Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 321328, 10 pages
http://dx.doi.org/10.1155/2014/321328

Research Article

C^n-Almost Periodic Functions and an Application to a Lasota-Wazewska Model on Time Scales

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Received 10 February 2014; Accepted 4 May 2014; Published 19 May 2014

Academic Editor: Shiping Lu

We first give the definition and some properties of C^n-almost periodic functions on time scales. Then, as an application, we are concerned with a class of Lasota-Wazewska models on time scales. By means of the fixed point theory and differential inequality techniques on time scales, we obtain some sufficient conditions ensuring the existence and global exponential stability of C^1-almost periodic solutions for the considered model. Our results are essentially new when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. Finally, we present a numerical example to show the feasibility of obtained results.

1. Introduction

Between the years 1923 and 1926, Harald Bohr found a theory of almost periodic real (and complex) functions. Several generalizations and classes of almost periodic functions have been introduced in the literature, including pseudo-almost periodic functions and almost automorphic functions [1, 2].

The author of [3] initiated the study on C^n-almost periodic functions, which turns out to be one of the most important generalizations of the concept of almost periodic functions in the sense of Bohr. This generalization relies on the requirement that a given function and its derivatives up to the nth order inclusively are almost periodic in the sense of Bohr. Many properties of such functions with real values are given in [3, 4]. Recently, C^n-almost periodic functions have attracted more and more attention. For example, the authors of [5] extended the study on C^n-almost periodicity to functions $\mathbb{R} \to \mathbb{X}$, where $\mathbb{X}$ is a Banach space; in [6], the authors proved the existence of C^n-almost periodic solutions for some ordinary differential equations by using the exponential dichotomy approach. For more results on C^n-almost periodic functions, we refer readers to [7–9] and references therein.

On the other hand, the theory of time scales, which was introduced by Hilger [10] in his Ph.D. thesis in order to unify continuous and discrete analysis, has recently received lots of attention. The study of dynamic equations on time scales helps avoid proving results twice, once for differential equations and once for difference equations. Many authors obtained a lot of good results on the study on dynamic equations on time scales (see [11–17] and reference therein). In [18], the authors proposed the concept of almost periodic time scales and the definition of almost periodic functions. They extended the study on almost periodicity to functions $\mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is an almost periodic time scale. However, to the best of our knowledge, there is no paper published on the existence of C^n-almost periodic solutions of dynamic equations on time scales.

Motivated by the above discussion, in this paper, we first give the definition and some properties of C^n-almost periodic functions on time scales. As an application, we are concerned with the existence and global exponential stability of C^1-almost periodic solutions for the following Lasota-Wazewska model on time scales:

$$x_i^{\Delta}(t) = -a_i(t)\,x_i(t) + \sum_{j=1}^{m} b_{ij}(t)\exp\left\{-c_i(t)\,x_i\left(t-\tau_{ij}(t)\right)\right\}, \quad i = 1, 2, \ldots, m, \tag{1}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an almost periodic time scale, $x_i(t)$ denotes the number of red blood cells at time t, a_i is the rate of the red blood cells, b_{ij} and c_i describe the production of red blood cells per unite time, and $\tau_{ij} > 0$ is the time required to produce a red blood cell and satisfies $t - \tau_{ij}(t) \in \mathbb{T}$ for $t \in \mathbb{T}$, $i, j = 1, 2, \ldots, m$. There is extensive literature concerning oscillation, global attractivity, periodicity, almost periodicity, and Hopf bifurcation of Lasota-Wazewska model, which was proposed to describe the survival of red blood cells in animals [19]. We refer readers to [20–25] and references therein for results on Lasota-Wazewska models.

Due to the biological meaning of (1), we just consider the following initial conditions:

$$x_i(s) = \varphi_i(s), \quad s \in [t_0 - \theta, t_0]_{\mathbb{T}}, \ t_0 \in \mathbb{T}, \tag{2}$$

where $\varphi_i \in C^1([t_0 - \theta, t_0]_{\mathbb{T}}, \mathbb{R}^+)$ is bounded, $\theta = \max_{(i,j)} \sup_{t \in \mathbb{T}}\{\tau_{ij}(t)\}$. Throughout this paper, we denote $[a, b]_{\mathbb{T}} = \{t \mid t \in [a, b] \cap \mathbb{T}\}$.

2. Preliminaries

In this section, we introduce some definitions and state some preliminary results.

Definition 1 (see [10]). Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to [0, \infty)$ are defined, respectively, by

$$\begin{aligned} \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\}, \\ \rho(t) &= \sup\{s \in \mathbb{T} : s < t\}, \\ \mu(t) &= \sigma(t) - t. \end{aligned} \tag{3}$$

Definition 2 (see [10]). A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 3 ([26]). A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 4 ([26]). A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\,\Delta\tau\right\}, \quad \text{for } s, t \in \mathbb{T}, \tag{4}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\operatorname{Log}(1 + hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases} \tag{5}$$

Definition 5 (see [26]). A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$; $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive provided $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ and the set of all positively regressive functions and rd-continuous functions will be denoted by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$.

Lemma 6 (see [26]). *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions; then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;
(iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(iv) $(e_p(t, s))^{\Delta} = p(t)e_p(t, s)$.

Lemma 7 (see [26]). *Let f, g be Δ-differentiable functions on $\mathbb{T}$; then*

(i) $(\nu_1 f + \nu_2 g)^{\Delta} = \nu_1 f^{\Delta} + \nu_2 g^{\Delta}$ *for any constants* ν_1, ν_2;
(ii) $(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$.

Lemma 8 (see [26] (Leibniz formula)). *Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n - k$ times Δ. If $f^{\wedge}$ exists for all $\wedge \in S_k^{(n)}$, then*

$$(fg)^{\Delta^n} = \sum_{k=0}^{n}\left(\sum_{\wedge \in S_k^{(n)}} f^{\wedge}\right) g^{\Delta^k}. \tag{6}$$

Lemma 9 (see [26]). *Suppose that $p \in \mathcal{R}^+$; then*

(i) $e_p(t, s) > 0$, *for all* $t, s \in \mathbb{T}$;
(ii) *if* $p(t) \leq q(t)$ *for all* $t \geq s$, $t, s \in \mathbb{T}$, *then* $e_p(t, s) \leq e_q(t, s)$ *for all* $t \geq s$.

Lemma 10 (see [26]). *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$\begin{aligned} \left[e_p(c, \cdot)\right]^{\Delta} &= -p\left[e_p(c, \cdot)\right]^{\sigma}, \\ \int_a^b p(t)\,e_p(c, \sigma(t))\,\Delta t &= e_p(c, a) - e_p(c, b). \end{aligned} \tag{7}$$

Lemma 11 (see [26]). *Let $r : \mathbb{T} \to \mathbb{R}$ be right-dense continuous and regressive, $a \in \mathbb{T}$ and $y_a \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$\begin{aligned} y^{\Delta}(t) &= r(t)\,y(t) + h(t), \\ y(a) &= y_a \end{aligned} \tag{8}$$

is given by

$$y(t) = e_r(t, a)\,y_a + \int_a^t e_r(t, \sigma(s))\,h(s)\,\Delta s. \tag{9}$$

Definition 12 (see [18]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t + \tau \in \mathbb{T}, \ \forall t \in \mathbb{T}\} \neq \{0\}. \tag{10}$$

Throughout this paper, we restrict our results on almost periodic time scales. We first recall some definitions and lemmas on almost periodic functions on almost periodic time scales, which can be found in [18].

Definition 13 (see [18]). Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{R})$ is said to be almost periodic on $\mathbb{T}$, if, for any $\varepsilon > 0$, the set

$$E(\varepsilon, f) = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon,\ \forall t \in \mathbb{T}\} \tag{11}$$

is relatively dense in $\mathbb{T}$; that is, for any $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one $\tau \in E(\varepsilon, f)$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, \quad \forall t \in \mathbb{T}. \tag{12}$$

The set $E(\varepsilon, f)$ is called the ε-translation set of $f(t)$, and τ is called the ε-translation number of $f(t)$.

Lemma 14 (see [18]). *If $f \in C(\mathbb{T}, \mathbb{R})$ is an almost periodic function, then f is bounded on $\mathbb{T}$.*

Lemma 15 (see [18]). *If $f, g \in C(\mathbb{T}, \mathbb{R})$ are almost periodic functions, then $f + g$, fg are also almost periodic.*

Lemma 16 (see [18]). *If $f \in C(\mathbb{T}, \mathbb{R})$ is almost periodic, then $F(t) = \int_0^t f(s)\Delta s$ is almost periodic if and only if $F(t)$ is bounded.*

Lemma 17 (see [18]). *If $f \in C(\mathbb{T}, \mathbb{R})$ is almost periodic and $F(\cdot)$ is uniformly continuous on the value field of $f(t)$, then $F \circ f$ is almost periodic.*

Definition 18 (see [27]). Let $X \in \mathbb{R}^m$ and $A(t)$ be an $m \times m$ rd-continuous matrix on $\mathbb{T}$; the linear system

$$X^{\Delta}(t) = A(t) X(t), \quad t \in \mathbb{T}, \tag{13}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants k, α, projection P, and the fundamental solution matrix $X(t)$ of (13) satisfying

$$\begin{aligned} \left|X(t) P X^{-1}(\sigma(s))\right|_0 &\le k e_{\ominus\alpha}(t, \sigma(s)), \quad s, t \in \mathbb{T},\ t \ge s, \\ \left|X(t)(I-P) X^{-1}(\sigma(s))\right|_0 &\le k e_{\ominus\alpha}(\sigma(s), t), \quad s, t \in \mathbb{T},\ t \le s, \end{aligned} \tag{14}$$

where $|\cdot|_0$ is a matrix norm on $\mathbb{T}$; that is, if $A = (a_{ij})_{m\times m}$, then we can take $|A|_0 = (\sum_{i=1}^m \sum_{j=1}^m |a_{ij}|^2)^{1/2}$.

Lemma 19 (see [18]). *If the linear system* (13) *admits an exponential dichotomy, then the following system*

$$X^{\Delta}(t) = A(t) X(t) + f(t), \quad t \in \mathbb{T}, \tag{15}$$

has a solution as follows:

$$\begin{aligned} X(t) = &\int_{-\infty}^{t} X(t) P X^{-1}(\sigma(s)) f(s) \Delta s \\ &- \int_{t}^{+\infty} X(t)(I-P) X^{-1}(\sigma(s)) f(s) \Delta s, \end{aligned} \tag{16}$$

where $X(t)$ is the fundamental solution matrix of (13).

Lemma 20 (see [18]). *Let $c_i(t)$ be a function on $\mathbb{T}$, where $c_i(t) > 0$, $-c_i(t) \in \mathscr{R}^+$, for all $t \in \mathbb{T}$ and $\min_{1\le i\le m} \inf_{t\in\mathbb{T}}\{c_i(t)\} > 0$. Then the linear system*

$$X^{\Delta}(t) = \operatorname{diag}(-c_1(t), -c_2(t), \ldots, -c_m(t)) X(t) \tag{17}$$

admits an exponential dichotomy on $\mathbb{T}$.

3. C^n-Almost Periodic Functions on Time Scales

In this section, we will state the definition and prove some properties of C^n-almost periodic functions on time scales.

We denote by $C^n(\mathbb{T}, \mathbb{R})$ the space of all functions $\mathbb{T} \to \mathbb{R}$ which have a continuous nth Δ-derivative on $\mathbb{T}$ and by $C_B^n(\mathbb{T}, \mathbb{R})$ the subspace of $C^n(\mathbb{T}, \mathbb{R})$ consisting of such functions satisfying $\sum_{i=0}^n \sup_{t\in\mathbb{T}} |f^{\Delta^i}(t)| < \infty$, where $f^{\Delta^i}(t)$ denotes the ith Δ-derivative of f and $f^{\Delta^0} = f$. It is not difficult to verify that $C_B^n(\mathbb{T}, \mathbb{R})$ is a Banach space with the norm $\|f\|_2 = \sum_{i=0}^n \sup_{t\in\mathbb{T}} |f^{\Delta^i}(t)|$.

Definition 21. Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C_B^n(\mathbb{T}, \mathbb{R})$ is said to be C^n-almost periodic on $\mathbb{T}$, if, for any $\varepsilon > 0$, the set

$$T(\varepsilon, f) = \{\tau \in \Pi : \|f(t+\tau) - f(t)\|_n < \varepsilon, \forall t \in \mathbb{T}\} \tag{18}$$

is relatively dense in $\mathbb{T}$; that is, for any $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one $\tau \in T(\varepsilon, f)$ such that

$$\|f(t+\tau) - f(t)\|_n < \varepsilon, \quad \forall t \in \mathbb{T}. \tag{19}$$

Remark 22. We denote by $AP^n(\mathbb{T}, \mathbb{R})$ the set of all C^n-almost periodic functions from $\mathbb{T}$ to $\mathbb{R}$. In particular, we denote $AP^0(\mathbb{T}, \mathbb{R})$ by $AP(\mathbb{T}, \mathbb{R})$, which is the set of all almost periodic functions from $\mathbb{T}$ to $\mathbb{R}$.

Theorem 23. *$f \in AP^n(\mathbb{T}, \mathbb{R})$ if and only if $f^{\Delta^i}(t) \in AP(\mathbb{T}, \mathbb{R})$, $i = 0, 1, \ldots, n$.*

Proof. Assume that $f \in AP^n(\mathbb{T}, \mathbb{R})$; then, for any $\varepsilon > 0$, there exists a constant l such that in any interval of length l there exists $\tau \in T(\varepsilon, f)$ such that

$$\|f(t+\tau) - f(t)\|_n = \sup_{t\in\mathbb{T}} \sum_{i=0}^{n} \left|f^{\Delta^i}(t+\tau) - f^{\Delta^i}(t)\right| < \varepsilon, \quad \forall t \in \mathbb{T}. \tag{20}$$

Hence, for $i = 0, 1, \ldots, n$, we have that

$$\left|f^{\Delta^i}(t+\tau) - f^{\Delta^i}(t)\right| < \varepsilon, \quad \forall t \in \mathbb{T}, \tag{21}$$

which means that $f^{\Delta^i}(t) \in AP(\mathbb{T}, \mathbb{R})$, $i = 0, 1, \ldots, n$. On the other hand, if $f^{\Delta^i}(t) \in AP(\mathbb{T}, \mathbb{R})$, $i = 0, 1, \ldots, n$, then, for any $\varepsilon_i > 0$, there exists a constant l such that, in any interval of length l, there exists $\tau \in E(\varepsilon_i, f)$ such that

$$\left|f^{\Delta^i}(t+\tau) - f^{\Delta^i}(t)\right| < \varepsilon_i, \quad \forall t \in \mathbb{T},\ i = 0, 1, \ldots, n. \tag{22}$$

Therefore, for any $\varepsilon = \sum_{i=0}^{n} \varepsilon_i$, there exists a constant l such that in any interval of length l there exists $\tau \in T(\varepsilon, f)$ such that

$$\|f(t+\tau) - f(t)\|_n = \sum_{i=0}^{n} \sup_{t\in\mathbb{T}} \left| f^{\Delta^i}(t+\tau) - f^{\Delta^i}(t) \right| < \varepsilon, \quad \forall t \in \mathbb{T}; \tag{23}$$

that is, $f \in C^n(\mathbb{T}, \mathbb{R})$. This completes the proof. □

Theorem 24. *If $f \in AP^n(\mathbb{T}, \mathbb{R})$, then f is bounded on $\mathbb{T}$.*

Proof. Since $f \in AP^n(\mathbb{T}, \mathbb{R})$, it follows from Theorem 23 that $f^{\Delta^i}(t) \in AP(\mathbb{T}, \mathbb{R})$, $i = 0, 1, \ldots, n$. By Lemma 14, there exist positive constants M_i such that $\sup_{t\in\mathbb{T}} |f^{\Delta^i}(t)| \le M_i$. Hence, $\|f\|_n \le \sum_{i=0}^{n} M_i$, which implies that f is bounded on $\mathbb{T}$. This completes the proof. □

Theorem 25. *If $f, g \in AP^n(\mathbb{T}, \mathbb{R})$, $\alpha \in \mathbb{R}$, then $f + g, \alpha f, fg$ are all C^n-almost periodic on $\mathbb{T}$. Moreover, if $\inf_{t\in\mathbb{T}} |g^{\Delta^i}(t)| > 0$, $i = 0, 1, \ldots, n$, then $f/g \in AP^n(\mathbb{T}, \mathbb{R})$.*

Proof. Since the proofs of $\alpha f, fg, f/g$ are similar to that of $f + g$, we only prove that $f + g \in AP^n(\mathbb{T}, \mathbb{R})$. Since $f, g \in AP^n(\mathbb{T}, \mathbb{R})$, it follows from Theorem 23 that $f^{\Delta^i}(t), g^{\Delta^j}(t) \in AP(\mathbb{T}, \mathbb{R})$, $i, j = 0, 1, \ldots, n$. By Lemma 15, $f^{\Delta^i}(t) g^{\Delta^j}(t) \in AP(\mathbb{T}, \mathbb{R})$, $i, j = 0, 1, \ldots, n$. Hence, $(fg)^{\Delta^i} \in AP(\mathbb{T}, \mathbb{R})$, $i = 0, 1, \ldots, n$, which means $fg \in AP^n(\mathbb{T}, \mathbb{R})$. □

Theorem 26. *If $f \in AP^n(\mathbb{T}, \mathbb{R})$, then $F(t) = \int_0^t f(s)\Delta s \in AP^{n+1}(\mathbb{T}, \mathbb{R})$ if and only if $F(t)$ is bounded.*

Similar to the proofs of Theorem 2.7 in [5] and Theorems 3.13 and 3.14 in [18], we have the following theorem, which is an analogue of Bochner's criterion for the case of C^n-almost periodicity on time scales.

Theorem 27. *A function $f \in AP^n(\mathbb{T}, \mathbb{R})$ if and only if for every sequence $s_n \subset \Pi$ there exists a subsequence s_n' such that $f^{\Delta^i}(t + s_n')$ converges uniformly in $t \in \mathbb{T}$, $i = 0, 1, \ldots, n$.*

Definition 28. Let $\mathbb{T}$ be an almost periodic time scale. A functions set $\mathcal{F} \subseteq AP(\mathbb{T}, \mathbb{R})$ is said to be equi-almost periodic on $\mathbb{T}$, if, for any $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one τ such that for all $f \in \mathcal{F}$

$$|f(t+\tau) - f(t)| < \varepsilon, \quad \forall t \in \mathbb{T}. \tag{24}$$

Definition 29. Let $\mathbb{T}$ be an almost periodic time scale. A functions set $\mathcal{F} \subseteq AP^n(\mathbb{T}, \mathbb{R})$ is said to be equi-C^n-almost periodic on $\mathbb{T}$, if for any $\varepsilon > 0$ there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one τ such that for all $f \in \mathcal{F}$

$$\|f(t+\tau) - f(t)\|_n < \varepsilon, \quad \forall t \in \mathbb{T}. \tag{25}$$

Similar to Theorem 2.2 in [7], we have the following theorem.

Theorem 30. *Let $\mathbb{T}$ be an almost periodic time scale. For a functions set $\mathcal{F} \subseteq AP^n(\mathbb{T}, \mathbb{R})$ is precompact if and only if $\mathcal{F}^{\Delta^i} \subseteq AP(\mathbb{T}, \mathbb{R})$ is precompact, equicontinuous, and equi-almost periodic, where $\mathcal{F}^{\Delta^i} = \{f^{\Delta^i}(t) : f \in \mathcal{F}\}$, $i = 0, 1, \ldots, n$.*

Definition 31. Let $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_m^*(t))^T$ be a C^1-almost periodic solution of (1) with initial value $\varphi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \ldots, \varphi_m^*(s))^T$. If there exist positive constants λ with $\ominus\lambda \in \mathcal{R}^+$ and $M > 1$ such that any solution $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ of (1) with initial value $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_m(s))^T$ satisfies

$$\left|x(t) - x^*(t)\right|_1 \le M \|\varphi - \varphi^*\| e_{\ominus\lambda}(t, t_0), \quad t \in [t_0, \infty)_{\mathbb{T}}, \quad t_0 \in \mathbb{T}, \tag{26}$$

where $|x(t) - x^*(t)|_1 = \max_{1\le i\le m}\{|x_i(t) - x_i^*(t)| + |(x_i(t) - x_i^*(t))^\Delta|\}$, $\|\phi - \phi^*\| = \max_{1\le i\le m}\{\sup_{s\in[t_0-\theta,t_0]_{\mathbb{T}}} |\varphi_i(s) - \varphi_i^*(s)| + \sup_{s\in[t_0-\theta,t_0]_{\mathbb{T}}} |(\varphi_i(s) - \varphi_i^*(s))^\Delta|\}$, then the solution $x^*(t)$ is said to be globally exponentially stable.

4. C^1-Almost Periodic Solutions of (1)

In this section, we will state and prove the sufficient conditions for the existence and global exponential stability of C^1-almost periodic solutions of (1).

Set $\mathbb{X} = \{\phi = (\phi_1, \phi_2, \ldots, \phi_m)^T \mid \phi_i \in AP^1(\mathbb{T}, \mathbb{R}), i = 1, 2, \ldots, m\}$ with the norm $\|\phi\|_{\mathbb{X}} = \max_{1\le i\le m} \|\phi_i\|_2 = \max_{1\le i\le m}\{\sup_{t\in\mathbb{T}} |\phi_i(t)| + \sup_{t\in\mathbb{T}} |\phi_i^\Delta(t)|\}$; then $\mathbb{X}$ is a Banach space. For convenience, for a C^1-almost periodic function $f : \mathbb{T} \to \mathbb{R}$, we denote $f^+ = \sup_{t\in\mathbb{T}} |f(t)|$, $f^- = \inf_{t\in\mathbb{T}} |f(t)|$.

Lemma 32. *Assume that*

$$a_i, b_{ij}, c_i \in AP^1(\mathbb{T}, \mathbb{R}^+), \quad \tau_{ij} \in AP^1(\mathbb{T}, \mathbb{T}), \quad -a_i \in \mathcal{R}^+, \quad i, j = 1, 2, \ldots, m. \tag{H_1}$$

Then, for every solution $x(t, t_0, \varphi) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ of (1) and (2), we have that for $i = 1, 2, \ldots, m$, $x_i(t)$ are positive and bounded on $[t_0, +\infty)_{\mathbb{T}}$.

Proof. By Lemma 11, we have

$$x_i(t) = e_{-a_i}(t, t_0) x_i(t_0) + \int_{t_0}^{t} e_{-a_i}(t, \sigma(s)) \times \sum_{j=1}^{m} b_{ij}(s) \exp\left\{-c_i(s) x_i\left(s - \tau_{ij}(s)\right)\right\} \Delta s, \tag{27}$$

where $t \in [t_0, +\infty)_{\mathbb{T}}$, $i = 1, 2, \ldots, m$. First, we prove

$$x_i(t) > 0, \quad \forall t \in [t_0, +\infty)_{\mathbb{T}}, \ i = 1, 2, \ldots, m. \tag{28}$$

By way of contradiction, assume that (28) does not hold. Then, there exists $i_0 \in \{1, 2, \dots, m\}$ and the first time $t_1 \in [t_0, +\infty)_{\mathbb{T}}$ such that

$$\begin{gathered} x_{i_0}(t_1) \leq 0, \quad x_{i_0}(t) > 0, \quad t \in [-\theta, t_1)_{\mathbb{T}}, \\ x_k(t) > 0, \quad \text{for } k \neq i_0,\ t \in [-\theta, t_1]_{\mathbb{T}},\ k = 1, 2, \dots, m. \end{gathered} \tag{29}$$

Then, we can obtain

$$\begin{aligned} x_{i_0}(t_1) &= e_{-a_{i_0}}(t_1, t_0)\, x_{i_0}(t_0) \\ &\quad + \int_{t_0}^{t_1} e_{-a_{i_0}}(t, \sigma(s)) \\ &\quad\quad \times \sum_{j=1}^{m} b_{i_0 j}(s) \exp\left\{-c_i(s)\, x_{i_0}\left(s - \tau_{i_0 j}(s)\right) \exp\right\} \Delta s \\ &> 0, \end{aligned} \tag{30}$$

which is a contradiction and hence (28) holds. On the other hand, for $i = 1, 2, \dots, m$, we have that

$$\begin{aligned} x_i(t) &= e_{-a_i}(t, t_0)\, x_i(t_0) \\ &\quad + \int_{t_0}^{t} e_{-a_i}(t, \sigma(s)) \\ &\quad\quad \times \sum_{j=1}^{m} b_{ij}(s) \exp\left\{-c_i(s)\, x_i\left(s - \tau_{ij}(s)\right)\right\} \Delta s \\ &\leq e_{-a_i^-}(t, t_0)\, x_i(t_0) \\ &\quad + \sum_{j=1}^{m} b_{ij}^+ \int_{t_0}^{t} e_{-a_i^-}(t, \sigma(s))\, \Delta s \\ &\leq e_{-a_i^-}(t, t_0)\, \varphi_i(t_0) + \frac{\sum_{j=1}^{m} b_{ij}^+}{a_i^-}, \end{aligned} \tag{31}$$

which implies that $x_i(t)$ is bounded on $[t_0, +\infty)_{\mathbb{T}}$. This completes the proof. □

Theorem 33. *Let (H_1) hold. Suppose further that*

$$\left(1 + \frac{c_i^+}{a_i^-} + \frac{a_i^+}{a_i^-}\right) \sum_{j=1}^{m} b_{ij}^+ < 1, \quad i = 1, 2, \dots, m; \tag{H_2}$$

then there is a unique C^1-almost periodic solution of (1) in $\mathbb{X}^* = \{\phi \in \mathbb{X} \mid \|\phi\|_{\mathbb{X}} \leq L\}$, where L is a constant satisfying $L/2 \geq \max_{1\leq i\leq m}\{(\sum_{j=1}^{m} b_{ij}^+)/a_i^-, (1 + (a_i^+/a_i^-)) \sum_{j=1}^{m} b_{ij}^+\}$.

Proof. For any $\phi \in \mathbb{X}^*$, we consider the following C^1-almost periodic system:

$$x_i^{\Delta}(t) = -a_i(t)\, x_i(t) + \sum_{j=1}^{n} b_{ij}(t) \exp\left\{-c_i(t)\, \phi_i\left(t - \tau_{ij}(t)\right)\right\}, \tag{32}$$

where $i = 1, 2, \dots, m$. Since $\min_{1\leq i\leq m} \inf_{t\in\mathbb{T}}\{a_i(t)\} > 0$ and $-a_i \in \mathscr{R}^+$, it follows from Lemma 19 that the linear system

$$x_i^{\Delta}(t) = -a_i(t)\, x_i(t), \quad i = 1, 2, \dots, m \tag{33}$$

admits an exponential dichotomy on $\mathbb{T}$. Thus, (32) has a C^1-almost periodic solution $x^{\phi}(t) = (x_1^{\phi}(t), x_2^{\phi}(t), \dots, x_m^{\phi}(t))^T$, where

$$\begin{aligned} x_i^{\phi}(t) = \int_{-\infty}^{t} & e_{-a_i}(t, \sigma(s)) \\ & \times \left(\sum_{j=1}^{n} b_{ij}(s) \exp\left\{-c_i(s)\, \phi_i\left(s - \tau_{ij}(s)\right)\right\}\right) \Delta s, \\ & i = 1, 2, \dots, m. \end{aligned} \tag{34}$$

Define a map Φ on $\mathbb{X}^*$ by

$$(\Phi\phi)(t) = ((\Phi\phi)_1(t), \dots, (\Phi\phi)_m(t))^T, \tag{35}$$

where $(\Phi\phi)_i(t) = x_i^{\phi}(t),\ i = 1, 2, \dots, m$. It is obvious that $\mathbb{X}^*$ is a Banach space with the norm $\|\cdot\|_{\mathbb{X}}$. At first, we show that Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. For any $\phi \in \mathbb{X}^*$, we have

$$\begin{aligned} &\left|(\Phi\phi)_i(t)\right| \\ &\quad \leq \left| \int_{-\infty}^{t} e_{-a_i^-}(t, \sigma(s)) \times \left(\sum_{j=1}^{m} b_{ij}(s) \exp\left\{-c_i(s)\, \phi_i\left(s - \tau_{ij}(s)\right)\right\}\right) \Delta s \right| \\ &\quad \leq \sum_{j=1}^{m} b_{ij}^+ \left| \int_{-\infty}^{t} e_{-a_i^-}(t, \sigma(s))\, \Delta s \right| \\ &\quad \leq \frac{\sum_{j=1}^{m} b_{ij}^+}{a_i^-} \leq \frac{L}{2}, \quad i = 1, 2, \dots, m, \\ &\left|(\Phi\phi)_i^{\Delta}(t)\right| \\ &\quad = \left| \sum_{j=1}^{m} b_{ij}(t) \exp\left\{-c_i(t)\, \phi_i\left(t - \tau_{ij}(t)\right)\right\} - a_i(t) \right. \\ &\qquad \times \int_{-\infty}^{t} e_{-a_i}(t, \sigma(s)) \\ &\qquad\quad \times \left(\sum_{j=1}^{m} b_{ij}(s) \right. \\ &\qquad\qquad \left.\left. \times \exp\left\{-c_i(s)\, \phi_i\left(s - \tau_{ij}(s)\right)\right\}\right) \Delta s \right| \end{aligned}$$

$$\leq \sum_{j=1}^{m} b_{ij}^{+} + a_i^{+} \sum_{j=1}^{m} b_{ij}^{+} \left| \int_{-\infty}^{t} e_{-a_i^{-}} (t, \sigma (s)) \Delta s \right|$$
$$\leq \left(1 + \frac{a_i^{+}}{a_i^{-}} \right) \sum_{j=1}^{m} b_{ij}^{+}$$
$$\leq \frac{L}{2}, \quad i = 1, 2, \ldots, m. \tag{36}$$

Therefore, we have that $\|\Phi\phi\|_{\mathbb{X}} \leq L$. Hence, the mapping Φ is a self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$. Next, we prove that the mapping Φ is a contraction mapping on $\mathbb{X}^*$. For any $\phi = (\phi_1, \phi_2, \ldots, \phi_m)^T$, $\psi = (\psi_1, \psi_2, \ldots, \psi_m)^T \in \mathbb{X}^*$, since

$$\left| e^{-x} - e^{-y} \right| \leq \left| x - y \right|, \quad x, y \in [0, +\infty), \tag{37}$$

we have that

$$\left| (\Phi\phi)_i (t) - (\Phi\psi)_i (t) \right|$$
$$= \left| \int_{-\infty}^{t} e_{-a_i} (t, \sigma (s)) \times \sum_{j=1}^{m} b_{ij} (s) \left(\exp \left\{ -c_i (s) \phi_i \left(s - \tau_{ij} (s) \right) \right\} - \exp \left\{ -c_i (s) \psi_i \left(s - \tau_{ij} (s) \right) \right\} \right) \Delta s \right|$$
$$\leq \sum_{j=1}^{m} b_{ij}^{+} \int_{-\infty}^{t} e_{-a_i} (t, \sigma (s)) \left| c_i (s) \right| \times \left| \phi_i \left(s - \tau_{ij} (s) \right) - \psi_i \left(s - \tau_{ij} (s) \right) \right| \Delta s$$
$$\leq c_i^{+} \|\phi_i - \psi_i\|_{\mathbb{X}} \sum_{j=1}^{m} b_{ij}^{+} \int_{-\infty}^{t} e_{-a_i} (t, \sigma (s)) \Delta s$$
$$\leq \frac{c_i^{+} \sum_{j=1}^{m} b_{ij}^{+}}{a_i^{-}} \|\phi_i - \psi_i\|_{\mathbb{X}}, \quad i = 1, 2, \ldots, m,$$

$$\left| \left((\Phi\phi)_i (t) - (\Phi\psi)_i (t) \right)^{\Delta} \right|$$
$$= \left| \sum_{j=1}^{m} b_{ij} (t) \exp \left\{ -c_i (t) \phi_i \left(t - \tau_{ij} (t) \right) \right\} - a_i (t) \int_{-\infty}^{t} e_{-a_i} (t, \sigma (s)) \times \sum_{j=1}^{m} b_{ij} (s) \times \left(\exp \left\{ -c_i (s) \phi_i \left(s - \tau_{ij} (s) \right) \right\} - \exp \left\{ -c_i (s) \psi_i \left(s - \tau_{ij} (s) \right) \right\} \right) \Delta s \right|$$
$$\leq \sum_{j=1}^{m} b_{ij}^{+} + a_i^{+} \sum_{j=1}^{m} b_{ij}^{+} \int_{-\infty}^{t} e_{-a_i} (t, \sigma (s)) \left| c_i (s) \right| \times \left| \phi_i \left(s - \tau_{ij} (s) \right) - \psi_i \left(s - \tau_{ij} (s) \right) \right| \Delta s$$
$$\leq \left(1 + \frac{a_i^{+}}{a_i^{-}} \right) \sum_{j=1}^{m} b_{ij}^{+} \|\phi_i - \psi_i\|_{\mathbb{X}}, \quad i = 1, 2, \ldots, m. \tag{38}$$

Hence, we obtain

$$\|\Phi\phi - \Phi\psi\|_{\mathbb{X}} \leq \left(1 + \frac{c_i^{+}}{a_i^{-}} + \frac{a_i^{+}}{a_i^{-}} \right) \sum_{j=1}^{m} b_{ij}^{+} \|\phi_i - \psi_i\|_{\mathbb{X}}. \tag{39}$$

Noting that $(1 + (c_i^{+}/a_i^{-}) + (a_i^{+}/a_i^{-})) \sum_{j=1}^{m} b_{ij}^{+} < 1$, we see that Φ is a contraction mapping on $\mathbb{X}^*$. By the fixed point theorem in Banach space, Φ has a unique fixed point in $\mathbb{X}^*$, which implies that (1) has a unique C^1-almost periodic solution in $\mathbb{X}^*$. This completes the proof. □

Theorem 34. *Let (H_1) and (H_2) hold. Suppose further that*

$$c_i^{+} \left(a_i^{-} + a_i^{+} \right) \sum_{j=1}^{m} b_{ij}^{+} < a_i^{-}, \quad i = 1, 2, \ldots, m; \tag{H_3}$$

then the C^1-almost periodic solution of (1) *is globally exponentially stable.*

Proof. According to Theorem 33, we know that (1) has a C^1-almost periodic solution $x^*(t) = (x_1^*(t), x_1^*(t), \ldots, x_m^*(t))^T$ with initial condition $\varphi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \ldots, \varphi_m^*(s))^T$. Suppose that $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ is an arbitrary solution of (1) with initial condition $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_m(s))^T$. Denote $y_i(t) = x_i(t) - x_i^*(t)$, $i = 1, 2, \ldots, m$. Then, it follows from (1) that

$$y_i^{\Delta} (t) = - a_i (t) y_i (t) + \sum_{j=1}^{m} b_{ij} (t) \left[\exp \left\{ -c_i (t) \left(x_i \left(t - \tau_{ij} (t) \right) \right) \right\} - \exp \left\{ -c_i (t) \left(x_i^* \left(t - \tau_{ij} (t) \right) \right) \right\} \right], \tag{40}$$

where $i = 1, 2, \ldots, m$. The initial condition of (40) is

$$\psi_i(s) = \varphi_i(s) - \varphi_i^*(s), \quad s \in [t_0 - \theta, t_0]_{\mathbb{T}}. \tag{41}$$

By Lemma 11 and (40), for $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} y_i(t) &= e_{-a_i}(t, t_0)\, y(t_0) \\ &\quad + \int_{t_0}^{t} e_{-a_i}(t, \sigma(s)) \\ &\qquad \times \sum_{j=1}^{m} b_{ij}(s) \left[\exp\left\{-c_i(s)\, x_i\left(s - \tau_{ij}(s)\right)\right\}\right. \\ &\qquad\qquad \left. - \exp\left\{-c_i(s)\, x_i^*\left(s - \tau_{ij}(s)\right)\right\}\right] \Delta s. \end{aligned} \tag{42}$$

Take a constant $\lambda > 0$ with $-\lambda \in \mathscr{R}^+$ such that $\ominus\lambda \geq \max_{1\leq i\leq m}\{-a_i^-\}$. Let M be a constant satisfying

$$M > \max_{1\leq i\leq m}\left\{\frac{a_i^+ a_i^-}{a_i^- - (a_i^+ + a_i^-)\, c_i^+ \sum_{j=1}^{m} b_{ij}^+}, \frac{a_i^-}{a_i^- - c_i^+ \sum_{j=1}^{m} b_{ij}^+}\right\}. \tag{43}$$

By (H_2) and (H_3), it is easy to verify that $M > 1$ and in view of $e_{\ominus\lambda}(t, t_0) \geq 1$ for $t \leq t_0$, we have

$$|y(t)|_1 \leq M e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \quad \forall t \in (-\infty, t_0]_{\mathbb{T}}. \tag{44}$$

We claim that

$$|y(t)|_1 \leq M e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{45}$$

To prove this claim, we show that, for any $p > 1$, the following inequality holds:

$$|y(t)|_1 < pM e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}, \tag{46}$$

which implies that, for $i = 1, 2, \ldots, m$, we have

$$|y_i(t)| < pM e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}, \tag{47}$$

$$\left|y_i^{\Delta}(t)\right| < pM e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{48}$$

By way of contradiction, assume that (46) does not hold. Firstly, we consider the following two cases.

Case One. (47) is not true. Then there exists $t_1 \in (t_0, +\infty)_{\mathbb{T}}$ and $i_0 \in \{1, 2, \ldots, m\}$ such that

$$\begin{gathered} \left|y_{i_0}(t_1)\right| \geq pM e_{\ominus\lambda}(t_1, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \\ \left|y_{i_0}(t)\right| < pM e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \\ t \in (t_0, t_1)_{\mathbb{T}}, \\ |y_l(t)| < pM e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \\ \text{for } l \neq i_0, \quad t \in (t_0, t_1]_{\mathbb{T}}, \quad l = 1, 2, \ldots, m. \end{gathered} \tag{49}$$

Hence, there must be a constant $\alpha \geq 1$ such that

$$\begin{gathered} \left|y_{i_0}(t_1)\right| = \alpha pM e_{\ominus\lambda}(t_1, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \\ \left|y_{i_0}(t)\right| < \alpha pM e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \\ t \in (t_0, t_1)_{\mathbb{T}}, \\ |y_l(t)| < \alpha pM e_{\ominus\lambda}(t, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}}, \\ \text{for } l \neq i_0, \quad t \in (t_0, t_1]_{\mathbb{T}}, \quad l = 1, 2, \ldots, m. \end{gathered} \tag{50}$$

In view of (42), we have

$$\begin{aligned} \left|y_{i_0}(t_1)\right| &= \Bigg| e_{-a_{i_0}}(t_1, t_0)\, y_{i_0}(t_0) \\ &\qquad + \int_{t_0}^{t_1} e_{-\alpha}(t_1, \sigma(s)) \\ &\qquad\quad \times \sum_{j=1}^{m} b_{i_0 j}(s) \\ &\qquad\quad \times \left[e^{-c_{i_0}(s)(x_{i_0}(s - \tau_{i_0 j}(s)))}\right. \\ &\qquad\qquad \left. - e^{-c_{i_0}(s)(x_{i_0}^*(s - \tau_{i_0 j}(s)))}\right] \Delta s \Bigg| \\ &\leq e_{-a_{i_0}^-}(t_1, t_0)\left|y_{i_0}(t_0)\right| \\ &\quad + \int_{t_0}^{t_1} e_{-a_{i_0}^-}(t_1, \sigma(s)) \sum_{j=1}^{m} \left|b_{i_0 j}(s)\right| \\ &\qquad \times \left|e^{-c_{i_0}(s)(x_{i_0}(s - \tau_{i_0 j}(s)))} - e^{-c_{i_0}(s)(x_{i_0}^*(s - \tau_{i_0 j}(s)))}\right| \Delta s \\ &\leq e_{-a_{i_0}^-}(t_1, t_0)\, |y(t_0)| \\ &\quad + \int_{t_0}^{t_1} e_{-a_{i_0}^-}(t_1, \sigma(s)) \sum_{j=1}^{m} b_{i_0 j}^+ c_{i_0}^+ \left|y_{i_0}\left(s - \tau_j(s)\right)\right| \Delta s \\ &\leq e_{\ominus\lambda}(t_1, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}} \\ &\quad + \sum_{j=1}^{m} b_{i_0 j}^+ c_{i_0}^+ \alpha pM e_{\ominus\lambda}(t_1, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}} \\ &\quad \times \int_{t_0}^{t_1} e_{-a_{i_0}^-}(t_1, \sigma(s))\, \Delta s \\ &= e_{\ominus\lambda}(t_1, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}} \\ &\quad + \frac{1}{-a_{i_0}^-} \sum_{j=1}^{m} b_{i_0 j}^+ c_{i_0}^+ \alpha pM e_{\ominus\lambda}(t_1, t_0)\, \|\varphi - \varphi^*\|_{\mathbb{X}} \end{aligned}$$

$$\begin{aligned}
&\times \int_{t_0}^{t_1} \left(-a_{i_0}^{-}\right) e_{-a_{i_0}^{-}}\left(t_1, \sigma(s)\right) \Delta s \\
&\leq e_{\ominus\lambda}\left(t_1, t_0\right) \left\|\varphi-\varphi^*\right\|_{\mathbb{X}} \\
&\quad + \frac{\sum_{j=1}^{m} b_{i_0 j}^{+} c_{i_0}^{+} \alpha p M e_{\ominus\lambda}\left(t_1, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}}{a_{i_0}^{-}} \\
&= \alpha p M e_{\ominus\lambda}\left(t_1, t_0\right)\left\|\varphi-\varphi^*\right\|\left(\frac{1}{\alpha p M}+\frac{\sum_{j=1}^{m} b_{i_0 j}^{+} c_{i_0}^{+}}{a_{i_0}^{-}}\right) \\
&\leq \alpha p M e_{\ominus\lambda}\left(t_1, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}\left(\frac{1}{M}+\frac{\sum_{j=1}^{m} b_{i_0 j}^{+} c_{i_0}^{+}}{a_{i_0}^{-}}\right) \\
&< \alpha p M e_{\ominus\lambda}\left(t_1, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}},
\end{aligned} \tag{51}$$

which is a contradiction.

Case Two. (48) is not true. Then there exists $t_2 \in (t_0, +\infty)_{\mathbb{T}}$ and $i_1 \in \{1, 2, \ldots, m\}$ such that

$$\begin{gathered}
\left|y_{i_1}^{\Delta}\left(t_2\right)\right| \geq p M e_{\ominus\lambda}\left(t_2, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}, \\
\left|y_{i_1}^{\Delta}(t)\right| < p M e_{\ominus\lambda}\left(t, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}, \\
t \in \left(t_0, t_2\right)_{\mathbb{T}},
\end{gathered} \tag{52}$$

$$\begin{gathered}
\left|y_{l}^{\Delta}(t)\right| < p M e_{\ominus\lambda}\left(t, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}, \\
\text{for } l \neq i_1, \quad t \in \left(t_0, t_2\right]_{\mathbb{T}}, \quad l = 1, 2, \ldots, m.
\end{gathered}$$

Hence, there must be a constant $c \geq 1$ such that

$$\begin{gathered}
\left|y_{i_1}^{\Delta}\left(t_2\right)\right| = c p M e_{\ominus\lambda}\left(t_2, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}, \\
\left|y_{i_1}^{\Delta}(t)\right| < c p M e_{\ominus\lambda}\left(t, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}, \\
t \in \left(t_0, t_2\right)_{\mathbb{T}},
\end{gathered} \tag{53}$$

$$\begin{gathered}
\left|y_{l}^{\Delta}(t)\right| < c p M e_{\ominus\lambda}\left(t, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}, \\
\text{for } l \neq i_1, \quad t \in \left(t_0, t_2\right]_{\mathbb{T}}, \quad l = 1, 2, \ldots, m.
\end{gathered} \tag{54}$$

In view of (42), we have

$$\begin{aligned}
&\left|y_{i_1}^{\Delta}\left(t_2\right)\right| \\
&= \Bigg| - a_{i_1}(t) e_{-a_{i_1}}\left(t_2, t_0\right) y_{i_1}\left(t_0\right) - a_{i_1}(t) \\
&\quad \times \int_{t_0}^{t_2} e_{-a_{i_1}}\left(t_2, \sigma(s)\right) \sum_{j=1}^{m} b_{i_1 j}(s) \\
&\quad \times \left[e^{-c_{i_1}(s)(x_{i_1}(s-\tau_{i_1 j}(s)))}\right. \\
&\quad \left. - e^{-c_{i_1}(s)(x_{i_1}^*(s-\tau_{i_1 j}(s)))}\right] \Delta s \\
&\quad + \sum_{j=1}^{m} b_{i_1 j}(t)\left[e^{-c_{i_1}(t)(x_{i_1}(t-\tau_{i_1 j}(t)))} - e^{-c_{i_1}(t)(x_{i_1}^*(t-\tau_{i_1 j}(t)))}\right] \Bigg| \\
&\leq a_{i_1}^{+} e_{-a_{i_1}}\left(t_2, t_0\right)\left|y_{i_1}\left(t_0\right)\right| \\
&\quad + a_{i_1}^{+} \int_{t_0}^{t_2} e_{-a_{i_1}}\left(t_2, \sigma(s)\right) \sum_{j=1}^{m}\left|b_{i_1 j}(s)\right| \\
&\quad \times \left|e^{-c_{i_1}(s)(x_{i_1}(s-\tau_{i_1 j}(s)))} - e^{-c_{i_1}(s)(x_{i_1}^*(s-\tau_{i_1 j}(s)))}\right| \Delta s \\
&\quad + \sum_{j=1}^{m}\left|b_{i_1 j}(t)\right|\left|e^{-c_{i_1}(t)(x_{i_1}(t-\tau_{i_1 j}(t)))} - e^{-c_{i_1}(t)(x_{i_1}^*(t-\tau_{i_1 j}(t)))}\right| \\
&\leq a_{i_1}^{+} e_{\ominus\lambda}\left(t_1, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}} \\
&\quad + c_{i_1}^{+} \sum_{j=1}^{m} b_{i_1 j}^{+}\left|y_{i_1}\left(t-\tau_{i_1 j}(t)\right)\right| \\
&\quad + a_{i_1}^{+} c_{i_1}^{+} \sum_{j=1}^{m} b_{i_1 j}^{+} \int_{t_0}^{t_2} e_{-a_{i_1}}\left(t_2, \sigma(s)\right)\left|y_{i_1}\left(s-\tau_{i_1 j}(s)\right)\right| \Delta s \\
&\leq a_{i_1}^{+} e_{\ominus\lambda}\left(t_1, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}} \\
&\quad + c_{i_1}^{+} \sum_{j=1}^{m} b_{i_1 j}^{+} c p M\left\|\varphi-\varphi^*\right\|_{\mathbb{X}} \\
&\quad + a_{i_1}^{+} c_{i_1}^{+} \sum_{j=1}^{m} b_{i_1 j}^{+} c p M\left\|\varphi-\varphi^*\right\|_{\mathbb{X}} \int_{t_0}^{t_2} e_{-a_{i_1}}\left(t_2, \sigma(s)\right) \Delta s \\
&\leq a_{i_1}^{+} e_{\ominus\lambda}\left(t_1, t_0\right)\left\|\varphi-\varphi^*\right\|_{\mathbb{X}} \\
&\quad + c_{i_1}^{+}\left(1+\frac{a_{i_1}^{+}}{a_{i_1}^{-}}\right) \sum_{j=1}^{m} b_{i_1 j}^{+} c p M\left\|\varphi-\varphi^*\right\|_{\mathbb{X}} \\
&= c p M\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}\left(\frac{a_{i_1}^{+}}{c p M}+c_{i_1}^{+}\left(1+\frac{a_{i_1}^{+}}{a_{i_1}^{-}}\right) \sum_{j=1}^{m} b_{i_1 j}^{+}\right) \\
&< c p M\left\|\varphi-\varphi^*\right\|_{\mathbb{X}}\left(\frac{a_{i_1}^{+}}{M}+c_{i_1}^{+}\left(1+\frac{a_{i_1}^{+}}{a_{i_1}^{-}}\right) \sum_{j=1}^{m} b_{i_1 j}^{+}\right) \\
&< c p M\left\|\varphi-\varphi^*\right\|_{\mathbb{X}},
\end{aligned} \tag{55}$$

which is a contradiction. Therefore, (46) holds. Let $p \to 1$; then (45) holds. Hence, we have that

$$\left|x(t)-x^*(t)\right|_1 \leq M\left\|\varphi-\varphi^*\right\|_{\mathbb{X}} e_{\ominus\lambda}\left(t, t_0\right), \quad t \in \left[t_0, +\infty\right)_{\mathbb{T}}; \tag{56}$$

that is, the C^1-almost periodic solution $x^*(t)$ of (1) is globally exponentially stable. This completes the proof. □

5. An Example

In this section, we present an example to illustrate the feasibility of our results obtained in previous sections.

Example 1. Let $m = 2$. Consider the following Lasota-Wazewska model on an almost periodic time scale $\mathbb{T}$:

$$x_i^{\Delta}(t) = -a_i(t)\,x_i(t) + \sum_{j=1}^{2} b_{ij}(t)\exp\left\{-c_i(t)\,x_i\left(t-\tau_{ij}(t)\right)\right\}, \quad i = 1, 2, \tag{57}$$

in which we take coefficients as follows:

$$\begin{aligned} a_1(t) &= 0.4 + 0.1\sin\left(\frac{1}{2}t\right), \\ a_2(t) &= 0.5 + 0.2\sin\left(\frac{3}{4}t\right), \\ b_{11}(t) &= 0.07 + 0.01\cos\pi t, \\ b_{12}(t) &= 0.03 + 0.01\sin\sqrt{3}t, \\ b_{21}(t)(t) &= 0.05 + 0.01\sin\pi t, \\ b_{22}(t) &= 0.02 + 0.01\cos\sqrt{2}t, \\ c_1(t) &= 0.03 + 0.01\cos\sqrt{3}t, \\ c_2(t) &= 0.05 + 0.01\sin\left(\frac{4}{3}t\right). \end{aligned} \tag{58}$$

By calculating, we have

$$\begin{aligned} a_1^+ &= 0.5, \\ a_2^+ &= 0.7, \\ a_1^- &= 0.3, \\ a_2^- &= 0.3, \\ b_{11}^+ &= 0.08, \\ b_{12}^+ &= 0.04, \\ b_{21}^+ &= 0.06, \\ b_{22}^+ &= 0.03, \\ c_1^+ &= 0.04, \\ c_2^+ &= 0.06. \end{aligned} \tag{59}$$

It can be verified that all conditions of Theorems 33 and 34 are satisfied. Therefore, (57) has a C^1-almost periodic solution, which is globally exponentially stable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by the National Natural Sciences Foundation of China under Grant no. 11361072.

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