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Research Article

On a New Criterion for Meromorphic Starlike Functions

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The main purpose of this paper is to derive a new criterion for meromorphic starlike functions of order α.

1. Introduction and Preliminaries

Let Σ_n denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_{k-1} z^{k-1} \quad (n \in \mathbb{N} := \{1, 2, \ldots\}), \tag{1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}. \tag{2}$$

A function $f \in \Sigma_n$ is said to be in the class $\mathscr{MS}_n^*(\alpha)$ of *meromorphic starlike functions of order* α if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in \mathbb{U}; 0 \leqq \alpha < 1). \tag{3}$$

For simplicity, we write $\mathscr{MS}_n^*(0) =: \mathscr{MS}_n^*$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}), \tag{4}$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0, \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}), \tag{5}$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}). \tag{6}$$

Indeed, it is known that

$$\begin{gathered} f(z) \prec g(z) \quad (z \in \mathbb{U}) \\ \Longrightarrow f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \end{gathered} \tag{7}$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$\begin{gathered} f(z) \prec g(z) \quad (z \in \mathbb{U}) \\ \Longleftrightarrow f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \end{gathered} \tag{8}$$

In a recent paper, Miller et al. [1] proved the following result.

Theorem A. *Let $n \in \mathbb{N}$, $0 \leqq \lambda \leqq 1$, and*

$$M_0(\lambda, n) = \frac{n + 1 - \lambda}{\sqrt{(n + 1 - \lambda)^2 + \lambda^2} + 1 - \lambda}. \tag{9}$$

If $f \in \Sigma_n$ satisfies the condition

$$\left| z^2 f'(z) + (1 - \lambda) z f(z) + \lambda \right| < M_0(\lambda, n) \quad (z \in \mathbb{U}), \tag{10}$$

then $f \in \mathscr{MS}_n^*$.

More recently, Catas [2] improved Theorem A as follows.

Theorem B. *Let $n \in \mathbb{N}$, $0 \leqq \lambda < 1$, and*

$$M(\lambda, n) = \max\{M_0(\lambda, n), M_1(\lambda, n)\}, \tag{11}$$

where $M_0(\lambda, n)$ is given by (9) *and*

$$M_1(\lambda, n) = \frac{2(n+1-\lambda)(1-\lambda)}{(1-\lambda)(n-1)+\sqrt{(n+1-\lambda)^2(1-\lambda)+[(n-1)(1-\lambda)]^2}}. \quad (12)$$

If $f \in \Sigma_n$ *satisfies the condition*

$$\left|z^2 f'(z) + (1-\lambda) z f(z) + \lambda\right| < M(\lambda, n) \quad (z \in \mathbb{U}), \quad (13)$$

then $f \in \mathscr{MS}_n^*$.

In this paper, we aim at finding the conditions for starlikeness of the expression $|z^2 f'(z) + \lambda z f(z) + 1 - \lambda|$ for $\lambda > 1$.

For some recent investigations of meromorphic functions, see, for example, the works of [3–12] and the references cited therein.

In order to prove our main results, we require the following subordination result due to Hallenbeck and Ruscheweyh [13].

Lemma 1. *Let* ϕ *be a convex function with* $\phi(0) = 1$, *and let* $\gamma \neq 0$ *be a complex number with* $\Re(\gamma) \geqq 0$. *If a function*

$$\mathfrak{p}(z) = 1 + \mathfrak{p}_n z^n + \mathfrak{p}_{n+1} z^{n+1} + \cdots \quad (14)$$

satisfies the condition

$$\mathfrak{p}(z) + \frac{1}{\gamma} z \mathfrak{p}'(z) \prec \phi(z), \quad (15)$$

then

$$\mathfrak{p}(z) \prec \chi(z) := \frac{\gamma}{n z^{\gamma/n}} \int_0^z \phi(t) t^{(\gamma/n)-1} dt \prec \phi(z). \quad (16)$$

2. Main Results

We begin by stating the following result.

Theorem 2. *Let* $n \in \mathbb{N}$, $\lambda > 1$, *and* $0 \leqq \alpha < 1$. *If* $f \in \Sigma_n$ *satisfies the inequality*

$$\left|z^2 f'(z) + \lambda z f(z) + 1 - \lambda\right| < M, \quad (17)$$

where

$$M := M(\lambda, \alpha, n) = \frac{(1-\alpha)(\lambda+n-1)}{\lambda - \alpha + \sqrt{(1-\lambda)^2 + (\lambda+n-1)^2}}, \quad (18)$$

then $f \in \mathscr{MS}_n^*(\alpha)$.

Proof. Suppose that

$$q(z) := z f(z) \quad (z \in \mathbb{U}). \quad (19)$$

It follows from (19) that

$$z q'(z) = z f(z) + z^2 f'(z). \quad (20)$$

By combining (17), (19), and (20), we easily get

$$\left|q(z) + \frac{1}{\lambda-1} z q'(z) - 1\right| < \frac{M}{\lambda-1}, \quad (21)$$

or equivalently

$$q(z) + \frac{1}{\lambda-1} z q'(z) \prec 1 + \frac{M}{\lambda-1} z. \quad (22)$$

An application of Lemma 1 yields

$$\begin{aligned} q(z) &\prec \frac{\lambda-1}{n z^{(\lambda-1)/n}} \int_0^z \left(1 + \frac{M}{\lambda-1} t\right) t^{[(\lambda-1)/n]-1} dt \\ &= 1 + \frac{M}{\lambda+n-1} z. \end{aligned} \quad (23)$$

The subordination (23) is equivalent to

$$\left|q(z) - 1\right| < \frac{M}{\lambda+n-1} =: N. \quad (24)$$

From (18) and (24), we know that

$$N < \frac{1-\alpha}{\lambda-\alpha} < 1. \quad (25)$$

We suppose that

$$-\frac{z f'(z)}{f(z)} := (1-\alpha) p(z) + \alpha. \quad (26)$$

By virtue of (19) and (26), we get

$$z^2 f'(z) = -q(z)\left[(1-\alpha) p(z) + \alpha\right], \quad (27)$$

which implies that (17) can be written as

$$\left|q(z)\left[(1-\alpha) p(z) + \alpha - \lambda\right] + \lambda - 1\right| < M = (\lambda+n-1) N. \quad (28)$$

We now only need to show that (28) implies $\Re(p(z)) > 0$ in $\mathbb{U}$. Indeed, if this is false, since $p(0) = 1$, then there exists a point $z_0 \in \mathbb{U}$ such that $p(z_0) = \beta i$, where β is a real number. Thus, in order to show that (28) implies $\Re(p(z)) > 0$ in $\mathbb{U}$, it suffices to obtain the contradiction from the inequality

$$\begin{aligned} &\left|q(z_0)\left[(1-\alpha)\beta i + \alpha - \lambda\right] + \lambda - 1\right| \\ &\geqq (\lambda+n-1) N \quad (\beta \in \mathbb{R}). \end{aligned} \quad (29)$$

By setting

$$q(z_0) = u + iv \quad (u, v \in \mathbb{R}), \quad (30)$$

we have

$$\begin{aligned} E &= \left|q(z_0)\left[(1-\alpha)\beta i+\alpha-\lambda\right]+\lambda-1\right|^2 \\ &= \left(u^2+v^2\right)\left[(1-\alpha)^2\beta^2+(\alpha-\lambda)^2\right] \\ &\quad -2(1-\lambda)\Re\left((u+iv)\left[(1-\alpha)\beta i+\alpha-\lambda\right]\right)+(1-\lambda)^2 \\ &= \left(u^2+v^2\right)(1-\alpha)^2\beta^2+2(1-\lambda)(1-\alpha)\beta v \\ &\quad +\left|(u+iv)(\alpha-\lambda)-(1-\lambda)\right|^2. \end{aligned} \tag{31}$$

By means of (24), we obtain

$$\begin{aligned} &\left|(u+iv)(\alpha-\lambda)-(1-\lambda)\right| \\ &\quad = \left|(u+iv)(\alpha-\lambda)-(\alpha-\lambda)+\alpha-\lambda-1+\lambda\right| \\ &\quad = \left|(\alpha-\lambda)(u+iv-1)-(1-\alpha)\right| \\ &\quad \geqq 1-\alpha-(\lambda-\alpha)\left|u+iv-1\right| \\ &\quad \geqq 1-\alpha-(\lambda-\alpha)N. \end{aligned} \tag{32}$$

It follows from (31) and (32) that

$$\begin{aligned} E \geqq{} & \left(u^2+v^2\right)(1-\alpha)^2\beta^2+2(1-\lambda)(1-\alpha)\beta v \\ & +\left[1-\alpha-(\lambda-\alpha)N\right]^2. \end{aligned} \tag{33}$$

We now set

$$\begin{aligned} F(\beta) &:= E-M^2 \\ &\geqq \left(u^2+v^2\right)(1-\alpha)^2\beta^2+2(1-\lambda)(1-\alpha)v\beta \\ &\quad +\left[1-\alpha-(\lambda-\alpha)N\right]^2-(\lambda+n-1)^2N^2. \end{aligned} \tag{34}$$

If $F(\beta) \geqq 0$, then (29) holds true. Since $(u^2+v^2)(1-\alpha)^2 > 0$, the inequality $F(\beta) \geqq 0$ holds if the discriminant $\Delta \leqq 0$; that is,

$$\begin{aligned} \Delta &= (1-\alpha)^2 \\ &\quad \times\left\{(1-\lambda)^2v^2-\left(u^2+v^2\right)\right. \\ &\quad \left.\times\left[(1-\alpha-(\lambda-\alpha)N)^2-(\lambda+n-1)^2N^2\right]\right\} \leqq 0, \end{aligned} \tag{35}$$

and the last inequality is equivalent to

$$\begin{aligned} & v^2\left[(1-\lambda)^2-(1-\alpha-(\lambda-\alpha)N)^2+(\lambda+n-1)^2N^2\right] \\ &\quad \leqq u^2\left[(1-\alpha-(\lambda-\alpha)N)^2-(\lambda+n-1)^2N^2\right]. \end{aligned} \tag{36}$$

Furthermore, in view of (24) and (36), after a geometric argument, we deduce that

$$\begin{aligned} \frac{v^2}{u^2} &\leqq \frac{N^2}{1-N^2} \\ &\leqq \frac{(1-\alpha-(\lambda-\alpha)N)^2-(\lambda+n-1)^2N^2}{(1-\lambda)^2-(1-\alpha-(\lambda-\alpha)N)^2+(\lambda+n-1)^2N^2}. \end{aligned} \tag{37}$$

It follows from (37) that $\Delta \leqq 0$, which implies that $F(\beta) \geqq 0$. But this contradicts (28). Therefore, we know that $\Re(p(z)) > 0$ in $\mathbb{U}$. By virtue of (26), we conclude that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\Re\left((1-\alpha)p(z)+\alpha\right) < -\alpha. \tag{38}$$

This evidently completes the proof of Theorem 2. □

Taking $\alpha = 0$ in Theorem 2, we obtain the following result.

Corollary 3. *Let $n \in \mathbb{N}$ and $\lambda > 1$. If $f \in \Sigma_n$ satisfies the inequality*

$$\left|z^2f'(z)+\lambda zf(z)+1-\lambda\right| < \frac{(\lambda+n-1)}{\lambda+\sqrt{(1-\lambda)^2+(\lambda+n-1)^2}}, \tag{39}$$

then $f \in \mathcal{MS}_n^*$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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