \forall u \in (\operatorname{span}\{\phi_1, \phi_2, \dots, \phi_k\})^\perp. \tag{29}$$

Proposition 8. *Assume that (f_0) and (f_∞) hold, and suppose that $\lambda \ne \lambda_k$ for any $k \in N^+$. Then J satisfies the (P.S.) condition.*

Proof. For any sequence $\{u_n\} \subset H^1(\Omega)$ such that

$$|J(u_n)| < \infty \quad \forall n, \qquad J'(u_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{30}$$

we need to prove that $\{u_n\}$ has a convergent subsequence. By the standard argument, it suffices to show that $\{u_n\}$ is bounded in $H^1(\Omega)$. Suppose by contradiction that $\|u_n\|$ as $n \longrightarrow \infty$. Let $w_n = u_n/\|u_n\|$. Then $\|w_n\| = 1$, so we may suppose that there is $w \in H^1(\Omega)$ such that

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{weakly in } H^1(\Omega), \\ w_n &\longrightarrow w \quad \text{strongly in } L^p(\Omega), \text{ where } p \in [1, 2^*), \end{aligned} \tag{31}$$

as $n \longrightarrow \infty$. Then, for any $v \in H^1(\Omega)$, from (24) and (27), we have

$$\begin{aligned}\frac{1}{\|u_n\|}\int_\Omega f(x,u_n) v\, dx &\le \frac{1}{\|u_n\|}\int_\Omega \left(\varepsilon |u_n| + \left|L_{\widetilde{M}_\varepsilon}(x)\right|\right)|v|\, dx \\ &\le \frac{1}{\|u_n\|}\left(\varepsilon \|u_n\|_2 \|v\|_2 + \left\|L_{\widetilde{M}_\varepsilon}\right\|_2 \|v\|_2\right) \\ &\le \frac{\varepsilon}{\sqrt{\theta + \lambda_1}}\|v\|_2 + \frac{\left\|L_{\widetilde{M}_\varepsilon}\right\|_2 \|v\|_2}{\|u_n\|}\end{aligned} \tag{32}$$

which shows that

$$\lim_{n \to \infty} \frac{1}{\|u_n\|}\int_\Omega f(x,u_n) v\, dx \le \frac{\varepsilon}{\sqrt{\theta+\lambda_1}}\|v\|_2. \tag{33}$$

By the arbitrariness of ε, one has

$$\lim_{n \to \infty} \frac{1}{\|u_n\|}\int_\Omega f(x,u_n) v\, dx = 0. \tag{34}$$

Like in the proof of (34), from (26), it follows that

$$\lim_{n \to \infty} \frac{1}{\|u_n\|^2}\int_\Omega F(x,u_n) v\, dx = 0. \tag{35}$$

Thus, for any $v \in H^1(\Omega)$, by (30), (31), and (34), passing to the limit in

$$\begin{aligned}\frac{\left\langle J'(u_n), v\right\rangle}{\|u_n\|} = &\int_\Omega (\nabla w_n \nabla v + \beta(x) w_n v)\, dx \\ &- \lambda \int_\Omega w_n v\, dx - \int_\Omega f(x,u_n) v\, dx\end{aligned} \tag{36}$$

gives

$$\int_\Omega (\nabla w_n \nabla v\, dx + \beta(x) w_n v)\, dx - \lambda \int_\Omega w_n v\, dx = 0, \tag{37}$$

which implies that

$$w \in \operatorname{Ker}(-\Delta + \beta - \lambda). \tag{38}$$

In addition, by (30), (31), and (35), passing to the limit in

$$\begin{aligned}\frac{J(u_n)}{\|u_n\|^2} = &\frac{1}{2}\int_\Omega \left(|\nabla w_n|^2 + \beta(x) w_n^2 + \theta w_n^2\right) dx \\ &- \frac{\theta+\lambda}{2}\int_\Omega w_n^2 dx - \frac{1}{\|u_n\|^2}\int_\Omega F(x,u_n)\, dx\end{aligned} \tag{39}$$

gives

$$\frac{1}{2} - \frac{\theta+\lambda}{2}\int_\Omega w^2 dx = 0. \tag{40}$$

If $\theta + \lambda = 0$, then, from (40), we have $1/2 = 0$, which is a contradiction.

If $\theta + \lambda \ne 0$, then, from (40), we have $w \ne 0$. From this and (38) it follows that λ is an eigenvalue of operator $-\Delta + \beta$, a contradiction; the proof is completed. □

Hereafter, we set

$$V = \operatorname{span}\{\phi_1, \dots, \phi_{k-1}\},$$
$$Z = \operatorname{span}\{\phi_k, \dots, \phi_{k+m-1}\} = E_k, \tag{41}$$
$$W = (V \oplus Z)^{\perp},$$

and we define

$$B_V = \{u \in V : \|u\| \le 1\}, \qquad B_{VZ} = \{u \in V \oplus Z : \|u\| \le 1\},$$
$$B_{ZW} = \{u \in Z \oplus W : \|u\| \le 1\}, \tag{42}$$

and S_V, S_{VZ}, and S_{ZW}, respectively, are their relative boundaries.

Theorems 5 and 6 will be a consequence of the geometry in Propositions 9 and 10 stated below, whose proofs will be postponed to Sections 4 and 5.

Proposition 9. *If $\lambda \in (\lambda_{k-1}, \lambda_k)$ and hypotheses (f_0) and (f_∞) are satisfied, then there exist constants D_λ and ρ_λ such that*

$$J(u) \ge D_\lambda, \quad \text{for } u \in Z \oplus W, \tag{43}$$

$$J(u) < D_\lambda, \quad \text{for } u \in \rho_\lambda S_V. \tag{44}$$

Moreover, if one of the sets of hypotheses (f_1) and (f_2) is satisfied, then there exists δ_0 such that for $\lambda \in (\lambda_k - \delta_0, \lambda_k)$ there exist $D_W, D_\lambda \in R$, $\rho_\lambda > R_1 > 0$ such that, in addition to (43) *and* (44),

$$J(u) \ge D_W, \quad \text{for } u \in W, \tag{45}$$

$$J(u) < D_W, \quad \text{for } u \in R_1 S_{VZ}, \tag{46}$$

$$J(u) < D_W, \quad \text{for } u \in V,\ \|z\| \ge R_1. \tag{47}$$

(The values with index λ depend on λ; the others may be fixed uniformly.)

Proposition 10. *If $\lambda \in (\lambda_k, \lambda_{k+m})$ and hypotheses (f_0) and (f_∞) are satisfied, then there exist constants K_λ and β_λ such that*

$$J(z) \ge K_\lambda, \quad \text{for } z \in W, \tag{48}$$

$$J(z) < K_\lambda, \quad \text{for } z \in \beta_\lambda S_{VZ}. \tag{49}$$

Moreover, if one of the sets of hypotheses (f_3) and (f_4) is satisfied, then there exists δ_1 such that for $\lambda \in (\lambda_k, \lambda_k + \delta_1)$ there exist $K_\lambda, K_V, E \in R$, $\beta_\lambda > R_2 > 0$, and $\xi > 0$ such that, in addition to (48) *and* (49),

$$J(u) < K_V, \quad \text{for } u \in V, \tag{50}$$

$$J(u) > K_V, \quad \text{for } u \in R_2 S_{ZW}, \tag{51}$$

$$J(u) > K_V, \quad \text{for } u \in W,\ \|u\| \ge R_2, \tag{52}$$

$$J(u) > E, \quad \text{for } u \in R_2 B_{ZW}, \tag{53}$$

$$J(u) < E, \quad \text{for } u \in \xi S_V. \tag{54}$$

(The values with index λ depend on λ; the others may be fixed uniformly.)

Proof of Theorem 5. Since the functional J satisfies the (P.S.) condition, we can apply two times the saddle point theorem (see, e.g., [17]); let

$$\Gamma_{k-1} = \left\{\gamma \in C^0(\rho_\lambda B_V; E) \text{ s.t. } \gamma|_{\rho_{\lambda S_V}} = \mathrm{id}\right\},$$
$$\Gamma_k = \left\{\gamma \in C^0(R_1 B_{VZ}; E) \text{ s.t. } \gamma|_{R_1 S_{VZ}} = \mathrm{id}\right\}. \tag{55}$$

The first solution, which we denote by u_{k-1} and may be obtained for any $\lambda \in (\lambda_{k-1}, \lambda_k)$ with just hypotheses (f_0) and (f_∞), corresponds to a critical point at the level

$$c_{k-1} = \inf_{\gamma \in \Gamma_{k-1}} \sup_{w \in \rho_{\lambda B_V}} J(\gamma(w)). \tag{56}$$

The criticality of this level is guaranteed by the estimates (43) and (44), since $\rho_\lambda S_V$ and $Z \oplus W$ link; that is, the image of any map in Γ_{k-1} intersects $Z \oplus W$.

The second solution, which we denote by u_k, corresponds to a critical point at the critical level

$$c_k = \inf_{\gamma \in \Gamma_k} \sup_{w \in R_1 B_{VZ}} J(\gamma(w)). \tag{57}$$

Actually, this is a critical level because of the estimates (45) and (46), since $R_1 S_{VZ}$ and W link.

To conclude the proof, we need to show that these two solutions are distinct.

We observe first that by estimate (45) we have that $c_k \ge D_W$, then we observe that we may build a map $\gamma_0 \in \Gamma_{k-1}$ in such a way that its image is the union between the annulus $\{u \in V : \|u\| \in [R_1, \rho_\lambda]\}$ and the image of a $(k-1)$-dimensional ball in $R_1 S_{VZ}$ whose boundary is $R_1 S_V$. By the estimates (46) and (47), we deduce that $\sup_{w \in \rho_\lambda B_V} J(\gamma_0(w)) < D_W$, and as a consequence $c_{k-1} < D_W$, proving that the two solutions are distinct, for being at different critical levels. □

Proof of Theorem 6. Since the functional J satisfies the (P.S.) condition, we can apply the saddle point theorem and Lemma 4.

The first solution, which we denote by w_k and may be obtained for any $\lambda \in (\lambda_k, \lambda_{k+m})$ with just hypotheses (f_0) and (f_∞), is again obtained through the saddle point theorem and corresponds to a critical point at the critical level

$$d_k = \inf_{\gamma \in \Gamma_k} \sup_{w \in \beta_\lambda B_{VZ}} J(\gamma(w)), \tag{58}$$

where now

$$\Gamma_k = \left\{\gamma \in C^0(\beta_\lambda B_{VZ}; E) \text{ s.t. } \gamma|_{\beta_\lambda S_{VZ}} = \mathrm{id}\right\}. \tag{59}$$

The criticality is guaranteed by estimates (48) and (49), since $\beta_\lambda S_{VZ}$ and W link.

The second solution, which we denote by w_{k-1}, comes from Lemma 3, where we set $X_1 = V$ and $X_2 = Z \oplus W$; actually we have the structure

$$\sup_{\xi S_V} J(u) < E = \inf_{R_2 B_{ZW}} J(u) \le \sup_{\xi B_V} J(u) < K_V < \inf_{R_2 S_{ZW}} J(u) \tag{60}$$

and then we have a critical point w_{k-1} at the level $d_{k-1} \le K_V$.

Finally, in order to prove that these two solutions are distinct, we need a sharper estimate for d_k than that given by (49). For this we use Lemma 3 to guarantee that, for any map $\gamma \in \Gamma_k$, since $\beta_\lambda > R_2$, one has that the image of γ either intersects $R_2 S_{ZW}$ or has a point $u \in W$ with $\|u\| \geq R_2$. This implies that

$$\sup_{w \in \beta_\lambda B_{VZ}} J(\gamma(w)) > K_V, \tag{61}$$

by estimates (51) and (52), and then $d_k > K_V$ proving that the two solutions are distinct, for being at different critical levels. □

Proof of Theorem 7. The proof will be divided into four steps.

Step 1. For $\lambda < \lambda_1$, from the definition of λ_1, (26), and (27), we get

$$J(u) \geq \frac{1}{2}\left(\frac{\lambda_1 - \lambda}{\theta + \lambda_1} - \varepsilon S^2\right)\|u\|^2 - \widetilde{L}_\varepsilon \|u\|. \tag{62}$$

Letting $\varepsilon = (\lambda_1 - \lambda)/2S^2(\theta + \lambda_1)$, it follows that J is coercive in $H^1(\Omega)$.

Similarly, from (26) and (29), we obtain

$$\begin{aligned} J(w) &\geq \frac{1}{2}\left(\frac{\lambda_2 - \lambda}{\theta + \lambda_2} - \varepsilon S^2\right)\|w\|^2 - \widetilde{L}_\varepsilon \|w\| \\ &\geq \frac{1}{2}\left(\frac{\lambda_2 - \lambda_1}{\theta + \lambda_2} - \varepsilon S^2\right)\|w\|^2 - \widetilde{L}_\varepsilon \|w\|, \end{aligned} \tag{63}$$

for all $w \in (\operatorname{span}\{\phi_1\})^\perp$. Let $\varepsilon = (\lambda_2 - \lambda_1)/2S^2(\theta + \lambda_2)$; hence J is coercive in $(\operatorname{span}\{\phi_1\})^\perp$ and J is bounded from below on $(\operatorname{span}\{\phi_1\})^\perp$, and, moreover, there is a constant M, independent of λ, such that $\inf_{(\operatorname{span}\{\phi_1\})^\perp} J \geq M$.

Step 2. If $\lambda < \lambda_1$ is sufficiently close to λ_1, we have $t^- < 0 < t^+$ such that $J(t^\pm \phi_1) < M$. In fact, for $\lambda < \lambda_1$, we have

$$J(t\phi_1) = \frac{\lambda_1 - \lambda}{2(\theta + \lambda_1)}\|t\phi_1\|^2 - \int_\Omega F(x, t\phi_1)\, dx. \tag{64}$$

For any fixed $\widetilde{u} \in R$ with $|\widetilde{u}| = 1$, from (f_2), we get $\lim_{|t| \to \infty} F(x, t\widetilde{u}) = +\infty$ uniformly in $x \in \Omega$. From Fatou's lemma and $\phi_1 > 0$ in Ω, it follows that

$$\lim_{t \to +\infty} \int_\Omega F(x, t\phi_1)\, dx \geq \int_\Omega \lim_{t \to +\infty} F(x, t\phi_1)\, dx = +\infty, \tag{65}$$

and hence, taking $t^+ (> 0)$ large enough, we get

$$\int_\Omega F(x, t^+\phi_1)\, dx > -M + 1. \tag{66}$$

For $\lambda_1 - 2(\theta + \lambda_1)/(t^+)^2 < \lambda < \lambda_1$, combining (64) and (66) yields $J(t^+\phi_1) < M$. A similar conclusion holds for some $t^- < 0$.

Step 3. If $\lambda < \lambda_1$, let

$$\begin{aligned} \Sigma_\pm = \{ z \in H^1(\Omega) : z = \pm t\phi_1 + w \text{ with } t > 0, \\ w \in (\operatorname{span}\{\phi_1\})^\perp \}. \end{aligned} \tag{67}$$

The functional J satisfies the $(\text{P.S.})_{c,\Sigma_+}$ and $(\text{P.S.})_{c,\Sigma_-}$ condition for all $c < M$.

In fact, let $\{z_n\} \subset \sum_+$ satisfy $J(z_n) \longrightarrow c < M$ and $J'(z_n) \longrightarrow 0$ as $n \longrightarrow \infty$. From Proposition 8, there is $z \in H^1(\Omega)$ such that $z_n \longrightarrow z$ strongly in $H^1(\Omega)$. If $z \in \partial\Sigma_+ = (\operatorname{span}\{\phi_1\})^\perp$, from the second conclusion of Step 1, we get $J(z_n) \longrightarrow c \geq M$, which is impossible. Hence, $c \in \Sigma_+$ and J satisfies the $(\text{P.S.})_{c,\Sigma_+}$. Similarly we have that $(\text{P.S.})_{c,\Sigma_-}$ holds for all $c < M$.

Step 4. Three solutions are obtained.

If $\lambda < \lambda_1$ is sufficiently close to λ_1, from Steps 1 and 2, we get $-\infty < \inf_{\Sigma_\pm} J < M$, which implies that J is bounded below in Σ_+. Consequently, from Ekeland's variational principle, there exists $\{z_n\} \subset \Sigma_+$ such that $J(z_n) \longrightarrow \inf_{\Sigma_+} J$ and $J'(z_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since J satisfies $(\text{P.S.})_{c,\Sigma_+}$ for all $c < M$, there is $z^+ \in \Sigma_+$ such that $J(z^+) = \inf_{\Sigma_+} J$; that is, the infimum is attained in Σ_+. A similar conclusion holds in Σ_-. So J has two distinct critical points, denoted by z^+, z^-.

Suppose that $J(z^+) \geq J(z^-)$. Letting

$$\psi(z) = J(z + z^+) - J(z^+), \qquad e = z^- - z^+, \tag{68}$$

we have $\psi(0) = 0$, $\psi(e) \leq 0$ and since z^+ is the local minimum of J, there are $r, \rho > 0$ such that $\psi(z) \geq \rho$ for $\|z\| = r$. Since $\psi' = J'$ and ψ satisfies the (P.S.) condition, from the mountain pass theorem, the number

$$c = \inf_{h \in \Gamma} \max_{t \in [0,1]} J(h(t)), \tag{69}$$

where $\Gamma = \{h \in C([0,1], H^1(\Omega)) : h(0) = z^+, h(1) = z^-\}$, is a critical value of J. From the definition of c, we have $c \geq M$ and obtain a third critical point of J. Hence, the proof is completed. □

4. Proof of Estimates

In this section we will prove all the estimates in Propositions 9 and 10.

4.1. Estimates of the Saddle Geometry

Lemma 11. *Under hypotheses (f_0) and (f_∞), one gets the following:*

(i) *for $\lambda \in (\lambda_{k-1}, \lambda_k)$, there exists D_λ satisfying* (43) *and $D_W \in R$ satisfying* (45)*;*

(ii) *for $\lambda \in (\lambda_k, \lambda_{k+m})$ one has the following:*

(a) *there exists $K_\lambda \in R$ satisfying* (48),

(b) *for a given $R_2 > 0$, there exists $E \in R$ satisfying* (53).

Proof. Let $u \in W$; using estimates (26) and (29) we get

$$J(u) \geq \frac{1}{2}\left(\frac{\lambda_{k+m} - \lambda}{\theta + \lambda_{k+m}} - \varepsilon S^2\right)\|u\|^2 - \widetilde{L}_\varepsilon \|u\|. \tag{70}$$

For $\lambda \in (\lambda_{k-1}, \lambda_k)$, letting $\varepsilon < (\lambda_{k+m} - \lambda_k)/S^2(\theta + \lambda_{k+m}) < (\lambda_{k+m} - \lambda)/S^2(\theta + \lambda_{k+m})$, it follows that J is bounded below in W; that is, there exists a D_W as in (45).

For $\lambda \in (\lambda_k, \lambda_{k+m})$, then the same estimate holds but the constant cannot be made independent of λ, giving (48).

In the same way, let $u \in Z \oplus W$ and set $\delta = \lambda_k - \lambda > 0$; we get

$$\begin{aligned} J(u) &\geq \frac{1}{2}\left(\frac{\lambda_k - \lambda}{\theta + \lambda_k} - \varepsilon S^2\right)\|u\|^2 - \widetilde{L}_\varepsilon \|u\| \\ &\geq \frac{1}{2}\left(\frac{\delta}{\theta + \lambda_k} - \varepsilon S^2\right)\|u\|^2 - \widetilde{L}_\varepsilon \|u\|. \end{aligned} \tag{71}$$

Letting $\varepsilon < \delta/S^2(\theta + \lambda_k)$, it follows that J is bounded below in $Z \oplus W$; that is, there exists a D_λ such that for all $u \in Z \oplus W$ we have (43), where again the constant D_λ depends on δ, that is, on λ.

Finally, (71) with $\lambda \in (\lambda_k, \lambda_{k+m})$ implies

$$J(u) \geq \frac{1}{2}\left(\frac{\lambda_k - \lambda_{k+m}}{\theta + \lambda_k} - \varepsilon S^2\right)\|u\|^2 - \widetilde{L}_\varepsilon \|u\|. \tag{72}$$

Then, no matter the value of λ, J is bounded from below in any bounded subset of $Z \oplus W$, giving (53) for a suitable value of E. □

Lemma 12. *Under hypotheses (f_0) and (f_∞), one gets the following:*

(i) *for $\lambda \in (\lambda_{k-1}, \lambda_k)$, given the constant $D_\lambda \in R$, there exists $\rho_\lambda > 0$ satisfying* (44)*;*

(ii) *for $\lambda \in (\lambda_k, \lambda_{k+m})$ one has the following:*

- (a) *there exists $K_V \in R$ satisfying* (50),
- (b) *for a given $K_\lambda \in R$, there exists $\beta_\lambda > 0$ satisfying* (49),
- (c) *for a given $E \in R$, there exists $\xi > 0$ satisfying* (54).

Moreover, given the values R_1, R_2, one may always choose $\rho_\lambda > R_1$, $\beta_\lambda > R_2$ as claimed in Propositions 9 and 10.

Proof. Let $u \in V$, by estimates (26) and (28); we get

$$J(u) \leq \frac{1}{2}\left(\frac{\lambda_{k-1} - \lambda}{\theta + \lambda_{k-1}} + \varepsilon S^2\right)\|u\|^2 + \widetilde{L}_\varepsilon \|u\|. \tag{73}$$

For $\lambda \in (\lambda_{k-1}, \lambda_k)$, letting $\varepsilon < (\lambda - \lambda_{k-1})/S^2(\theta + \lambda_{k-1})$, then one obtains (44) for suitably large $\rho_\lambda > R_1$.

For $\lambda \in (\lambda_k, \lambda_{k+m})$, letting $\varepsilon < (\lambda_k - \lambda_{k-1})/S^2(\theta + \lambda_{k-1})$, one obtains, for suitable K_V and $\xi > 0$, (50) and (54).

Finally, letting $u \in V \oplus Z$ and setting $\delta = \lambda - \lambda_k > 0$, we get

$$\begin{aligned} J(u) &\leq \frac{1}{2}\left(\frac{\lambda_k - \lambda}{\theta + \lambda_k} + \varepsilon S^2\right)\|u\|^2 + \widetilde{L}_\varepsilon \|u\| \\ &\leq \frac{1}{2}\left(-\frac{\delta}{\theta + \lambda_k} + \varepsilon S^2\right)\|u\|^2 + \widetilde{L}_\varepsilon \|u\|. \end{aligned} \tag{74}$$

Letting $\varepsilon < \delta/S^2\lambda_k$, it is clear that (once δ is fixed) this goes to $-\infty$ and then we may find the claimed $\beta_\lambda > R_2$ such that (49) holds.

Observe that K_V and E can be chosen uniformly for $\lambda \in (\lambda_k, \lambda_{k+m})$, while ρ_λ, β_λ will in fact depend on λ. □

4.2. Estimating the Effect of the Nontrivial Perturbation. In this section we will prove the remaining inequalities in Propositions 9 and 10, those which rely on the hypothesis (f_1) or (f_2) or (f_3) or (f_4), which, roughly speaking, say that the perturbation F is nontrivial in such a way that a new solution arises when λ is sufficiently near to the eigenvalue λ_k. The proof is simpler for Theorem 5, since we need to estimate the functional in the compact set S_{VZ}, while for Theorem 6 the same kind of estimate is required in the noncompact set S_{ZW}.

4.2.1. Estimating J in S_{VZ}. For the next estimates, we will need the following lemma.

Lemma 13. *Hypothesis (f_2) implies that there exists a nondecreasing function $D : (0, +\infty) \longrightarrow R$ such that*

$$\lim_{R \longrightarrow +\infty} D(R) = +\infty, \qquad \inf_{u \in RS_{VZ}} \int_\Omega F(x, u)\, dx > D(R). \tag{75}$$

Proof. First we claim that there exists a constant $\eta > 0$ such that the sets $\Omega_u = \{x \in \Omega : |u(x)| > \eta\}$ have measure $|\Omega_u| > \eta$, for all $u \in S_{VZ}$.

Actually, $V \oplus Z$ is a finite-dimensional subspace and the functions $u \in S_{VZ}$ are smooth; they are uniformly bounded; that is, there exists $M > 0$ such that $|u(x)| \leq M$ for all $x \in \Omega$. Suppose that for $\eta_n \longrightarrow 0 (\eta_n < 1)$ there exists $\{u_n\} \subset S_{VZ}$ such that $|\Omega_{u_n}| \leq \eta_n$.

On one hand, by (28), one obtains

$$\frac{1}{\theta + \lambda_k} \leq \int_\Omega u^2 dx. \tag{76}$$

On the other hand,

$$\begin{aligned} \int_\Omega u^2 dx &= \int_{\Omega_{u_n}} |u_n|^2 dx + \int_{\Omega - \Omega_{u_n}} |u_n|^2 dx \\ &\leq M^2 \left|\Omega_{u_n}\right| + \eta_n^2 \left|\Omega - \Omega_{u_n}\right| \\ &\leq \eta_n \left(M^2 + |\Omega|\right) \\ &\longrightarrow 0. \end{aligned} \tag{77}$$

That is a contradiction.

Now, for any $H > 0$, we will show that we can find a $\widetilde{R}$ large enough so that $\int_\Omega F(x, Ru)dx \geq H$ for any $u \in S_{VZ}$ and $R \geq \widetilde{R}$; this means that

$$\lim_{R \longrightarrow \infty} \inf_{u \in RS_{VZ}} \int_\Omega F(x, u)\, dx = +\infty. \tag{78}$$

Actually, it follows from (f_2) that for any $M > 0$ there exists $t_0(> 0)$ such that $F(x,t) > M$ for $|t| > t_0$. For $R > t_0/\eta$, one has $\Omega_u \subseteq \{x \in \Omega : |Ru(x)| > t_0\}$, and then one gets

$$\int_{|Ru|\geq t_0} F(x, Ru)\,dx \geq M\eta. \tag{79}$$

For $R \leq t_0/\eta$, by (f_0) and (f_2), there exist $\widetilde{C} > 0$ and $L_{\widetilde{C}} \in L^2(\Omega)$ such that $F(x,t) \geq -\widetilde{C}(1 + L_{\widetilde{C}}(x))$, for $t \in R$ and a.e. $x \in \Omega$.

Let $M = (H+\widetilde{C}|\Omega|+\widetilde{C}|\Omega|^{1/2}\|L_{\widetilde{C}}\|_2)\eta^{-1}$; one finally obtains

$$\begin{aligned}\int_\Omega F(x,Ru)\,dx &= \int_{|Ru|\geq t_0} F(x,Ru)\,dx + \int_{|Ru|\leq t_0} F(x,Ru)\,dx\\ &\geq M\eta - \widetilde{C}|\Omega| - \widetilde{C}|\Omega|^{1/2}\|L_{\widetilde{C}}\|_2\\ &= H.\end{aligned} \tag{80}$$

It is elementary that

$$D(R) = \inf_{\rho\geq R}\inf_{u\in RS_{VZ}} \int_\Omega F(x,u)\,dx \tag{81}$$

is well defined and satisfies the claim. □

Now we may prove the following.

Lemma 14. *Consider Theorem 5 with one of the sets of hypotheses (f_1) and (f_2). Given the constant $D_W \in R$, there exist R_1, $\delta_0 > 0$ such that, for any $\lambda \in (\lambda_k - \delta_0, \lambda_k)$, (46) and (47) hold.*

Proof. We consider the two sets of hypotheses separately.

(i) In case (f_1), it follows from (f_0) and (f_1) that for any $M > 0$ there exist C_M and $L_M \in L^2(\Omega)$ such that

$$F(x,t) \geq M|t| - C_M(1 + L_M(x)), \tag{82}$$

for $t \in R$ and $x \in \Omega$; in particular we set $M = 1$. Let $u \in RS_{VZ}$; for being in a finite-dimensional subspace, all the norms are equivalent, so that (set $\delta = \lambda_k - \lambda > 0$ and use estimates (28) and (82))

$$\begin{aligned}J(u) &\leq \frac{\lambda_k - \lambda}{2(\theta + \lambda_k)}\|u\|^2 - \|u\| + C_0\\ &\leq \frac{\delta}{2(\theta + \lambda_k)}\|u\|^2 - \|u\| + C_0\\ &\leq \frac{\delta}{2(\theta + \lambda_k)}R^2 - R + C_0,\end{aligned} \tag{83}$$

where $C_0 = C_1|\Omega|^{1/2}\|L_1\|_2 + C_1|\Omega|$.

(ii) In case (f_2), let $D(R)$ be as in Lemma 13, for $\|u\| = R$; let $u = w + \phi$ with $w \in V$ and $\phi \in Z = E_k$:

$$\begin{aligned}J(u) &\leq \frac{\lambda_{k-1} - \lambda}{2(\theta + \lambda_{k-1})}\|w\|^2 + \frac{\lambda_k - \lambda}{2(\theta + \lambda_k)}\|\phi\|^2 - \int_\Omega F(x,u)\,dx\\ &\leq \frac{\lambda_{k-1} - \lambda_k + \delta}{2(\theta + \lambda_{k-1})}\|w\|^2 + \frac{\delta}{2(\theta + \lambda_k)}\|\phi\|^2 - \int_\Omega F(x,u)\,dx\\ &\leq \frac{\delta}{2(\theta + \lambda_k)}\|\phi\|^2 - \frac{\lambda_k - \lambda_{k-1}}{4(\theta + \lambda_{k-1})}\|w\|^2 - \int_\Omega F(x,u)\,dx.\end{aligned} \tag{84}$$

We assume that $\delta \leq (\lambda_k - \lambda_{k-1})/2$; it is easy to see that $(\lambda_{k-1} - \lambda_k)\|w\|^2/4(\theta + \lambda_{k-1}) \leq C$ for some constant C; we estimate

$$\begin{aligned}J(z) &\leq \frac{\delta}{2(\theta + \lambda_k)}\|\phi\|^2 + C - D(R)\\ &\leq \frac{\delta}{2(\theta + \lambda_k)}R^2 - D(R) + C.\end{aligned} \tag{85}$$

Considering (83) and (85), we see that since $\lim_{R\to\infty}D(R) = +\infty$ by Lemma 13, we may fix R_1 so that $C - D(R_1) < D_W - 1$ (or $C_0 - R_1 < D_W - 1$ for the case (f_1)) and then for $0 < \delta < \min\{2(\theta + \lambda_k)/R_1^2, (\lambda_k - \lambda_{k-1})/2\}$ one gets (46).

To obtain (47), we observe that (since $\lambda > \lambda_{k-1}$) if $\phi = 0$, that is, if $u \in V$, then in estimates (83) and (85) we may avoid the term $\delta R^2/2(\theta + \lambda_k)$ so that (remember that $D(R)$ is nondecreasing) $J(u) < D_W - 1$ for $\|u\| > R_1$. □

4.2.2. Estimating J in S_{ZW}. We consider the corresponding estimates of the previous lemma, for Theorem 6.

Lemma 15. *Consider Theorem 6 with one of the sets of hypotheses (f_3) and (f_4). Given the constant $K_V \in R$, there exist R_2, $\delta_1 > 0$ such that, for any $\lambda \in (\lambda_k, \lambda_k + \delta_1)$, (51) and (52) hold.*

Proof. Letting $\lambda = \lambda_k + \delta$, we see from (70) that property (52) will be satisfied provided that R_2 is large enough (say $R_2 > \widetilde{R}$); observe that this value can be made independent from λ once δ is small enough.

Now we consider the two sets of hypotheses separately.

(i) In case (f_3), suppose $u \in E_k \oplus W$; we can assume that $u = w + \phi$, with $w \in W$ and $\phi \in E_k$. Since E_k is a finite dimension subspace, all the norms are equivalent, so that there exists $K > 0$ such that for all $\phi \in E_k$ we have $\|\phi\| \leq K\|\phi\|_1$. In addition, by (f_0) and (f_3), there exist C_2 and $L_2 \in L^2(\Omega)$ such that

$$-F(x,s) \geq K|s| - C_2(1 + L_2(x)), \tag{86}$$

uniformly in $x \in \Omega$. So by (29) and (86),

$$\begin{aligned} J(w+\phi) \geq & \frac{\lambda_{k+m}-(\lambda_k+\delta)}{2(\theta+\lambda_{k+m})}\|w\|^2 - \frac{\delta}{2(\theta+\lambda_k)}\|\phi\|^2 \\ & + K\|w+\phi\|_1 - C_4 \\ \geq & \frac{\lambda_{k+m}-(\lambda_k+\delta)}{2(\theta+\lambda_{k+m})}\|w\|^2 - \frac{\delta}{2(\theta+\lambda_k)}\|\phi\|^2 \\ & + K\|\phi\|_1 - K\|-w\|_1 - C_4 \\ \geq & \frac{\lambda_{k+m}-(\lambda_k+\delta)}{2(\theta+\lambda_{k+m})}\|w\|^2 - \frac{\delta}{2(\theta+\lambda_k)}\|\phi\|^2 \\ & + \|\phi\| - C_3\|w\| - C_4, \end{aligned} \tag{87}$$

where $C_3 = |\Omega|^{1/2}SK$, $C_4 = C_2|\Omega|^{1/2}\|L_2\|_2 + C_2|\Omega|$. Since

$$\begin{aligned} \left(1 - \frac{\delta}{2(\theta+\lambda_k)}\|u\|\right)\|u\| \leq & \|w\| + \|\phi\| \\ & - \frac{\delta}{2(\theta+\lambda_k)}\left(\|\phi\|^2 + \|w\|^2\right) \\ \leq & \left(1 - \frac{\delta}{2(\theta+\lambda_k)}\|\phi\|\right)\|\phi\| \\ & + \|w\|, \end{aligned} \tag{88}$$

supposing $\delta \leq (\lambda_{k+m} - \lambda_k)/2$, (87) becomes

$$\begin{aligned} J(w+\phi) \geq & \frac{\lambda_{k+m}-(\lambda_k+\delta)}{2(\theta+\lambda_{k+m})}\|w\|^2 - C_3\|w\| - C_4 \\ & - \|w\| + \left(1 - \frac{\delta}{2(\theta+\lambda_k)}\|u\|\right)\|u\| \\ \geq & \frac{\lambda_{k+m}-\lambda_k}{4(\theta+\lambda_{k+m})}\|w\|^2 - (C_3+1)\|w\| \\ & - C_4 + \left(1 - \frac{\delta}{2(\theta+\lambda_k)}\|u\|\right)\|u\|, \end{aligned} \tag{89}$$

since $(\lambda_{k+m} - \lambda_k)/4(\theta + \lambda_{k+m}) > 0$, so

$$h(w) = \frac{\lambda_{k+m}-\lambda_k}{4(\theta+\lambda_{k+m})}\|w\|^2 - (C_3+1)\|w\| - C_4 \tag{90}$$

is bounded below for all $w \in W$; that is, there exists $C_5 \in R$ such that $h(w) \geq C_5$; by (89) one gets

$$\begin{aligned} J(u) \geq & \left(1 - \frac{\delta}{2(\theta+\lambda_k)}\|u\|\right)\|u\| + C_5 \\ = & -\frac{\delta}{2(\theta+\lambda_k)}\|u\|^2 + \|u\| + C_5. \end{aligned} \tag{91}$$

(ii) In case (f_4), first we give some conclusions which are similar to Lemma 3 of [18]. Under the property of F, there exists a const C, and $G \in C(R, R)$ which is subadditive, that is,

$$G(s+t) \leq G(s) + G(t) \tag{92}$$

for all $s, t \in R$, and coercive, that is,

$$G(s) \longrightarrow +\infty \tag{93}$$

as $|s| \longrightarrow \infty$, and satisfies that

$$G(s) \leq |s| + 4 \tag{94}$$

for all $s \in R$, such that

$$-F(x,s) \geq G(s) - C \tag{95}$$

for all $s \in R$ and $x \in \overline{\Omega}$.

In fact, since $-F(x, s) \longrightarrow +\infty$ as $|s| \longrightarrow \infty$ uniformly for all $x \in \overline{\Omega}$, there exists a sequence of positive integers n_k with $n_{k+1} > 2n_k$ for all positive integers k such that

$$-F(x,s) \geq k \tag{96}$$

for all $|s| \geq n_k$ and all $x \in \overline{\Omega}$. Let $n_0 = 0$ and define

$$G(s) = k + 2 + \frac{|s| - n_{k-1}}{n_k - n_{k-1}} \tag{97}$$

for $n_{k-1} \leq |s| < n_k$, where $k \in N$.

By the definition of G we have

$$k + 2 \leq G(s) \leq k + 3 \tag{98}$$

for all $n_{k-1} \leq |s| < n_k$. By (f_4) and $F \in C^1(\overline{\Omega} \times R, R)$, there exists $C_F > 0$ such that

$$-F(x,s) \geq -C_F, \quad \forall (x,s) \in \left(\Omega, R^2\right). \tag{99}$$

It follows that

$$-F(x,s) \geq G(s) - C, \tag{100}$$

where $C = C_F + 4$. In fact, when $n_{k-1} \leq |s| < n_k$, for some $k \geq 2$, one has, by (96) and (98),

$$-F(x,s) \geq k - 1 \geq G(s) - 4 \geq G(s) - C \tag{101}$$

for all $x \in \overline{\Omega}$. When $|s| < n_1$, we have, by (98) and (99),

$$-F(x,s) \geq -C_F = 4 - C \geq G(s) - C \tag{102}$$

for all $x \in \overline{\Omega}$.

It is obvious that G is continuous and coercive. Moreover, one has

$$G(s) \leq |s| + 4 \tag{103}$$

for all $s \in R$. In fact, for every $s \in R$, there exists $k \in N$ such that

$$n_{k-1} \leq |s| < n_k, \tag{104}$$

which implies that

$$G(s) \le (k-1) + 4 \le n_{k-1} + 4 \le |s| + 4 \tag{105}$$

for all $s \in R$ by (98) and the fact that $n_k \ge k$ for all integers $k \ge 0$.

Now we only need to prove the subadditivity of G. Let

$$n_{k-1} \le |s| < n_k, \qquad n_{j-1} \le |t| < n_j \tag{106}$$

and $m = \max\{k, j\}$. Then we have

$$|s+t| \le |s| + |t| < n_k + n_j \le 2n_m < n_{m+1}. \tag{107}$$

Hence we obtain, by (98),

$$G(s+t) \le m + 4 \le k + 2 + j + 2 \le G(s) + G(t), \tag{108}$$

which shows that G is subadditive.

For $u \in E_k \oplus W$, we assume that $u = w + \phi$, with $w \in W$ and $\phi \in E_k$; letting $0 < \delta < (\lambda_{k+m} - \lambda_k)/2$, by (28), (92), (94), and (95), one gets

$$\begin{aligned} J(w+\phi) \ge{} & \frac{\lambda_{k+m} - (\lambda_k + \delta)}{2(\theta + \lambda_{k+m})}\|w\|^2 - \frac{\delta}{2(\theta+\lambda_k)}\|\phi\|^2 \\ & + \int_\Omega G(\phi + w)\,dx - C|\Omega| \\ \ge{} & \frac{\lambda_{k+m} - (\lambda_k + \delta)}{2(\theta + \lambda_{k+m})}\|w\|^2 - \frac{\delta}{2(\theta+\lambda_k)}\|\phi\|^2 \\ & + \int_\Omega G(\phi)\,dx \\ & - \int_\Omega G(-w)\,dx - C|\Omega| \\ \ge{} & \frac{\lambda_{k+m} - \lambda_k}{4(\theta + \lambda_{k+m})}\|w\|^2 - \frac{\delta}{2(\theta+\lambda_k)}\|\phi\|^2 \\ & + \int_\Omega G(\phi)\,dx \\ & - \int_\Omega (|w| + 4)\,dx - C|\Omega| \\ \ge{} & \frac{\lambda_{k+m} - \lambda_k}{4(\theta + \lambda_{k+m})}\|w\|^2 - \frac{\delta}{2(\theta+\lambda_k)}\|z\|^2 \\ & + \int_\Omega G(\phi)\,dx - S|\Omega|\,\|w\| - C_1 \\ ={} & g(w) + \int_\Omega G(\phi)\,dx - \frac{\delta}{2(\theta+\lambda_k)}\|u\|^2, \end{aligned} \tag{109}$$

where $g(w) = ((\lambda_{k+m} - \lambda_k)/4(\theta + \lambda_{k+m}))\|w\|^2 - S|\Omega|\|w\| - C_1$, $C_1 = (4 + C)|\Omega|$. Since $\phi \in E_k$, E_k is a finite-dimensional subspace, and G is coercive, from the proof of (75), one can get

$$\lim_{\|\phi\| \to \infty} \int_\Omega G(\phi)\,dx = +\infty. \tag{110}$$

That is, $\int_\Omega G(\phi)dx$ is coercive on E_k. Since $(\lambda_{k+m} - \lambda_k)/4(\theta + \lambda_{k+m}) > 0$, so $g(w)$ is coercive on W, and $\int_\Omega G(\phi)dx$ and $g(w)$ are bounded below, so it is obvious that

$$\lim_{\|u\| \to \infty} \left(g(w) + \int_\Omega G(\phi)\,dx \right) = +\infty \tag{111}$$

for all $u \in Z \oplus W$.

Considering (91), (109), and (111), we can choose R_2 large enough such that for all $\|u\| \ge R_2$ one gets

$$g(w) + \int_\Omega G(\phi)\,dx > K_V + 1 \tag{112}$$

(or $R_2 + C_5 > K_V + 1$ for the case (f_3)) and property (52) holds; then for $0 < \delta < \min\{2(\theta + \lambda_k)/R_2^2, (\lambda_{k+m} - \lambda_k)/2\} = \delta_1$ and $u \in R_2 S_{ZW}$ one gets $J(u) > K_V$; that is, the property (51) holds. □

5. Proof of the Geometry in Propositions 9 and 10

We finally give the proof of Propositions 9 and 10, which is nothing but a resume of the lemmata above, verifying that all the constants can be chosen sequentially without contradictions.

Proof of Proposition 9. Under hypotheses (f_0) and (f_∞), if we fix a value λ, then we obtain the constant D_λ from Lemma 11 and with this we get ρ_λ from Lemma 12. If we consider also one of the two sets of hypotheses (f_1) and (f_2), then we proceed as follows. First of all, we determine (once forever) the constant D_W from Lemma 11; with this we obtain from Lemma 14 the values R_1 and δ_0. Then, for any (now fixed) $\lambda \in (\lambda_k - \delta_0, \lambda_k)$, we obtain from Lemma 11 the value D_λ. Finally, we can get from Lemma 12 the corresponding value of $\rho_\lambda > R_1$. □

Proof of Proposition 10. Under hypotheses (f_0) and (f_∞), if we fix a value $\lambda \in (\lambda_k, \lambda_{k+m})$, then we obtain the constant K_λ from Lemma 11 and with this we get β_λ from Lemma 12. If we consider also one of the two sets of hypotheses (f_3) and (f_4), then we proceed as follows. First of all, we determine (once forever) the constant K_V from Lemma 12; with this we obtain from Lemma 15 the values R_2 and δ_1. Since we have R_2, we can get from Lemma 11 the constant E and with this obtain ξ from Lemma 12.

Finally, for any (now fixed) $\lambda \in (\lambda_k, \lambda_k + \delta_1)$, we obtain from Lemma 11 the constant K_λ and with this we get from Lemma 12 the corresponding value of $\beta_\lambda > R_2$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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