

Research Article

On Sufficient Conditions for the Existence of Past-Present-Future Dependent Fixed Point in the Razumikhin Class and Application

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We introduce the new type of nonself mapping and study sufficient conditions for the existence of past-present-future (for short PPF) dependent fixed point for such mapping in the Razumikhin class. Also, we apply our result to prove the PPF dependent coincidence point theorems. Finally, we use PPF dependence techniques to obtain solution for a nonlinear integral problem with delay.

1. Introduction

It is well known that many problems in many branches of mathematics, such as optimization problems, equilibrium problems, and variational problems, can be transformed to fixed point problem of the form $Tx = x$ for self-mapping T defined on framework of metric space (X, d) or vector space $(X, \|\cdot\|)$. Therefore, the applications of fixed point theory are very important in diverse disciplines of mathematics. The famous Banach's contraction mapping principle is one of the cornerstones in the development of fixed point theory. From inspiration of this work, several researchers heavily studied this field. For example, see works of Kannan [1], Chatterjea [2], Berinde [3], Ćirić [4], Geraghty [5], Meir and Keeler [6], Suzuki [7], Mizogushi and Takahashi [8], Dass and Gupta [9], Jaggi [10], Lou [11], and so forth.

On the other hand, Bernfeld et al. [12] introduced the concept of Past-Present-Future (for short PPF) dependent fixed point or the fixed point with PPF dependence which is one type of fixed points for mappings that have different domains and ranges. They also established the existence of PPF dependent fixed point theorems in the Razumikhin class for

Banach type contraction mappings. These results are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data, and future consideration. The generalizations of this result have been investigated heavily by many mathematicians (see [13–18] and references therein).

In this paper, we will introduce the new type of nonself mapping called Ciric-rational type contraction mapping. Also, we will study the sufficient conditions for the existence of PPF dependent fixed point theorems for such mapping in Razumikhin class. Furthermore, we apply the main result to the existence of PPF dependence coincidence point theorems. In the last section, an application to an integral problem with delay is also given.

2. Preliminaries

In this section, we recall some concepts and definitions that will be required in the sequel. Throughout this paper, let E denote a Banach space with the norm $\|\cdot\|_E$, I denote a closed interval $[a, b]$ in \mathbb{R} , and $E_0 = C(I, E)$ denote the set of all

continuous E -valued functions on I equips with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t \in I} \|\phi(t)\|_E. \tag{1}$$

A point $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of a nonself mapping $T : E_0 \rightarrow E$ if $T\phi = \phi(c)$ for some $c \in I$.

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in E_0 is defined by

$$\mathcal{R}_c := \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\}. \tag{2}$$

It is easy to see that constant functions are member of \mathcal{R}_c .

The class \mathcal{R}_c is algebraically closed with respect to difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Similarly, \mathcal{R}_c is topologically closed if it is closed with respect to the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Definition 1 (see Bernfeld et al. [12]). The mapping $T : E_0 \rightarrow E$ is said to be Banach type contraction if there exists a real number $\alpha \in [0, 1)$ such that

$$\|T\phi - T\xi\|_E \leq \alpha \|\phi - \xi\|_{E_0} \tag{3}$$

for all $\phi, \xi \in E_0$.

The following PPF dependent fixed point theorem is proved by Bernfeld et al. [12].

Theorem 2 (see Bernfeld et al. [12]). *Let $T : E_0 \rightarrow E$ be a Banach type contraction. If \mathcal{R}_c is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in \mathcal{R}_c .*

3. PPF Dependent Fixed Point Theorems

In this section, we introduce the concept of the Ciric-rational type contraction mappings. Also, we study sufficient condition for the existence of PPF dependent fixed point for such mapping.

Definition 3. The mapping $T : E_0 \rightarrow E$ is called Ciric-rational type contraction if there exist real numbers $\alpha, \beta, \gamma, \delta, \kappa \in [0, 1)$ with $\alpha + \beta + \gamma + 2\delta + \kappa < 1$ and $c \in I$ such that

$$\begin{aligned} & \|T\phi - T\xi\|_E \\ & \leq \alpha \max \left\{ \|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E \|\xi(c) - T\xi\|_E, \right. \\ & \quad \left. \frac{\|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}{2} \right\} \\ & + \frac{\beta \|\phi(c) - T\phi\|_E \|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\ & + \frac{\gamma \|\phi(c) - T\xi\|_E \|\xi(c) - T\phi\|_E}{1 + \|\phi - \xi\|_{E_0}} \end{aligned}$$

$$\begin{aligned} & + \frac{\delta \|\phi(c) - T\phi\|_E \|\phi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\ & + \frac{\kappa \|\xi(c) - T\xi\|_E \|\xi(c) - T\phi\|_E}{1 + \|\phi - \xi\|_{E_0}} \end{aligned} \tag{4}$$

for all $\phi, \xi \in E_0$.

Remark 4. (i) All Banach type, Kannan type, and Chatterjea type mappings are Ciric-rational type contraction mapping.

(ii) If $\beta = \gamma = \delta = \kappa = 0$, then Ciric-rational type contraction mapping reduces to Ciric-type contraction.

(iii) If $\alpha = 0$, then T is a generalization and improvement of rational type contraction mapping.

Here, we prove PPF dependent fixed point theorems for Ciric-rational type contraction mappings.

Theorem 5. *Let $T : E_0 \rightarrow E$ be a Ciric-rational type contraction mapping. If \mathcal{R}_c is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in \mathcal{R}_c .*

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in \mathcal{R}_c is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{5}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in \mathcal{R}_c .

Proof. Let ϕ_0 be an arbitrary function in $\mathcal{R}_c \subseteq E_0$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c). \tag{6}$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c). \tag{7}$$

By continuing this process, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c) \tag{8}$$

for all $n \in \mathbb{N}$. Since \mathcal{R}_c is algebraically closed with respect to difference, we have

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E \tag{9}$$

for all $n \in \mathbb{N}$.

Next, we will show that $\{\phi_n\}$ is a Cauchy sequence in \mathcal{R}_c .

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
 & \|\phi_n - \phi_{n+1}\|_{E_0} \\
 &= \|\phi_n(c) - \phi_{n+1}(c)\|_E \\
 &= \|T\phi_{n-1} - T\phi_n\|_E \\
 &\leq \alpha \max \left\{ \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - T\phi_{n-1}\|_E, \right. \\
 &\quad \left. \|\phi_n(c) - T\phi_n\|_E, \right. \\
 &\quad \left. \frac{\|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_{n-1}\|_E}{2} \right\} \\
 &+ \frac{\beta \|\phi_{n-1}(c) - T\phi_{n-1}\|_E \|\phi_n(c) - T\phi_n\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
 &+ \frac{\gamma \|\phi_{n-1}(c) - T\phi_n\|_E \|\phi_n(c) - T\phi_{n-1}\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
 &+ \frac{\delta \|\phi_{n-1}(c) - T\phi_{n-1}\|_E \|\phi_{n-1}(c) - T\phi_n\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
 &+ \frac{\kappa \|\phi_n(c) - T\phi_n\|_E \|\phi_n(c) - T\phi_{n-1}\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
 &= \alpha \max \left\{ \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - \phi_n(c)\|_E, \right. \\
 &\quad \left. \|\phi_n(c) - \phi_{n+1}(c)\|_E, \frac{\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E}{2} \right\} \\
 &+ \frac{\beta \|\phi_{n-1}(c) - \phi_n(c)\|_E \|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
 &+ \frac{\delta \|\phi_{n-1}(c) - \phi_n(c)\|_E \|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}} \\
 &= \alpha \max \left\{ \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - \phi_n(c)\|_E, \right. \\
 &\quad \left. \|\phi_n(c) - \phi_{n+1}(c)\|_E, \frac{\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E}{2} \right\} \\
 &+ \beta \|\phi_{n-1}(c) - \phi_n(c)\|_E \left(\frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|\phi_{n-1}(c) - \phi_n(c)\|_E} \right) \\
 &+ \delta \|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E \left(\frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|\phi_{n-1}(c) - \phi_n(c)\|_E} \right) \\
 &\leq \alpha \max \left\{ \|\phi_{n-1}(c) - \phi_n(c)\|_E, \|\phi_n(c) - \phi_{n+1}(c)\|_E, \right. \\
 &\quad \left. \frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E + \|\phi_n(c) - \phi_{n+1}(c)\|_E}{2} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &+ \beta \|\phi_{n-1}(c) - \phi_n(c)\|_E + \delta \|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E \\
 &\leq \alpha \max \{ \|\phi_{n-1}(c) - \phi_n(c)\|_E, \|\phi_n(c) - \phi_{n+1}(c)\|_E \} \\
 &+ \beta \|\phi_{n-1}(c) - \phi_n(c)\|_E + \delta \|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E \\
 &= \alpha \max \{ \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0} \} \\
 &+ \beta \|\phi_{n-1} - \phi_n\|_{E_0} + \delta \|\phi_{n-1} - \phi_{n+1}\|_{E_0} \\
 &\leq \alpha \max \{ \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0} \} \\
 &+ \beta \|\phi_{n-1} - \phi_n\|_{E_0} + \delta \|\phi_{n-1} - \phi_n\|_{E_0} + \delta \|\phi_n - \phi_{n+1}\|_{E_0}. \tag{10}
 \end{aligned}$$

For fixed $n \in \mathbb{N}$, if $\max\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\} = \|\phi_{n-1} - \phi_n\|_{E_0}$, then we get

$$\begin{aligned}
 \|\phi_n - \phi_{n+1}\|_{E_0} &\leq \alpha \|\phi_{n-1} - \phi_n\|_{E_0} + \beta \|\phi_{n-1} - \phi_n\|_{E_0} \\
 &+ \delta \|\phi_{n-1} - \phi_n\|_{E_0} + \delta \|\phi_n - \phi_{n+1}\|_{E_0}. \tag{11}
 \end{aligned}$$

This implies that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq \left(\frac{\alpha + \beta + \delta}{1 - \delta} \right) \|\phi_{n-1} - \phi_n\|_{E_0}. \tag{12}$$

On the other hand, if $\max\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\} = \|\phi_n - \phi_{n+1}\|_{E_0}$, then we get

$$\begin{aligned}
 \|\phi_n - \phi_{n+1}\|_{E_0} &\leq \alpha \|\phi_n - \phi_{n+1}\|_{E_0} + \beta \|\phi_{n-1} - \phi_n\|_{E_0} \\
 &+ \delta \|\phi_{n-1} - \phi_n\|_{E_0} + \delta \|\phi_n - \phi_{n+1}\|_{E_0}. \tag{13}
 \end{aligned}$$

This implies that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq \left(\frac{\beta + \delta}{1 - \alpha - \delta} \right) \|\phi_{n-1} - \phi_n\|_{E_0}. \tag{14}$$

Now, we let

$$k := \max \left\{ \frac{\alpha + \beta + \delta}{1 - \delta}, \frac{\beta + \delta}{1 - \alpha - \delta} \right\}. \tag{15}$$

From (12) and (14), we get

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k \|\phi_{n-1} - \phi_n\|_{E_0} \tag{16}$$

for all $n \in \mathbb{N}$. Repeated application of the above relation yields

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n \|\phi_0 - \phi_1\|_{E_0} \tag{17}$$

for all $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}
 \|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} \\
 &+ \dots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
 &\leq (k^n + k^{n+1} + \dots + k^{m-1}) \|\phi_0 - \phi_1\|_{E_0} \tag{18} \\
 &\leq \frac{k^n}{1 - k} \|\phi_0 - \phi_1\|_{E_0}.
 \end{aligned}$$

Since $0 \leq \alpha + \beta + \gamma + 2\delta + \kappa < 1$, we have $0 \leq k < 1$. This shows that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By the completeness of E_0 , we get $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$. Therefore, $\lim_{n \rightarrow \infty} \phi_n = \phi^*$; that is,

$$\lim_{n \rightarrow \infty} \|\phi_n - \phi^*\|_{E_0} = 0_{E_0}. \quad (19)$$

Further, since \mathcal{R}_c is topologically closed, we have $\phi^* \in \mathcal{R}_c$ and thus

$$\lim_{n \rightarrow \infty} \|\phi_n(c) - \phi^*(c)\|_E = 0_E. \quad (20)$$

Now we prove that ϕ^* is a PPF dependent fixed point of T . From the assumption of Ciric-rational type contraction of T , we get

$$\begin{aligned} & \|T\phi^* - \phi^*(c)\|_E \\ & \leq \|T\phi^* - \phi_{n+1}(c)\|_E + \|\phi_{n+1}(c) - \phi^*(c)\|_E \\ & = \|T\phi^* - T\phi_n\|_E + \|\phi_{n+1} - \phi^*\|_{E_0} \\ & \leq \alpha \max \left\{ \|\phi^* - \phi_n\|_{E_0}, \|\phi^*(c) - T\phi^*\|_E, \right. \\ & \quad \|\phi_n(c) - T\phi_n\|_E, \\ & \quad \left. \frac{\|\phi^*(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi^*\|_E}{2} \right\} \\ & + \frac{\beta \|\phi^*(c) - T\phi^*\|_E \|\phi_n(c) - T\phi_n\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \\ & + \frac{\gamma \|\phi^*(c) - T\phi_n\|_E \|\phi_n(c) - T\phi^*\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \\ & + \frac{\delta \|\phi^*(c) - T\phi^*\|_E \|\phi^*(c) - T\phi_n\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \\ & + \frac{\kappa \|\phi_n(c) - T\phi_n\|_E \|\phi_n(c) - T\phi^*\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \\ & + \|\phi_{n+1} - \phi^*\|_{E_0} \\ & = \alpha \max \left\{ \|\phi^* - \phi_n\|_{E_0}, \|\phi^*(c) - T\phi^*\|_E, \right. \\ & \quad \|\phi_n(c) - \phi_{n+1}(c)\|_E, \\ & \quad \left. \frac{\|\phi^*(c) - \phi_{n+1}(c)\|_E + \|\phi_n(c) - T\phi^*\|_E}{2} \right\} \\ & + \frac{\beta \|\phi^*(c) - T\phi^*\|_E \|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \end{aligned}$$

$$\begin{aligned} & + \frac{\gamma \|\phi^*(c) - \phi_{n+1}(c)\|_E \|\phi_n(c) - T\phi^*\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \\ & + \frac{\delta \|\phi^*(c) - T\phi^*\|_E \|\phi^*(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \\ & + \frac{\kappa \|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_n(c) - T\phi^*\|_E}{1 + \|\phi^* - \phi_n\|_{E_0}} \\ & + \|\phi_{n+1} - \phi^*\|_{E_0} \end{aligned} \quad (21)$$

for all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ in the above inequality, by (19) and (20), we have

$$\|T\phi^* - \phi^*(c)\|_E \leq \alpha \|T\phi^* - \phi^*(c)\|_E. \quad (22)$$

This implies that

$$\|T\phi^* - \phi^*(c)\|_E = 0 \quad (23)$$

and then

$$T\phi^* = \phi^*(c). \quad (24)$$

Therefore, ϕ^* is a PPF dependent fixed point of T in \mathcal{R}_c .

Finally, we prove the uniqueness of PPF dependent fixed point of T in \mathcal{R}_c . Let ϕ^* and ξ^* be two PPF dependent fixed points of T in \mathcal{R}_c . Therefore,

$$\begin{aligned} & \|\phi^* - \xi^*\|_{E_0} \\ & = \|\phi^*(c) - \xi^*(c)\|_E \\ & = \|T\phi^* - T\xi^*\|_E \\ & \leq \alpha \max \left\{ \|\phi^* - \xi^*\|_{E_0}, \|\phi^*(c) - T\phi^*\|_E, \right. \\ & \quad \|\xi^*(c) - T\xi^*\|_E, \\ & \quad \left. \frac{\|\phi^*(c) - T\xi^*\|_E + \|\xi^*(c) - T\phi^*\|_E}{2} \right\} \\ & + \frac{\beta \|\phi^*(c) - T\phi^*\|_E \|\xi^*(c) - T\xi^*\|_E}{1 + \|\phi^* - \xi^*\|_{E_0}} \\ & + \frac{\gamma \|\phi^*(c) - T\xi^*\|_E \|\xi^*(c) - T\phi^*\|_E}{1 + \|\phi^* - \xi^*\|_{E_0}} \\ & + \frac{\delta \|\phi^*(c) - T\phi^*\|_E \|\phi^*(c) - T\xi^*\|_E}{1 + \|\phi^* - \xi^*\|_{E_0}} \\ & + \frac{\kappa \|\xi^*(c) - T\xi^*\|_E \|\xi^*(c) - T\phi^*\|_E}{1 + \|\phi^* - \xi^*\|_{E_0}} \\ & = \alpha \max \left\{ \|\phi^* - \xi^*\|_{E_0}, \right. \\ & \quad \left. \frac{\|\phi^*(c) - \xi^*(c)\|_E + \|\xi^*(c) - \phi^*(c)\|_E}{2} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{\gamma \|\phi^*(c) - T\xi^*\|_E \|\xi^*(c) - T\phi^*\|_E}{1 + \|\phi^* - \xi^*\|_{E_0}} \\
 = & \alpha \max \left\{ \|\phi^* - \xi^*\|_{E_0}, \frac{\|\phi^* - \xi^*\|_{E_0} + \|\xi^* - \phi^*\|_{E_0}}{2} \right\} \\
 & + \frac{\gamma \|\phi^*(c) - \xi^*(c)\|_E \|\xi^*(c) - \phi^*(c)\|_E}{1 + \|\phi^* - \xi^*\|_{E_0}} \\
 = & \alpha \|\phi^* - \xi^*\|_{E_0} + \frac{\gamma \|\phi^* - \xi^*\|_{E_0} \|\xi^* - \phi^*\|_{E_0}}{1 + \|\phi^* - \xi^*\|_{E_0}} \\
 = & \alpha \|\phi^* - \xi^*\|_{E_0} + \gamma \|\phi^* - \xi^*\|_{E_0} \left(\frac{\|\phi^* - \xi^*\|_{E_0}}{1 + \|\phi^* - \xi^*\|_{E_0}} \right) \\
 \leq & \alpha \|\phi^* - \xi^*\|_{E_0} + \gamma \|\phi^* - \xi^*\|_{E_0}. \tag{25}
 \end{aligned}$$

Since $0 \leq \alpha + \gamma < 1$, we have $\|\phi^* - \xi^*\|_{E_0} = 0$ and hence $\phi^* = \xi^*$. Therefore, T has a unique PPF dependent fixed point in \mathcal{R}_c . This completes the proof. \square

Remark 6. If the Razumikhin class \mathcal{R}_c is not topologically closed, then the limit of the sequence $\{\phi_n\}$ in Theorem 5 may be outside of \mathcal{R}_c . Therefore, a PPF dependent fixed point of T may not be unique.

By applying Theorem 5, we obtain the following result.

Corollary 7. Let $T : E_0 \rightarrow E$ be a nonself mapping and there exists a real number $\alpha \in [0, 1)$ such that

$$\|T\phi - T\xi\|_E \leq \alpha \|\phi - \xi\|_{E_0} \tag{26}$$

for all $\phi, \xi \in E_0$.

If there exists $c \in I$ such that \mathcal{R}_c is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in \mathcal{R}_c .

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in \mathcal{R}_c is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{27}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in \mathcal{R}_c .

If we set $\beta = \gamma = \delta = \kappa = 0$ in Theorem 5, we get the PPF dependent fixed point result for Ciric-type contraction mapping.

Corollary 8. Let $T : E_0 \rightarrow E$ be a nonself mapping and there exist real number $0 \leq \alpha < 1$ and $c \in I$ such that

$$\begin{aligned}
 & \|T\phi - T\xi\|_E \\
 \leq & \alpha \max \left\{ \|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E \|\xi(c) - T\xi\|_E, \right. \\
 & \left. \frac{\|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}{2} \right\} \tag{28}
 \end{aligned}$$

for all $\phi, \xi \in E_0$.

If \mathcal{R}_c is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in \mathcal{R}_c .

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in \mathcal{R}_c is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{29}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in \mathcal{R}_c .

If we set $\alpha = 0$ in Theorem 5, we get the PPF dependent fixed point result for generalized ratio type contraction mapping.

Corollary 9. Let $T : E_0 \rightarrow E$ be a nonself mapping and there exist real numbers $\beta, \gamma, \delta, \kappa \in [0, 1)$ with $\beta + \gamma + 2\delta + \kappa < 1$ and $c \in I$ such that

$$\begin{aligned}
 \|T\phi - T\xi\|_E \leq & \frac{\beta \|\phi(c) - T\phi\|_E \|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\
 & + \frac{\gamma \|\phi(c) - T\xi\|_E \|\xi(c) - T\phi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\
 & + \frac{\delta \|\phi(c) - T\phi\|_E \|\phi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}} \\
 & + \frac{\kappa \|\xi(c) - T\xi\|_E \|\xi(c) - T\phi\|_E}{1 + \|\phi - \xi\|_{E_0}} \tag{30}
 \end{aligned}$$

for all $\phi, \xi \in E_0$.

If \mathcal{R}_c is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in \mathcal{R}_c .

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in \mathcal{R}_c is defined by

$$T\phi_{n-1} = \phi_n(c) \tag{31}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in \mathcal{R}_c .

4. PPF Dependent Coincidence Point Theorems

Definition 10. Let $T : E_0 \rightarrow E$ and $S : E_0 \rightarrow E_0$ be two nonself mappings. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point or a coincidence point with PPF dependence of T and S if $T\phi = (S\phi)(c)$ for some $c \in I$.

Next, we introduce the condition of the Ciric-rational type contraction for a pair of two nonself mappings.

Definition 11. Let $T : E_0 \rightarrow E$ and $S : E_0 \rightarrow E_0$ be two nonself mappings. The ordered pair (T, S) is said to satisfy the condition of Ciric-rational type contraction if there exist real

numbers $\alpha, \beta, \gamma, \delta, \kappa \in [0, 1)$ with $\alpha + \beta + \gamma + 2\delta + \kappa < 1$ and $c \in I$ such that

$$\begin{aligned} & \|T\phi - T\xi\|_E \\ & \leq \alpha \max \left\{ \|S\phi - S\xi\|_{E_0}, \right. \\ & \quad \left. \begin{aligned} & \| (S\phi)(c) - T\phi \|_E \| (S\xi)(c) - T\xi \|_E, \\ & \frac{\| (S\phi)(c) - T\xi \|_E + \| (S\xi)(c) - T\phi \|_E}{2} \end{aligned} \right\} \\ & + \frac{\beta \| (S\phi)(c) - T\phi \|_E \| (S\xi)(c) - T\xi \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\gamma \| (S\phi)(c) - T\xi \|_E \| (S\xi)(c) - T\phi \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\delta \| (S\phi)(c) - T\phi \|_E \| (S\phi)(c) - T\xi \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\kappa \| (S\xi)(c) - T\xi \|_E \| (S\xi)(c) - T\phi \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \end{aligned} \tag{32}$$

for all $\phi, \xi \in E_0$.

Remark 12. It is easy to see that

(T, S) satisfies the condition of Ciric-rational type contraction and S is identity mapping

$$\Downarrow$$

T is a Ciric-rational type contraction mapping.

Now, we apply our result to the previous section to the PPF dependent coincidence point theorem.

Theorem 13. Let $T : E_0 \rightarrow E$ and $S : E_0 \rightarrow E_0$ be two nonself mappings. Suppose that the following conditions hold:

- (\star_1) (T, S) satisfies the condition of Ciric-rational type contraction;
- (\star_2) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point.

Proof. For self-mapping $S : E_0 \rightarrow E_0$, it is well know that there exists $F_0 \subset E_0$ such that $S(F_0) = S(E_0)$ and $S|_{F_0}$ is one-to-one. Since

$$T(F_0) \subseteq T(E_0) \subseteq E, \tag{33}$$

we can define a nonself mapping $\mathcal{H} : S(F_0) \rightarrow E$ by

$$\mathcal{H}(S\phi) = T\phi \tag{34}$$

for all $\phi \in F_0$. Since S_{F_0} is one-to-one mapping, we have \mathcal{H} is well-defined.

By the condition of Ciric-rational type contraction of (T, S) and the construction of \mathcal{H} , we get

$$\begin{aligned} & \| \mathcal{H}(S\phi) - \mathcal{H}(S\xi) \|_E \\ & \leq \alpha \max \left\{ \|S\phi - S\xi\|_{E_0}, \right. \\ & \quad \left. \begin{aligned} & \| (S\phi)(c) - \mathcal{H}(S\phi) \|_E \| (S\xi)(c) - \mathcal{H}(S\xi) \|_E, \\ & \frac{\| (S\phi)(c) - \mathcal{H}(S\xi) \|_E + \| (S\xi)(c) - \mathcal{H}(S\phi) \|_E}{2} \end{aligned} \right\} \\ & + \frac{\beta \| (S\phi)(c) - \mathcal{H}(S\phi) \|_E \| (S\xi)(c) - \mathcal{H}(S\xi) \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\gamma \| (S\phi)(c) - \mathcal{H}(S\xi) \|_E \| (S\xi)(c) - \mathcal{H}(S\phi) \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\delta \| (S\phi)(c) - \mathcal{H}(S\phi) \|_E \| (S\phi)(c) - \mathcal{H}(S\xi) \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\kappa \| (S\xi)(c) - \mathcal{H}(S\xi) \|_E \| (S\xi)(c) - \mathcal{H}(S\phi) \|_E}{1 + \|S\phi - S\xi\|_{E_0}} \end{aligned} \tag{35}$$

for all $S\phi, S\xi \in S(E_0)$. This implies that \mathcal{H} is a Ciric-rational type contraction mapping.

Using Theorem 5 with a mapping \mathcal{H} , we can find a unique PPF dependent fixed point of \mathcal{H} . Let a unique PPF dependent fixed point of \mathcal{H} be $\zeta \in S(F_0)$; that is, $\mathcal{H}\zeta = \zeta(c)$. Since $\zeta \in S(F_0)$, we can find $\omega \in F_0$ such that $\zeta = S\omega$. Now, we have

$$T\omega = \mathcal{H}(S\omega) = \mathcal{H}\zeta = \zeta(c) = (S\omega)(c). \tag{36}$$

Therefore, ω is a PPF dependent coincidence point of T and S . This completes the proof. \square

By applying Theorem 13, we obtain the following corollaries.

Corollary 14. Let $T : E_0 \rightarrow E$ and $S : E_0 \rightarrow E_0$ be two nonself mappings. Suppose that the following conditions hold:

- (\star_1) there exists a real number $\alpha \in [0, 1)$ such that

$$\|T\phi - T\xi\|_E \leq \alpha \|S\phi - S\xi\|_{E_0} \tag{37}$$

for all $\phi, \xi \in E_0$;

- (\star_2) there exists $c \in I$ such that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point in \mathcal{R}_c .

Corollary 15. Let $T : E_0 \rightarrow E$ and $S : E_0 \rightarrow E_0$ be two nonself mappings. Suppose that the following conditions hold:

(\star_1) there exist real numbers $0 \leq \alpha < 1$ and $c \in I$ such that

$$\begin{aligned} & \|T\phi - T\xi\|_E \\ & \leq \alpha \max \left\{ \|S\phi - S\xi\|_{E_0}, \right. \\ & \quad \|(S\phi)(c) - T\phi\|_E \|(S\xi)(c) - T\xi\|_E, \\ & \quad \left. \frac{\|(S\phi)(c) - T\phi\|_E + \|(S\xi)(c) - T\phi\|_E}{2} \right\} \end{aligned} \tag{38}$$

for all $\phi, \xi \in E_0$;
 (\star_2) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point in \mathcal{R}_c .

Corollary 16. Let $T : E_0 \rightarrow E$ and $S : E_0 \rightarrow E_0$ be two nonself mappings. Suppose that the following conditions hold:

(\star_1) there exist real numbers $\beta, \gamma, \delta, \kappa \in [0, 1)$ with $\beta + \gamma + 2\delta + \kappa < 1$ and $c \in I$ such that

$$\begin{aligned} \|T\phi - T\xi\|_E & \leq + \frac{\beta \|(S\phi)(c) - T\phi\|_E \|(S\xi)(c) - T\xi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\gamma \|(S\phi)(c) - T\xi\|_E \|(S\xi)(c) - T\phi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\delta \|(S\phi)(c) - T\phi\|_E \|(S\phi)(c) - T\xi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \\ & + \frac{\kappa \|(S\xi)(c) - T\xi\|_E \|(S\xi)(c) - T\phi\|_E}{1 + \|S\phi - S\xi\|_{E_0}} \end{aligned} \tag{39}$$

for all $\phi, \xi \in E_0$;
 (\star_2) $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point in \mathcal{R}_c .

5. Application to a Nonlinear Integral Equation

In this section, we apply our result to study the existence and uniqueness of solution of a nonlinear integral equation.

Given a closed interval $J := [j, 0]$ such that $j \in \mathbb{R}^-$, let Ω_0 denote the space of continuous real-valued functions defined on J . We equip the space Ω_0 with supremum normed $\|\cdot\|_{\Omega_0}$ defined by

$$\|\phi\|_{\Omega_0} = \sup_{t \in J} |\phi(t)|. \tag{40}$$

It well known that Ω_0 is a Banach space with this normed.

For fixed $T \in \mathbb{R}^+$, for each $t \in I := [0, T]$, define a function $t \mapsto \phi_t$ by

$$\phi_t(a) = \phi(t+a), \quad \text{for } a \in J, \tag{41}$$

where the argument a represents the delay in the argument of solutions.

Given $\varsigma \in C(I, \mathbb{R})$, we will consider the following nonlinear integral problem:

$$\phi(t) = \varsigma(0) + \int_0^T G(T, s) f(s, \phi_s) ds \tag{42}$$

for all $t \in I$, where $\phi \in C(I, \mathbb{R})$, $f \in C(I \times C(J, \mathbb{R}), \mathbb{R})$, and $G \in C(I \times I, \mathbb{R}_+)$.

Theorem 17. Problem (42) has only one solution defined on $J \cup I$ if the following conditions hold:

$$(\heartsuit_1) \sup_{t \in I} \left(\int_0^t G(t, s) ds \right) \leq 1,$$

(\heartsuit_2) there exist nonnegative real number $\alpha < 1$ such that, for all $t \in I$ and $\phi, \xi \in C(I, \mathbb{R})$, one has

$$|f(t, \phi) - f(t, \xi)| \leq \alpha |\phi(0) - \xi(0)|. \tag{43}$$

Proof. Define the following set:

$$\widehat{E} := \{ \widehat{\phi} = (\phi_t)_{t \in I} : \phi_t \in \Omega_0, \phi \in C(I, \mathbb{R}) \}. \tag{44}$$

Also, define the normed $\|\cdot\|_{\widehat{E}}$ in \widehat{E} by

$$\|\widehat{\phi}\|_{\widehat{E}} := \sup_{t \in I} \|\phi_t\|_{\Omega_0}. \tag{45}$$

We obtain that $\widehat{\phi} \in C(J, \mathbb{R})$. Next, we show that \widehat{E} is complete. Consider a Cauchy sequence $\{\widehat{\phi}_n\}$ in \widehat{E} . It is easy to see that $\{\phi_{n_t}\}_{t \in I}$ is a Cauchy sequence in $C(J, \mathbb{R})$ for all $t \in I$. This implies that $\{\phi_{n_t}(s)\}$ is a Cauchy sequence in \mathbb{R} for each $s \in J$. So $\phi_{n_t}(s)$ converges to $\phi_t(s)$ for each $s \in J$. Since $\{\phi_{n_t}\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $\phi_t(s)$ is also continuous in $s \in J$. Thus going backwards we get that $\widehat{\phi}_n$ converges to $\widehat{\phi}$ in \widehat{E} . Therefore, \widehat{E} is complete.

Next, we define the function $T : \widehat{E} \rightarrow \mathbb{R}$ by

$$T\widehat{\phi} \equiv T(\phi_t)_{t \in I} := \varsigma(0) + \int_0^T G(T, s) f(s, \phi_s) ds. \tag{46}$$

For $\widehat{\phi}, \widehat{\xi} \in \widehat{E}$, we have

$$\begin{aligned}
 |T\widehat{\phi} - T\widehat{\xi}| &= \left| \int_0^T G(T, s) f(s, \phi_s) ds - \int_0^T G(T, s) f(s, \xi_s) ds \right| \\
 &= \left| \int_0^T (G(T, s) f(s, \phi_s) - G(T, s) f(s, \xi_s)) ds \right| \\
 &= \int_0^T (G(T, s) f(s, \phi_s) - G(T, s) f(s, \xi_s)) ds \\
 &\leq \int_0^T |G(T, s) f(s, \phi_s) - G(T, s) f(s, \xi_s)| ds \\
 &= \int_0^T G(T, s) |f(s, \phi_s) - f(s, \xi_s)| ds \\
 &\leq \int_0^T G(T, s) \alpha |\phi_s(0) - \xi_s(0)| ds \\
 &= \int_0^T G(T, s) \alpha |\phi(s) - \xi(s)| ds \\
 &\leq \int_0^T G(T, s) \alpha \|\widehat{\phi} - \widehat{\xi}\|_{\widehat{E}} ds \\
 &= \alpha \|\widehat{\phi} - \widehat{\xi}\|_{\widehat{E}} \left(\int_0^T G(T, s) ds \right) \\
 &\leq \alpha \|\widehat{\phi} - \widehat{\xi}\|_{\widehat{E}} \left[\sup_{t \in I} \left(\int_0^t G(t, s) ds \right) \right] \\
 &\leq \alpha \|\widehat{\phi} - \widehat{\xi}\|_{\widehat{E}}.
 \end{aligned} \tag{47}$$

This implies that T is a Ciric-rational type contraction.

Moreover, the Razumikhin \mathcal{R}_0 is $C(I, \mathbb{R})$ which is topologically closed and algebraically closed with respect to difference. Now all hypotheses of Theorem 5 are automatically satisfied with $c = 0$. Therefore, there exists PPF dependence fixed point $\widehat{\phi}^*$ of T ; that is, $T\widehat{\phi}^* = \widehat{\phi}^*(0)$. This implies that

$$\varsigma(0) + \int_0^T G(T, s) f(s, \widehat{\phi}_s^*) ds = (\widehat{\phi}_t^*(0))_{t \in I} = (\widehat{\phi}^*(t))_{t \in I}. \tag{48}$$

Hence, the integral equation (42) has a solution. This completes the proof. \square

Conflict of Interests

The authors declare that they have no competing interests.

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