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Research Article

Generalized Reflexive and Generalized Antireflexive Solutions to a System of Matrix Equations

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Two efficient iterative algorithms are presented to solve a system of matrix equations $A_1X_1B_1 + A_2X_2B_2 = E$, $C_1X_1D_1 + C_2X_2D_2 = F$ over generalized reflexive and generalized antireflexive matrices. By the algorithms, the least norm generalized reflexive (antireflexive) solutions and the unique optimal approximation generalized reflexive (antireflexive) solutions to the system can be obtained, too. For any initial value, it is proved that the iterative solutions obtained by the proposed algorithms converge to their true values. The given numerical examples demonstrate that the iterative algorithms are efficient.

1. Introduction

Throughout the paper, we denote the set of all $m \times n$ real matrices by $R^{m\times n}$, the transpose matrix of A by A^T, the identity matrix of order n by I_n, the Kronecker product of A and B by $A \otimes B$, the $mn \times 1$ vector formed by the vertical concatenation of the respective columns of a matrix $A \in R^{m\times n}$ by $\mathrm{vec}(A)$, the trace of a matrix A by $\mathrm{tr}(A)$, and the Frobenius norm of a matrix A by $\|A\|$, where $\|A\| = \sqrt{\mathrm{tr}(A^TA)}$. Let $P \in R^{m\times m}$ and $Q \in R^{n\times n}$ be two real generalized reflection matrices, that is; $P^T = P$, $P^2 = I_m$ and $Q^T = Q$, $Q^2 = I_n$. A matrix $A \in R^{m\times n}$ is called generalized reflexive (antireflexive) matrix with respect to the matrix pair (P, Q) if $A = PAQ$ ($A = -PAQ$). The set of all $m \times n$ real generalized reflexive (generalized antireflexive) matrices with respect to matrix pair (P, Q) is denoted by $R_r^{m\times n}(P, Q)$ ($R_a^{m\times n}(P, Q)$). The generalized reflexive and generalized antireflexive matrices have been widely used in engineering, scientific computations, and various other fields.

The linear matrix equation pair $A_1X_1B_1 + A_2X_2B_2 = E$, $C_1X_1D_1 + C_2X_2D_2 = F$ is encountered in many systems and control applications. In this paper, we consider it with real generalized reflexive and generalized antireflexive constraint on the solutions. Let T_1, T_2, T_3, and T_4 be four real generalized reflection matrices; the following four problems are studied.

Problem 1. For given matrices $A_1 \in R^{p\times k}$, $A_2 \in R^{p\times m}$, $B_1 \in R^{r\times q}$, $B_2 \in R^{n\times q}$, $C_1 \in R^{s\times k}$, $C_2 \in R^{s\times m}$, $D_1 \in R^{r\times t}$, $D_2 \in R^{n\times t}$, $E \in R^{p\times q}$, and $F \in R^{s\times t}$, find $X_1 \in S_1 \subset R_r^{k\times r}(T_1, T_2)$ and $X_2 \in S_2 \subset R_r^{m\times n}(T_3, T_4)$ such that

$$\begin{aligned} A_1X_1B_1 + A_2X_2B_2 &= E, \\ C_1X_1D_1 + C_2X_2D_2 &= F. \end{aligned} \tag{1}$$

Problem 2. When Problem 1 is consistent, let S_r denote the set of its solutions. For the given matrices $\widetilde{X}_1 \in R^{k\times r}$, $\widetilde{X}_2 \in R^{m\times n}$, find $\{\widehat{X}_1, \widehat{X}_2\} \in S_r$ such that

$$\begin{aligned} &\left\|\widehat{X}_1 - \widetilde{X}_1\right\|^2 + \left\|\widehat{X}_2 - \widetilde{X}_2\right\|^2 \\ &\quad = \min_{\{X_1, X_2\} \in S_r} \left(\left\|X_1 - \widetilde{X}_1\right\|^2 + \left\|X_2 - \widetilde{X}_2\right\|^2\right). \end{aligned} \tag{2}$$

Problem 3. For given matrices $A_1 \in R^{p\times k}$, $A_2 \in R^{p\times m}$, $B_1 \in R^{r\times q}$, $B_2 \in R^{n\times q}$, $C_1 \in R^{s\times k}$, $C_2 \in R^{s\times m}$, $D_1 \in R^{r\times t}$, $D_2 \in R^{n\times t}$,

$E \in R^{p\times q}$ and $F \in R^{s\times t}$, find $X_1 \in S_3 \subset R_a^{k\times r}(T_3, T_4)$ and $X_2 \in S_4 \subset R_a^{m\times n}(T_3, T_4)$ such that

$$\begin{aligned} A_1X_1B_1 + A_2X_2B_2 &= E, \\ C_1X_1D_1 + C_2X_2D_2 &= F. \end{aligned} \tag{3}$$

Problem 4. When Problem 3 is consistent, let S_a denote the set of its solutions. For the given matrices $\widetilde{X}_1 \in R^{k\times r}$, $\widetilde{X}_2 \in R^{m\times n}$, find $\{\widehat{X}_1, \widehat{X}_2\} \in S_a$ such that

$$\begin{aligned} &\left\|\widehat{X}_1 - \widetilde{X}_1\right\|^2 + \left\|\widehat{X}_2 - \widetilde{X}_2\right\|^2 \\ &\quad = \min_{\{X_1, X_2\}\in S_a} \left(\left\|X_1 - \widetilde{X}_1\right\|^2 + \left\|X_2 - \widetilde{X}_2\right\|^2\right). \end{aligned} \tag{4}$$

Problem 2 (4) is to find the optimal approximation generalized reflexive (antireflexive) solution to the given generalized reflexive (antireflexive) matrices $\widetilde{X}_1$ and $\widetilde{X}_2$ in the solution set of Problem 1 (3); it is occurs frequently in experiment design (see, e.g., [1]). In recent years, the matrix optimal approximation problem has been studied extensively (e.g., [2–12]).

Research on solving matrix equation pair has been actively ongoing for the last 30 years or more. For instance, Mitra [13] gave conditions for the existence of a solution and a representation of the general common solution to $AXB = E$, $CXD = F$. Shinozaki and Sibuya [14] and van der Woude [15] discussed conditions for the existence of a common solution to $AXB = E, CXD = F$. Navarra et al. [5] derived sufficient and necessary conditions for the existence of a common solution to $AXB = E, CXD = F$. Yuan [11] obtained an analytical expression of the least-squares solutions of $AXB = E, CXD = F$ by using the generalized singular value decomposition (GSVD) of matrices. Dehghan and Hajarian [32] presented some examples to show a motivation for studying the general coupled matrix equations $\sum_{j=1}^{l} A_{ij}X_jB_{ij} = C_i$, $i = 1, 2, \ldots, l$, and they [17] constructed an iterative algorithm to solve the general coupled matrix equations $\sum_{j=1}^{p} A_{ij}X_jB_{ij} = M_i$, $i = 1, 2, \ldots, p$. Wang [18, 19] gave the centrosymmetric solution to the system of quaternion matrix equations $A_1X = C_1$, $A_3XB_3 = C_3$. Wang [16] also solved a system of matrix equations over arbitrary regular rings with identity.

Recently, some finite iterative algorithms have also been developed to solve matrix equations. Ding et al. [20–23] studied the iterative solutions of matrix equations $AXB = F$ and $A_iXB_i = F_i$, generalized Sylvester matrix equations $AXB + CXD = F$ and $AXB + CX^TD = F$. They presented gradient based and least-squares based iterative algorithms for the solution. Duan et al. [24–27] considered iterative method for some coupled linear matrix equations. Deng et al. [28] studied the consistent conditions and the general expressions about the Hermitian solutions of the matrix equations $(AX, XB) = (C, D)$ and designed an iterative method for its Hermitian minimum norm solutions. Li and Wu [29] gave symmetric and skew-antisymmetric solutions to certain matrix equations $A_1X = C_1$, $XB_3 = C_3$ over the real quaternion algebra H. For more studies on iterative algorithms on coupled matrix equations, we refer to [3, 8–10, 30–36]. Peng et al. [12] presented iterative methods to obtain the symmetric solutions of $AXB = E, CXD = F$. Sheng and Chen [7] presented a finite iterative method for solving $AXB = E, CXD = F$. Liao and Lei [37] presented an analytical expression of the least-squares solution and an algorithm for $AXB = E, CXD = F$ with the minimum norm. Peng et al. [6] presented an algorithm for the least-squares reflexive solution. Dehghan and Hajarian [2] presented an iterative algorithm for solving a pair of matrix equations $AXB = E$, $CXD = F$ over generalized matrices; they [31] also presented an algorithm for solving a pair of matrix equations $AXB + CYD = J$, $EXF + GYH = K$ over generalized reflexive (antireflexive) matrices. Cai and Chen [38] presented an iterative algorithm for the least-squares bisymmetric solutions of the matrix equations $AXB = E$, $CXD = F$. Yin and Huang [39] presented an iterative algorithm to solve the least-squares generalized reflexive solutions of the matrix equations $AXB = E, CXD = F$. Lin and Wang [40] presented an iterative algorithm to solve a system of linear matrix equations $A_1X_1B_1 + A_2X_2B_2 = E$, $C_1X_1D_1 + C_2X_2D_2 = F$ with real matrices X_1 and X_2.

However, to our knowledge, there has been little information on finding the solutions to the Problems 1–4 by iterative algorithm. In this paper, two efficient iterative algorithms are presented to solve the Problems 1–4. The suggested iterative algorithms automatically determine the solvability of equations pair (1) with the constraint. When the pair of equations is consistent, then, for any initial generalized reflexive (antireflexive) matrices X_1^1 and X_2^1, the solution can be obtained in the absence of round errors, and the least norm solution can be obtained by choosing a special kind of initial matrix. In addition, the unique optimal approximation solution pair $\widehat{X}_1, \widehat{X}_2$ to given matrix pair $\widetilde{X}_1, \widetilde{X}_2$ in Frobenius norm can be obtained by finding the least norm solution of a new pair of matrix equations $A_1\overline{X}_1B_1 + A_2\overline{X}_2B_2 = \overline{E}$, $C_1\overline{X}_1D_1 + C_2\overline{X}_2D_2 = \overline{F}$, where $\overline{X}_i = X_i - \widetilde{X}_i$, $(i = 1, 2)$, $\overline{E} = E - A_1\widetilde{X}_1B_1 + A_2\widetilde{X}_2B_2$, $\overline{F} = F - C_1\widetilde{X}_1D_1 + C_2\widetilde{X}_2D_2$. The given numerical examples demonstrate that our iterative algorithms are efficient. In particular, when the numbers of the parameter matrices $A_1, A_2, B_1, B_2, C_1, C_2, D_1$, and D_2 are large, our algorithms are efficient as well while the algorithm of [31] is not convergent. That is, our algorithms have merits of good numerical stability and ease to program.

The rest of this paper is outlined as follows. In Section 2, we first propose an efficient iterative algorithm for solving Problems 1 and 2; then we give some properties of this iterative algorithm. We show that the algorithm can obtain a solution group for any (special) initial generalized reflexive matrix group in the absence of round-off errors. In Section 3, Problems 3 and 4 are solved similarly. In Section 4 numerical examples are given to illustrate that our algorithms are quite efficient.

2. Iterative Algorithm for Solving Problems 1 and 2

In this section, we present an iterative algorithm for solving Problems 1 and 2.

Algorithm 5. (1) Input matrices $A_1 \in R^{p\times k}$, $A_2 \in R^{p\times m}$, $B_1 \in R^{r\times q}$, $B_2 \in R^{n\times q}$, $C_1 \in R^{s\times k}$, $C_2 \in R^{s\times m}$, $D_1 \in R^{r\times t}$, $D_2 \in R^{n\times t}$, $E \in R^{p\times q}$, $F \in R^{s\times t}$, $X_1^1 \in R_r^{k\times r}$, and $X_2^1 \in R_r^{m\times n}$ (where X_1^1, X_2^1 are any initial generalized reflexive matrices).

(2) Calculate

$$\begin{gathered} E^1 = E; \qquad F^1 = F; \\ Q_1^1 = A_1^T E^1 B_1^T + C_1^T F^1 D_1^T; \qquad Q_2^1 = A_2^T E^1 B_2^T + C_2^T F^1 D_2^T; \\ P_1^1 = Q_1^1 + T_1 Q_1^1 T_2; \qquad P_2^1 = Q_2^1 + T_3 Q_2^1 T_4; \\ \beta^1 = \left(\operatorname{tr}\left[\left(E^1\right)^T\left(A_1 P_1^1 B_1 + A_2 P_2^1 B_2\right)\right] + \operatorname{tr}\left[\left(F^1\right)^T\left(C_1 P_1^1 D_1 + C_2 P_2^1 D_2\right)\right]\right) \\ \times\left(\left\|A_1 P_1^1 B_1 + A_2 P_2^1 B_2\right\|^2 + \left\|C_1 P_1^1 D_1 + C_2 P_2^1 D_2\right\|^2\right)^{-1}; \\ \Delta X_1^1 = \beta^1 P_1^1; \qquad \Delta X_2^1 = \beta^1 P_2^1; \qquad k = 1. \end{gathered} \tag{5}$$

(3) If $\Delta X = \operatorname{diag}(\Delta X_1^k, \Delta X_2^k) = 0$ $(k = 1, 2, \ldots)$, then stop. Otherwise,

$$\begin{gathered} X_1^{k+1} = X_1^k + \Delta X_1^k; \\ X_2^{k+1} = X_2^k + \Delta X_2^k. \end{gathered} \tag{6}$$

(4) Calculate

$$\begin{gathered} E^{k+1} = E^k - \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2\right); \\ F^{k+1} = F^k - \left(C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2\right); \\ Q_1^{k+1} = A_1^T E^{k+1} B_1^T + C_1^T F^{k+1} D_1^T; \\ Q_2^{k+1} = A_2^T E^{k+1} B_2^T + C_2^T F^{k+1} D_2^T; \\ P_1^{k+1} = Q_1^{k+1} + T_1 Q_1^{k+1} T_2; \qquad P_2^{k+1} = Q_2^{k+1} + T_3 Q_2^{k+1} T_4; \\ \beta^{k+1} = \left(\operatorname{tr}\left[\left(E^{k+1}\right)^T\left(A_1 P_1^{k+1} B_1 + A_2 P_2^{k+1} B_2\right)\right] + \operatorname{tr}\left[\left(F^{k+1}\right)^T\left(C_1 P_1^{k+1} D_1 + C_2 P_2^{k+1} D_2\right)\right]\right) \\ \times\left(\left\|A_1 P_1^{k+1} B_1 + A_2 P_2^{k+1} B_2\right\|^2 + \left\|C_1 P_1^{k+1} D_1 + C_2 P_2^{k+1} D_2\right\|^2\right)^{-1}; \\ \Delta X_1^{k+1} = \beta^{k+1} P_1^{k+1}; \qquad \Delta X_2^{k+1} = \beta^{k+1} P_2^{k+1}; \\ k = k + 1. \end{gathered} \tag{7}$$

Go to (3).

Lemma 6. *In Algorithm 5, the choice of β^k makes $\|\operatorname{diag}(E^{k+1}, F^{k+1})\|$ reach a minimum and $\operatorname{diag}(E^{k+1}, F^{k+1})$ and $\operatorname{diag}(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2)$ orthogonal to each other.*

Proof. From Algorithm 5, we have

$$\begin{aligned} &\left\|\operatorname{diag}\left(E^{k+1}, F^{k+1}\right)\right\|^2 \\ &= \left\|\operatorname{diag}\left(E^k - \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2\right), F^k - \left(C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2\right)\right)\right\|^2 \\ &= \left\|\operatorname{diag}\left(E^k - \left(A_1 \beta^k P_1^k B_1 + A_2 \beta^k P_2^k B_2\right), F^k - \left(C_1 \beta^k P_1^k D_1 + C_2 \beta^k P_2^k D_2\right)\right)\right\|^2 \\ &= \left\|E^k - \left(A_1 \beta^k P_1^k B_1 + A_2 \beta^k P_2^k B_2\right)\right\|^2 + \left\|F^k - \left(C_1 \beta^k P_1^k D_1 + C_2 \beta^k P_2^k D_2\right)\right\|^2 \\ &= \left\|E^k\right\|^2 + \left\|F^k\right\|^2 - 2\left[\operatorname{tr}\left(\left(E^k\right)^T\left(A_1 P_1^k B_1 + A_2 P_2^k B_2\right)\right) + \operatorname{tr}\left(\left(F^k\right)^T\left(C_1 P_1^k D_1 + C_2 P_2^k D_2\right)\right)\right]\beta^k \\ &\quad + \left[\left\|A_1 P_1^k B_1 + A_2 P_2^k B_2\right\|^2 + \left\|C_1 P_1^k D_1 + C_2 P_2^k D_2\right\|^2\right]\left(\beta^k\right)^2. \end{aligned} \tag{8}$$

From the above, the condition of $\|\operatorname{diag}(E^{k+1}, F^{k+1})\|$ reaching a minimum is

$$\begin{aligned} \beta^k = &\left(\operatorname{tr}\left[\left(E^k\right)^T\left(A_1 P_1^k B_1 + A_2 P_2^k B_2\right)\right] + \operatorname{tr}\left[\left(F^k\right)^T\left(C_1 P_1^k D_1 + C_2 P_2^k D_2\right)\right]\right) \\ &\times\left(\left\|A_1 P_1^k B_1 + A_2 P_2^k B_2\right\|^2 + \left\|C_1 P_1^k D_1 + C_2 P_2^k D_2\right\|^2\right)^{-1}. \end{aligned} \tag{9}$$

On the other side, if the choice of β^k makes $\operatorname{diag}(E^{k+1}, F^{k+1})$ and $\operatorname{diag}(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2)$ orthogonal to each other, that is, $\operatorname{tr}[\operatorname{diag}(E^{k+1}, F^{k+1})^T \operatorname{diag}(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2)] = 0$, we can have the same β^k as (3). □

Theorem 7. *Algorithm 5 is bound to be convergent.*

Proof. From Algorithm 5 and Lemma 6 we have

$$\begin{aligned}
&\operatorname{diag}\left(E^k, F^k\right) \\
&\quad= \operatorname{diag}\left(E^{k+1}, F^{k+1}\right) \\
&\qquad+ \operatorname{diag}\left(A_1\Delta X_1^k B_1 + A_2\Delta X_2^k B_2, C_1\Delta X_1^k D_1 + C_2\Delta X_2^k D_2\right), \\
&\Longleftrightarrow \left\|\operatorname{diag}(E^k, F^k)\right\|^2 \\
&\quad= \left\|\operatorname{diag}\left(E^{k+1}, F^{k+1}\right)\right. \\
&\qquad+ \operatorname{diag}\left(A_1\Delta X_1^k B_1 + A_2\Delta X_2^k B_2,\right. \\
&\qquad\qquad \left.\left. C_1\Delta X_1^k D_1 + C_2\Delta X_2^k D_2\right)\right\|^2 \\
&\quad= \left\|\operatorname{diag}\left(E^{k+1}, F^{k+1}\right)\right\|^2 \\
&\qquad+ \left\|\operatorname{diag}\left(A_1\Delta X_1^k B_1 + A_2\Delta X_2^k B_2,\right.\right. \\
&\qquad\qquad \left.\left. C_1\Delta X_1^k D_1 + C_2\Delta X_2^k D_2\right)\right\|^2
\end{aligned} \tag{10}$$

such that

$$\begin{aligned}
&\left\|\operatorname{diag}\left(E^{k+1}, F^{k+1}\right)\right\|^2 \\
&\quad= \left\|\operatorname{diag}\left(E^k, F^k\right)\right\|^2 \\
&\qquad- \left\|\operatorname{diag}\left(A_1\Delta X_1^k B_1 + A_2\Delta X_2^k B_2,\right.\right. \\
&\qquad\qquad \left.\left. C_1\Delta X_1^k D_1 + C_2\Delta X_2^k D_2\right)\right\|^2 \\
&\quad\le \left\|\operatorname{diag}\left(E^k, F^k\right)\right\|^2.
\end{aligned} \tag{11}$$

From (11), we know that Algorithm 5 is convergent. □

Lemma 8 (see [41]). *Suppose that the consistent system of linear equations $My = b$ has a solution $y_0 \in R(M^T)$; then y_0 is the least Frobenius norm solution of the system of linear equations.*

Theorem 9. *Assume that the system* (1) *is consistent. Let $X_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T + T_1(A_1^T Y B_1^T + C_1^T Z D_1^T)T_2$, $X_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T + T_3(A_2^T Y B_2^T + C_2^T Z D_2^T)T_4$ be initial matrices, where $Y \in R^{p\times q}$, $Z \in R^{s\times t}$ are any initial matrices, or, especially, $X_1^1 = 0$, $X_2^1 = 0$; then the solution generated by Algorithm 5 is the least Frobenius norm solution to* (1).

Proof. If (1) is consistent, from $X_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T + T_1(A_1^T Y B_1^T + C_1^T Z D_1^T)T_2$, $X_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T + T_3(A_2^T Y B_2^T + C_2^T Z D_2^T)T_4$, using Algorithm 5, we have the generalized reflexive iterative solution pair X_1^k, X_2^k of (1) as the following:

$$\begin{aligned}
X_1^k &= X_1^{k-1} + \Delta X_1^{k-1} \\
&= X_1^1 + \Delta X_1^1 + \cdots + \Delta X_1^{k-1} \\
&= A_1^T Y B_1^T + C_1^T Z D_1^T + T_1\left(A_1^T Y B_1^T + C_1^T Z D_1^T\right)T_2 \\
&\quad+ A_1^T\left[\beta^1 E^1 + \cdots + \beta^{k-1}E^{k-1}\right]B_1^T \\
&\quad+ C_1^T\left[\beta^1 F^1 + \cdots + \beta^{k-1}F^{k-1}\right]D_1^T \\
&\quad+ T_1 A_1^T\left[\beta^1 E^1 + \cdots + \beta^{k-1}E^{k-1}\right]B_1^T T_2 \\
&\quad+ T_1 C_1^T\left[\beta^1 F^1 + \cdots + \beta^{k-1}F^{k-1}\right]D_1^T T_2 \\
&= A_1^T M B_1^T + C_1^T N D_1^T + T_1\left(A_1^T M B_1^T + C_1^T N D_1^T\right)T_2, \\
X_2^k &= X_2^{k-1} + \Delta X_2^{k-1} \\
&= X_2^1 + \Delta X_2^1 + \cdots + \Delta X_2^{k-1} \\
&= A_2^T Y B_2^T + C_2^T Z D_2^T + T_3\left(A_2^T Y B_2^T + C_2^T Z D_2^T\right)T_4 \\
&\quad+ A_2^T\left[\beta^1 E^1 + \cdots + \beta^{k-1}E^{k-1}\right]B_2^T \\
&\quad+ C_2^T\left[\beta^1 F^1 + \cdots + \beta^{k-1}F^{k-1}\right]D_2^T \\
&\quad+ T_3 A_2^T\left[\beta^1 E^1 + \cdots + \beta^{k-1}E^{k-1}\right]B_2^T T_4 \\
&\quad+ T_3 C_2^T\left[\beta^1 F^1 + \cdots + \beta^{k-1}F^{k-1}\right]D_2^T T_4 \\
&= A_2^T M B_2^T + C_2^T N D_2^T + T_3\left(A_2^T M B_2^T + C_2^T N D_2^T\right)T_4.
\end{aligned} \tag{12}$$

We know that (1) is equivalent to the system

$$\begin{pmatrix}
B_1^T \otimes A_1 & B_2^T \otimes A_2 \\
D_1^T \otimes C_1 & D_2^T \otimes C_2 \\
(T_2 B_1)^T \otimes (A_1 T_1) & (T_4 B_2)^T \otimes (A_2 T_3) \\
(T_2 D_1)^T \otimes (C_1 T_1) & (D_2 T_4)^T \otimes (C_2 T_3)
\end{pmatrix}
\begin{pmatrix}
\operatorname{vec}(X_1) \\
\operatorname{vec}(X_2)
\end{pmatrix}
= \begin{pmatrix}
\operatorname{vec}(E) \\
\operatorname{vec}(F) \\
\operatorname{vec}(E) \\
\operatorname{vec}(F)
\end{pmatrix}. \tag{13}$$

From (12) and (13) we have

$$\begin{aligned}
&\begin{pmatrix} \operatorname{vec}\left(A_1^T M B_1^T + C_1^T N D_1^T + T_1\left(A_1^T M B_1^T + C_1^T N D_1^T\right) T_2\right) \\ \operatorname{vec}\left(A_2^T M B_2^T + C_2^T N D_2^T + T_3\left(A_2^T M B_2^T + C_2^T N D_2^T\right) T_4\right) \end{pmatrix} \\
&= \begin{pmatrix} B_1 \otimes A_1^T & D_1 \otimes C_1^T & (T_2 B_1) \otimes \left(T_1 A_1^T\right) & (T_2 D_1) \otimes \left(T_1 C_1^T\right) \\ B_2 \otimes A_2^T & D_2 \otimes C_2^T & (T_4 B_2) \otimes \left(T_3 A_2^T\right) & (T_4 D_2) \otimes \left(T_3 C_2^T\right) \end{pmatrix} \begin{pmatrix} \operatorname{vec}(M) \\ \operatorname{vec}(N) \\ \operatorname{vec}(M) \\ \operatorname{vec}(N) \end{pmatrix} \\
&= \begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ D_1^T \otimes C_1 & D_2^T \otimes C_2 \\ \left(B_1^T T_2\right) \otimes (A_1 T_1) & \left(B_2^T T_4\right) \otimes (A_2 T_3) \\ \left(D_1^T T_2\right) \otimes (C_1 T_1) & \left(D_2^T T_4\right) \otimes (C_2 T_3) \end{pmatrix}^T \begin{pmatrix} \operatorname{vec}(M) \\ \operatorname{vec}(N) \\ \operatorname{vec}(M) \\ \operatorname{vec}(N) \end{pmatrix} \\
&\in R\left(\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ D_1^T \otimes C_1 & D_2^T \otimes C_2 \\ \left(B_1^T T_2\right) \otimes (A_1 T_1) & \left(B_2^T T_4\right) \otimes (A_2 T_3) \\ \left(D_1^T T_2\right) \otimes (C_1 T_1) & \left(D_2^T T_4\right) \otimes (C_2 T_3) \end{pmatrix}^T\right),
\end{aligned} \tag{14}$$

where $R(*)$ is the column space of matrix $*$.

Considering Lemma 8, with the initial matrices $X_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T + T_1(A_1^T Y B_1^T + C_1^T Z D_1^T) T_2$, $X_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T + T_3(A_2^T Y B_2^T + C_2^T Z D_2^T) T_4$, where $Y \in R^{p\times q}$, $Z \in R^{s\times t}$ are arbitrary, or, especially, $X_1^1 = 0$ and $X_2^1 = 0$, then the solution pair X_1^k, X_2^k, generated by Algorithm 5, is the least Frobenius norm solution of the matrix equations (1). □

Suppose that Problem 1 is consistent. Obviously the solution set S_r of (1) is nonempty. For given matrices pair $\widetilde{X}_1 \in R^{k\times r}$, $\widetilde{X}_2 \in R^{m\times n}$, we consider two case:

Case 1. Consider $\widetilde{X}_1 \in R_r^{k\times r}(T_1, T_2)$, $\widetilde{X}_2 \in R_r^{m\times n}(T_3, T_4)$.

We have

$$\begin{aligned} A_1 X_1 B_1 + A_2 X_2 B_2 &= E, \\ C_1 X_1 D_1 + C_2 X_2 D_2 &= F, \end{aligned} \tag{15}$$

$$\Longleftrightarrow \begin{cases} A_1\left(X_1 - \widetilde{X}_1\right) B_1 + A_2\left(X_2 - \widetilde{X}_2\right) B_2 \\ \quad = E - A_1 \widetilde{X}_1 B_1 - A_2 \widetilde{X}_2 B_2, \\ C_1\left(X_1 - \widetilde{X}_1\right) D_1 + C_2\left(X_2 - \widetilde{X}_2\right) D_2 \\ \quad = F - C_1 \widetilde{X}_1 D_1 - C_2 \widetilde{X}_2 D_2. \end{cases} \tag{16}$$

Let $\overline{X}_1 = X_1 - \widetilde{X}_1$, $\overline{X}_2 = X_2 - \widetilde{X}_2$, $\overline{E} = E - A_1 \widetilde{X}_1 B_1 - A_2 \widetilde{X}_2 B_2$, and $\overline{F} = F - C_1 \widetilde{X}_1 D_1 - C_2 \widetilde{X}_2 D_2$; then Problem 2 is equivalent to finding the least Frobenius norm solution pair of the system

$$\begin{aligned} A_1 \overline{X}_1 B_1 + A_2 \overline{X}_2 B_2 &= \overline{E}, \\ C_1 \overline{X}_1 D_1 + C_2 \overline{X}_2 D_2 &= \overline{F}, \end{aligned} \tag{17}$$

which can be obtained using Algorithm 5 with the initial matrix pair $X_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T + T_1(A_1^T Y B_1^T + C_1^T Z D_1^T) T_2$, $X_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T + T_3(A_2^T Y B_2^T + C_2^T Z D_2^T) T_4$, where $Y \in R^{p\times q}$, $Z \in R^{s\times t}$ are arbitrary, or, especially, $\overline{X}_1^1 = 0$, $\overline{X}_2^1 = 0$, and the solution of the matrix optimal approximation Problem 2 can be represented as $\widehat{X}_1 = \overline{X}_1^k + \widetilde{X}_1$, $\widehat{X}_2 = \overline{X}_2^k + \widetilde{X}_2$.

Case 2. $\widetilde{X}_1 \in R^{k\times r}$, $\widetilde{X}_2 \in R^{m\times n}$, and $\widetilde{X}_1, \widetilde{X}_2$ are not generalized reflexive matrices.

Lemma 10. *Suppose $H_1 \in R_r^{k\times r}(T_1, T_2)$, $H_2 \in R_a^{m\times n}(T_1, T_2)$; then $tr(H_1^T H_2) = 0$.*

Proof. Since $H_1 = T_1 H_1 T_2$, $H_2 = -T_1 H_2 T_2$, we have

$$\begin{aligned} \operatorname{tr}\left(H_1^T H_2\right) &= \operatorname{tr}\left[\left(T_1 H_1 T_2\right)^T\left(-T_1 H_2 T_2\right)\right] \\ &= -\operatorname{tr}\left(T_2^T H_1^T T_1^T T_1 H_2 T_2\right) \\ &= -\operatorname{tr}\left(T_2^T H_1^T T_1^T T_1 H_2 T_2\right) = -\operatorname{tr}\left(T_2^T H_1^T H_2 T_2\right) \\ &= -\operatorname{tr}\left(H_1^T H_2 T_2 T_2^T\right) = -\operatorname{tr}\left(H_1^T H_2\right). \end{aligned} \tag{18}$$

So, $\operatorname{tr}(H_1^T H_2) = 0$. □

For $\{X_1, X_2\} \in S_r$, by Lemma 10, we have

$$\begin{aligned} \left\|X_1 - \widetilde{X}_1\right\|^2 &= \left\|X_1 - \left(\frac{\widetilde{X}_1 + T_1 \widetilde{X}_1 T_2}{2} + \frac{\widetilde{X}_1 - T_1 \widetilde{X}_1 T_2}{2}\right)\right\|^2 \\ &= \left\|\left(X_1 - \frac{\widetilde{X}_1 + T_1 \widetilde{X}_1 T_2}{2}\right) - \frac{\widetilde{X}_1 - T_1 \widetilde{X}_1 T_2}{2}\right\|^2 \\ &= \left\|X_1 - \frac{\widetilde{X}_1 + T_1 \widetilde{X}_1 T_2}{2}\right\|^2 + \left\|\frac{\widetilde{X}_1 - T_1 \widetilde{X}_1 T_2}{2}\right\|^2, \end{aligned}$$

$$\begin{aligned}\left\|X_2-\widetilde{X}_2\right\|^2&=\left\|X_2-\left(\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}+\frac{\widetilde{X}_2-T_3\widetilde{X}_2T_4}{2}\right)\right\|^2\\&=\left\|\left(X_2-\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}\right)-\frac{\widetilde{X}_2-T_3\widetilde{X}_2T_4}{2}\right\|^2\\&=\left\|X_2-\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}\right\|^2+\left\|\frac{\widetilde{X}_2-T_3\widetilde{X}_2T_4}{2}\right\|^2.\end{aligned}\tag{19}$$

For any $\{X_1,X_2\}\in S_r$, we have

$$\begin{aligned}&\left\|\widehat{X}_1-\widetilde{X}_1\right\|^2+\left\|\widehat{X}_2-\widetilde{X}_2\right\|^2\\&\quad=\min_{\{X_1,X_2\}\in S_r}\left(\left\|X_1-\widetilde{X}_1\right\|^2+\left\|X_2-\widetilde{X}_2\right\|^2\right)\\&\quad=\left\|\frac{\widetilde{X}_1-T_1\widetilde{X}_1T_2}{2}\right\|^2+\left\|\frac{\widetilde{X}_2-T_3\widetilde{X}_2T_4}{2}\right\|^2\\&\qquad+\min_{\{X_1,X_2\}\in S_r}\left(\left\|X_1-\frac{\widetilde{X}_1+T_1\widetilde{X}_1T_2}{2}\right\|^2\right.\\&\qquad\left.+\left\|X_2-\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}\right\|^2\right).\end{aligned}\tag{20}$$

That is, Problem 2 is equivalent to finding $\{\widehat{X}_1,\widehat{X}_2\}\in S_r$ such that $\|X_1-(\widetilde{X}_1+T_1\widetilde{X}_1T_2)/2\|^2+\|X_2-(\widetilde{X}_1+T_3\widetilde{X}_2T_4)/2\|^2$ reach a minimum.

For any $\{X_1,X_2\}\in S_r$, we have

$$\begin{aligned}&A_1X_1B_1+A_2X_2B_2=E,\\&C_1X_1D_1+C_2X_2D_2=F,\end{aligned}\tag{21}$$

$$\Longleftrightarrow\left\{\begin{aligned}&A_1\left(X_1-\frac{\widetilde{X}_1+T_1\widetilde{X}_1T_2}{2}\right)B_1\\&\quad+A_2\left(X_2-\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}\right)B_2\\&\quad=E-A_1\frac{\widetilde{X}_1+T_1\widetilde{X}_1T_2}{2}B_1-A_2\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}B_2,\\&C_1\left(X_1-\frac{\widetilde{X}_1+T_1\widetilde{X}_1T_2}{2}\right)D_1\\&\quad+C_2\left(X_2-\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}\right)D_2\\&\quad=F-C_1\frac{\widetilde{X}_1+T_1\widetilde{X}_1T_2}{2}D_1-C_2\frac{\widetilde{X}_2+T_3\widetilde{X}_2T_4}{2}D_2.\end{aligned}\right.\tag{22}$$

Let $\overline{X}_1=X_1-(\widetilde{X}_1+T_1\widetilde{X}_1T_2)/2$, $\overline{X}_2=X_2-(\widetilde{X}_2+T_3\widetilde{X}_2T_4)/2$, $\overline{E}=E-A_1((\widetilde{X}_1+T_1\widetilde{X}_1T_2)/2)B_1-A_2((\widetilde{X}_2+T_3\widetilde{X}_2T_4)/2)B_2$, and $\overline{F}=F-C_1((\widetilde{X}_1+T_1\widetilde{X}_1T_2)/2)D_1-C_2((\widetilde{X}_2+T_3\widetilde{X}_2T_4)/2)D_2$ and then to finding $\{\widehat{X}_1,\widehat{X}_2\}\in S$ such that $\|X_1-(\widetilde{X}_1+T_1\widetilde{X}_1T_2)/2\|^2+\|X_2-(\widetilde{X}_2+T_3\widetilde{X}_2T_4)/2\|^2$ reach a minimum is equivalent to finding the least Frobenius norm generalized reflexive solution pair $\overline{X}_1^k,\overline{X}_2^k$ of the system

$$\begin{aligned}A_1\overline{X}_1B_1+A_2\overline{X}_2B_2&=\overline{E},\\C_1\overline{X}_1D_1+C_2\overline{X}_2D_2&=\overline{F},\end{aligned}\tag{23}$$

which can be obtained using Algorithm 5.

So we can have the least Frobenius norm generalized reflexive solution of Problem 2 as $\widehat{X}_1=\overline{X}_1^k+(\widetilde{X}_1+T_1\widetilde{X}_1T_2)/2$, $\widehat{X}_2=\overline{X}_2^k+(\widetilde{X}_2+T_3\widetilde{X}_2T_4)/2$.

3. Iterative Algorithm for Solving Problems 3 and 4

In this section, we present an iterative algorithm for solving Problems 3 and 4.

Algorithm 11. (1) Input matrices $A_1\in R^{p\times k}$, $A_2\in R^{p\times m}$, $B_1\in R^{r\times q}$, $B_2\in R^{n\times q}$, $C_1\in R^{s\times k}$, $C_2\in R^{s\times m}$, $D_1\in R^{r\times t}$, $D_2\in R^{n\times t}$, $E\in R^{p\times q}$, $F\in R^{s\times t}$, $X_1^1\in R_a^{k\times r}$, and $X_2^1\in R_a^{m\times n}$ (where X_1^1,X_2^1 are any initial generalized antireflexive matrices).

(2) Calculate

$$\begin{aligned}&E^1=E;\qquad F^1=F;\\&Q_1^1=A_1^TE^1B_1^T+C_1^TF^1D_1^T;\qquad Q_2^1=A_2^TE^1B_2^T+C_2^TF^1D_2^T;\\&P_1^1=Q_1^1-T_1Q_1^1T_2;\qquad P_2^1=Q_2^1-T_3Q_2^1T_4;\\&\beta^1=\left(\operatorname{tr}\left[\left(E^1\right)^T\left(A_1P_1^1B_1+A_2P_2^1B_2\right)\right]\right.\\&\qquad\left.+\operatorname{tr}\left[\left(F^1\right)^T\left(C_1P_1^1D_1+C_2P_2^1D_2\right)\right]\right)\\&\qquad\times\left(\left\|A_1P_1^1B_1+A_2P_2^1B_2\right\|^2+\left\|C_1P_1^1D_1+C_2P_2^1D_2\right\|^2\right)^{-1};\\&\Delta X_1^1=\beta^1P_1^1;\qquad\Delta X_2^1=\beta^1P_2^1;\qquad k=1.\end{aligned}\tag{24}$$

(3) If $\Delta X=\operatorname{diag}(\Delta X_1^k,\Delta X_2^k)=0$ $(k=1,2,\ldots)$, then stop. Otherwise,

$$\begin{aligned}X_1^{k+1}&=X_1^k+\Delta X_1^k;\\X_2^{k+1}&=X_2^k+\Delta X_2^k.\end{aligned}\tag{25}$$

(4) Calculate

$$\begin{aligned}E^{k+1}&=E^k-\left(A_1\Delta X_1^kB_1+A_2\Delta X_2^kB_2\right);\\F^{k+1}&=F^k-\left(C_1\Delta X_1^kD_1+C_2\Delta X_2^kD_2\right);\\Q_1^{k+1}&=A_1^TE^{k+1}B_1^T+C_1^TF^{k+1}D_1^T;\end{aligned}$$

$$Q_2^{k+1} = A_2^T E^{k+1} B_2^T + C_2^T F^{k+1} D_2^T;$$

$$P_1^{k+1} = Q_1^{k+1} - T_1 Q_1^{k+1} T_2; \qquad P_2^{k+1} = Q_2^{k+1} - T_3 Q_2^{k+1} T_4;$$

$$\begin{aligned}\beta^{k+1} = &\Big(\mathrm{tr}\left[\left(E^{k+1}\right)^T \left(A_1 P_1^{k+1} B_1 + A_2 P_2^{k+1} B_2\right)\right] \\ &+ \mathrm{tr}\left[\left(F^{k+1}\right)^T \left(C_1 P_1^{k+1} D_1 + C_2 P_2^{k+1} D_2\right)\right]\Big) \\ &\times \Big(\left\|A_1 P_1^{k+1} B_1 + A_2 P_2^{k+1} B_2\right\|^2 \\ &+ \left\|C_1 P_1^{k+1} D_1 + C_2 P_2^{k+1} D_2\right\|^2\Big)^{-1};\end{aligned}$$

$$\Delta X_1^{k+1} = \beta^{k+1} P_1^{k+1}; \qquad \Delta X_2^{k+1} = \beta^{k+1} P_2^{k+1}; \qquad k = k+1. \tag{26}$$

Go to (3).

The properties of Algorithm 11 can be proposed similarly to Algorithm 5.

4. Examples

In this section, we show two numerical examples to illustrate the efficiency of Algorithms 5 and 11. All computations are performed by MATLAB 7. For the influence of the error of calculation, we consider the matrix R as a zero matrix if $\|R\| < 10^{-10}$.

Example 12. Consider the generalized reflexive solution of the linear matrix equations:

$$\begin{gathered} A_1 X_1 B_1 + A_2 X_2 B_2 = E, \\ C_1 X_1 D_1 + C_2 X_2 D_2 = F, \end{gathered} \tag{27}$$

where

$$A_1 = \begin{pmatrix} 139 & 105 & 54 \\ 124 & 176 & 50 \\ 159 & 35 & 175 \\ 191 & 196 & 147 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 13 & 117 & 103 & 87 & 116 \\ 198 & 85 & 67 & 45 & 152 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 27 & 60 \\ 2 & 132 \\ 179 & 57 \\ 40 & 94 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 106 & 76 & 92 & 12 & 83 \\ 128 & 157 & 114 & 121 & 61 \\ 42 & 136 & 159 & 10 & 175 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 3 & 88 & 192 \\ 154 & 100 & 145 \\ 194 & 43 & 82 \\ 198 & 129 & 149 \\ 158 & 64 & 54 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 88 & 31 & 140 \\ 71 & 135 & 146 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 4 & 63 \\ 115 & 3 \\ 90 & 77 \\ 9 & 137 \\ 5 & 19 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 96 & 90 & 55 \\ 111 & 143 & 51 \\ 24 & 179 & 173 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$E = \begin{pmatrix} 4079684 & 5931341 & 5257546 & 3760503 & 5978636 \\ 4507138 & 7309406 & 6800596 & 4204046 & 7209376 \\ 5983467 & 8290427 & 6916092 & 5829089 & 5964332 \\ 4470189 & 8583167 & 7694045 & 5535093 & 8046508 \end{pmatrix},$$

$$F = \begin{pmatrix} 1148659 & -415508 & 175592 \\ 5221721 & 3459626 & 5372914 \\ 6867542 & 6499561 & 7947258 \\ 6828512 & 5933355 & 8858444 \\ 3684631 & 3115265 & 5873214 \end{pmatrix}. \tag{28}$$

Let

$$X_1^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

Using Algorithm 5 and iterating 1214 steps, we obtain the least Frobenius norm solution pair of the matrix equation in Example 12 as follows:

$$\begin{gathered} X_1 = \begin{pmatrix} 122.0000 & 122.0000 \\ 86.0000 & -29.0000 \\ 29.0000 & -86.0000 \end{pmatrix}, \\ X_2 = \begin{pmatrix} 57.0000 & 126.0000 & -35.0000 \\ 126.0000 & 57.0000 & 35.0000 \end{pmatrix}. \end{gathered} \tag{30}$$

Example 13. Consider the generalized antireflexive solution of the linear matrix equations:

$$A_1 X_1 B_1 + A_2 X_2 B_2 = E, \qquad C_1 X_1 D_1 + C_2 X_2 D_2 = F, \tag{31}$$

where $A_1, B_1, A_2, B_2, C_1, D_1, C_2, D_2, T_1, T_2, T_3$, and T_4 are the same matrices of Example 12:

$$E = \begin{pmatrix} -743633 & 6710933 & 7426284 & 2563311 & 8338860 \\ 1535166 & 7945314 & 9163408 & 1986230 & 12338200 \\ 982308 & 14732018 & 17074320 & 3775802 & 18238112 \\ 1999993 & 12662336 & 13364050 & 5402942 & 16219108 \end{pmatrix},$$

$$F = \begin{pmatrix} 5228109 & 6926269 & 12710902 \\ 7508081 & 7671994 & 14810466 \\ 4201528 & 4853203 & 12104012 \\ 5302290 & 5127440 & 15650036 \\ 2867500 & -410942 & 5039552 \end{pmatrix}. \tag{32}$$

Let

$$X_1^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

Using Algorithm 11 and iterating 240 steps, we obtain the least Frobenius norm solution pair of the matrix equation in Example 13 as follows:

$$X_1 = \begin{pmatrix} 226.0000 & -226.0000 \\ 59.0000 & 191.0000 \\ 191.0000 & 59.0000 \end{pmatrix},$$
$$X_2 = \begin{pmatrix} 189.0000 & -63.0000 & 268.0000 \\ 63.0000 & -189.0000 & 268.0000 \end{pmatrix}. \tag{34}$$

The numbers of the parameter matrices $A_1, A_2, B_1, B_2, C_1, C_2, D_1$, and D_2 in our examples are larger than the numbers of the parameter matrices in the example of [31]. To our examples, the algorithm of [31] is not convergent. The numerical examples demonstrate that our algorithm has merits of good numerical stability and ease to program.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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