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Research Article

Blow-Up of Solutions for a Class of Sixth Order Nonlinear Strongly Damped Wave Equation

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We consider the blow-up phenomenon of sixth order nonlinear strongly damped wave equation. By using the concavity method, we prove a finite time blow-up result under assumptions on the nonlinear term and the initial data.

1. Introduction

It is well known that nonlinear strongly damped wave equation is proposed to describe all kinds of viscous vibration system. The global well-posedness of third order nonlinear strongly damped wave equation

$$u_{tt} - \alpha\Delta u_t - \Delta u = f(u), \quad \alpha > 0,\ x \in \Omega,\ t > 0, \tag{1}$$

was studied by Webb [1] firstly. He gave the existence and asymptotic behavior of strong solutions for the problem (1). Then this result was improved by Y. C. Liu and D. C. Liu [2]. The existence and uniqueness of strong solutions were proved under the hypothesis of the weaker conditions. For two classes of strongly damped nonlinear wave equation, the finite time blow-up of solutions was proved by Shang [3]. A number of authors (Chen et al. [4], Zhou [5], and Al'shin et al. [6]) have shown the existence of the global weak solutions and the global attractors for third order nonlinear strongly damped wave equation.

For the fourth order nonlinear strongly damped wave equation, there are also some results about initial boundary value problem or Cauchy problem [7–9]. In [7], Shang studied the initial boundary value problem of the following equation:

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u), \quad x \in \Omega,\ t > 0. \tag{2}$$

Under some assumptions on f and $n = 1, 2, 3$, he investigated the existence, uniqueness, asymptotic behavior, and blow-up phenomenon of the solutions.

In [8], Xu et al. considered the initial boundary value problem of fourth order wave equation with viscous damping term

$$u_{tt} - \alpha u_{xxt} + u_{xxxx} = f(u_x)_x, \quad x \in \Omega,\ t > 0. \tag{3}$$

They proved the global existence and nonexistence of the solution by argument related to the potential well-convexity method.

In order to investigate the water wave problem with surface tension, Schneider and Wayne [10] studied a class of Boussinesq equation as follows:

$$u_{tt} = u_{xx} + u_{xxtt} + \mu u_{xxxx} - u_{xxxxtt} + \left(u^2\right)_{xx}, \tag{4}$$

where $x, t, \mu \in \mathbf{R}$. This type of equations can be formally derived from the 2D water wave problem and models the water wave problem with surface tension. They proved that the long wave limit can be described approximately by two decoupled Kawahara equations. A more natural model seems to be an extension from the classical Boussinesq equation as follows (see [11]):

$$u_{tt} = u_{xx} + (\mu + 1)\, u_{xxxx} - u_{xxxxxx} + \left(u^2\right)_{xx}. \tag{5}$$

Wang and Mu [12] studied the Cauchy problem of the equation

$$u_{tt} = u_{xx} + u_{xxtt} - u_{xxxx} - u_{xxxxtt} + f(u)_{xx}. \tag{6}$$

They obtained the existence and uniqueness of the local solutions and proved the blow-up of solutions to the problem (6). Esfahani et al. [13] studied the solutions of

$$u_{tt} = u_{xx} + \beta u_{xxxx} + u_{xxxxxx} + \left(|u|^{\alpha} u\right)_{xx}, \tag{7}$$

where $\beta = \pm 1$ and $\alpha > 0$. They proved the local well-posedness in $L^2(R)$ and $H^1(R)$ and gave finite time blow-up results to the problem (7).

For the sixth order nonlinear wave equation with strong damping term

$$u_{tt} - \Delta u - \nu \Delta u_t + \Delta^2 u + \Delta^2 u_{tt} = \Delta f(u), \tag{8}$$

H. W. Wang and S. B. Wang [14] established a global existence result of small amplitude solutions of the Cauchy problem (8) for all space dimensions $n > 1$. When $\nu = 0$, H. W. Wang and S. B. Wang [15] studied the long-time behavior of small solutions of the Cauchy problem for a Rosenau equation. The decay and scattering for small amplitude solution are established.

In this paper, we study a class of sixth order nonlinear strongly damped wave equation:

$$\begin{gathered} u_{tt} - \Delta u_t = \mathbf{div} \vec{\sigma}(\nabla u) + \Delta u + \mu \Delta^2 u + \Delta^3 u, \quad x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ \left(\mathscr{B}_i u\right)(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \end{gathered} \tag{9}$$

where $\mu > 0$, Ω is a bounded domain of R^n $(n \ge 1)$ with a smooth boundary $\partial\Omega$, and $\mathscr{B}_i$, $i = 0, 1, 2$, are homogeneous boundary condition:

$$D = \left\{ u \in C^6\left(\overline{\Omega}\right) \mid \mathscr{B}_i u = \nabla^i u \equiv 0, i = 0, 1, 2, x \in \partial\Omega \right\}. \tag{10}$$

By using the ideas of the concavity theory introduced by Levine [17], we prove the finite time blow-up results under assumption on the nonlinear term $\mathbf{div}\,\vec{\sigma}(\nabla u)$ and the initial data u_0, u_1.

2. Preliminaries and Main Results

In this section, we introduce some notations, basic ideas, and important lemmas which will be needed in the course of the paper.

Let $H = \mathscr{L}^2(\Omega)$ be a Hilbert space which is equipped with the scalar product $(u, v) = \int_{\Omega} u(x) v(x) dx$.

Now, we define

$$Au = -\Delta u - \mu \Delta^2 u - \Delta^3 u, \tag{11}$$

where $A : D \to H$ is a symmetric linear operator and satisfies $(u, Av) = (Au, v)$ for all $u, v \in D \subseteq H$.

For the nonlinear term $\mathbf{div}\,\vec{\sigma}(\nabla u)$ of the problem (9), $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) : D^n \to H^n$ is a vector function which satisfies the following conditions.

(a) Assume that the Fréchet derivative $\vec{\sigma}_x$ is a symmetric, bounded, linear operator on H^n and that $x \to \vec{\sigma}_x$ is a continuous map from D^n to $\mathfrak{L}(H)^n$.

(b) The scalar valued function $G : D \to \mathrm{R}$ is defined by

$$\begin{aligned} G(u) &= \int_0^1 \left(\mathbf{div}\,\vec{\sigma}(\rho \nabla u), u\right) d\rho \\ &= \int_{\Omega} \left(\int_0^1 \mathbf{div}\,\vec{\sigma}(\rho \nabla u)\, u \, d\rho \right) dx \\ &= \int_{\Omega} \left(\int_0^1 \frac{d}{d\rho} \left(\int_0^{\rho u} \mathbf{div}\,\vec{\sigma}(\nabla z)\, dz \right) d\rho \right) dx \\ &= \int_{\Omega} \left(\int_0^{u} \mathbf{div}\,\vec{\sigma}(\nabla z)\, dz \right) dx, \end{aligned} \tag{12}$$

where $G(u)$ denotes the potential associated with $\mathbf{div}\,\vec{\sigma}(\nabla u)$. The Fréchet derivative of $G(u)$ is G_u which can be shown to act as follows:

$$G_u v = \left(\mathbf{div}\ \vec{\sigma}(\nabla u), v\right), \tag{13}$$

for all $u, v \in D$.

(c) Assume that for some $\alpha > 0$

$$\left(\mathbf{div}\,\vec{\sigma}(\nabla u), u\right) \ge 2(2\alpha + 1) G(u), \tag{14}$$

for all $u \in D$.

To obtain the finite time blow-up result, we need the following interpolation inequality of Evance [16] for function in $W_0^{k,p}(\Omega)$.

Lemma 1. *For all $u \in W_0^{k,p}(\Omega)$, if $1 \le p < \infty$, j and k are integers, and $1 \le j \le k$, $(j/k) \le \alpha \le 1$, then*

$$\left\| D^j u \right\|_{L^q(\Omega)} \le C \left\| D^k u \right\|_{L^p(\Omega)}^{\alpha} \| Du \|_{L^r(\Omega)}^{1-\alpha}, \quad \forall u \in W_0^{k,p}(\Omega), \tag{15}$$

where $(1/q) - (j/n) = \alpha((1/p) - (k/n)) + (1-\alpha)((1/r) - (1/n))$, with the constant C depending only on (p, r, j, k, α, n) and Ω.

In particular, if $j = 2$, $k = 3$, and $p = q = r = 2$, one has

$$\left\| D^2 u \right\|_{L^2(\Omega)} \le C \left\| D^3 u \right\|_{L^2(\Omega)}^{\alpha} \| Du \|_{L^2(\Omega)}^{1-\alpha}, \quad \forall u \in W_0^{3,2}(\Omega). \tag{16}$$

Lemma 2. *Assume that $u \in W_0^{3,2}(\Omega)$ and $0 < \mu \le (1/C_1)$ (with C_1 depending on the constant C of Sobolev's interpolation inequality); then $(Au, u) \ge 0$.*

Proof. By Lemma 1, we see that

$$\left\| D^2 u \right\|_{L^2(\Omega)} \le C \left\| D^3 u \right\|_{L^2(\Omega)}^{\alpha} \| Du \|_{L^2(\Omega)}^{1-\alpha}, \quad \forall u \in W_0^{3,2}(\Omega). \tag{17}$$

Using Young's inequality, we have

$$\begin{aligned} \left\| D^2 u \right\|_{L^2(\Omega)}^2 &\le C \left\| D^3 u \right\|_{L^2(\Omega)}^{2\alpha} \| Du \|_{L^2(\Omega)}^{2(1-\alpha)} \\ &\le C \left[\alpha \left\| D^3 u \right\|_{L^2(\Omega)}^2 + (1-\alpha) \| Du \|_{L^2(\Omega)}^2 \right] \\ &\le C_1 \left[\left\| D^3 u \right\|_{L^2(\Omega)}^2 + \| Du \|_{L^2(\Omega)}^2 \right]. \end{aligned} \tag{18}$$

For the operator A, using integration of parts, we have

$$\begin{aligned}(Au,u) &= \left(-\Delta u - \mu\Delta^2 u - \Delta^3 u, u\right) \\ &= \|\nabla u\|^2_{L^2(\Omega)} + \left\|\nabla^3 u\right\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 u\right\|^2_{L^2(\Omega)} \\ &\geq \frac{1}{C_1}\left\|\nabla^2 u\right\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 u\right\|^2_{L^2(\Omega)} \\ &= \left(\frac{1}{C_1} - \mu\right)\left\|\nabla^2 u\right\|^2_{L^2(\Omega)},\end{aligned} \tag{19}$$

where $0 < \mu \leq (1/C_1)$.

The verification of the action of G_x can be proved from the definition. The details, not being germane to this paper, are omitted here. But a formula will be useful in the sequel as follows. □

Lemma 3. *Let* $G(u) = \int_0^1 (\mathbf{div}\,\sigma(\nabla(\rho u)), u)d\rho$*; then one has*

$$G(u(t)) - G(u(0)) = \int_0^t \left(\mathbf{div}\,\sigma(\nabla u(\eta)), u_\eta(\eta)\right)d\eta, \tag{20}$$

for $u : [0,T) \to D$ *with a strongly continuous derivative* u_t.

Proof. By the chain rule and the action of G_x, we have

$$\begin{aligned}\frac{d}{dt}G(u) &= \frac{d}{dt}\int_0^1 (\mathbf{div}\,\vec{\sigma}(\nabla(\rho u)), u)\,d\rho \\ &= -\frac{d}{dt}\int_0^1 (\vec{\sigma}(\nabla(\rho u)), \nabla u)\,d\rho \\ &= -\int_0^1 \rho\left(\vec{\sigma}_{\rho\nabla u}\cdot\nabla u_t, \nabla u\right) + (\vec{\sigma}(\rho\nabla u), \nabla u_t)\,d\rho \\ &= -\int_0^1 \rho\left(\vec{\sigma}_{\rho\nabla u}\cdot\nabla u, \nabla u_t\right) + (\vec{\sigma}(\rho\nabla u), \nabla u_t)\,d\rho \\ &= -\int_0^1 \rho\frac{d}{d\rho}(\vec{\sigma}(\rho\nabla u), \nabla u_t) + (\vec{\sigma}(\rho\nabla u), \nabla u_t)\,d\rho \\ &= -\int_0^1 \frac{d}{d\rho}\left[\rho(\vec{\sigma}(\rho\nabla u), \nabla u_t)\right]d\rho \\ &= (\mathbf{div}\,\vec{\sigma}(\nabla u), u_t),\end{aligned} \tag{21}$$

where we have used the symmetry of $\vec{\sigma}_x$ in the fourth line. □

The following lemma contributing to the result of this paper is analogous to Corollary 1.1 of [17] with slight modification.

Lemma 4. *Assume that* $\vec{\sigma}$ *is homogenous of degree* $1+\gamma$ *for some* $\gamma > 0$ *(i.e.,* $\vec{\sigma}(sh) = s^{1+\gamma}\vec{\sigma}(h)$ *for all* $s > 0$ *and for all* $h \in D$*). Let* $(h_0, \mathbf{div}\,\vec{\sigma}(\nabla h_0)) > 0$ *for some* $h_0 \in D$*. Then there are infinitely many vectors* u_0 *such that*

$$\begin{aligned}G(u_0) &> \frac{1}{2}(u_0, Au_0) \\ &= \frac{1}{2}\left[\left\|\nabla^3 u_0\right\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 u_0\right\|^2_{L^2(\Omega)}\right].\end{aligned} \tag{22}$$

Proof. Let $u_0 = sh_0$, where s is large enough so that

$$\begin{aligned}s^\gamma G(h_0) &= s^\gamma\int_0^1 (\mathbf{div}\,\vec{\sigma}(\rho\nabla h_0), h_0)\,d\rho \\ &> \frac{1}{2}(h_0, Ah_0) \\ &= \frac{1}{2}\left[\left\|\nabla^3 h_0\right\|^2_{L^2(\Omega)} + \|\nabla h_0\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 h_0\right\|^2_{L^2(\Omega)}\right].\end{aligned} \tag{23}$$

Then for all

$$s > s_0 = \sqrt[\gamma]{\frac{(2+\gamma)\left[\left\|\nabla^3 h_0\right\|^2_{L^2(\Omega)} + \|\nabla h_0\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 h_0\right\|^2_{L^2(\Omega)}\right]}{2(\mathbf{div}\,\vec{\sigma}(\nabla h_0), h_0)}}, \tag{24}$$

we have

$$\begin{aligned}G(u_0) &= \int_0^1 (\mathbf{div}\,\vec{\sigma}(\rho s\nabla h_0), sh_0)\,d\rho \\ &= -\int_0^1 (\vec{\sigma}(\rho s\nabla h_0), s\nabla h_0)\,d\rho \\ &= s^{2+\gamma}G(h_0) > \frac{1}{2}(u_0, Ah_0).\end{aligned} \tag{25}$$

The local existence of solution for the problem (9) can be obtained by the standard Faedo-Galerkin approximation methods. The interested reader is referred to Lions [18] or Robinson [19] for details. □

Next, we are ready to state the blow-up result of this paper.

Theorem 5. *Let* $u : [0,T) \to D$ *be a strongly continuously differentiable solution of* (9) *in the* D *norm. Suppose that* $0 < \mu \leq (1/C_1)$ *with (*C_1 *depending on the constant* C *of Sobolev's interpolation inequality)* $(\mathbf{div}\,\vec{\sigma}(\nabla u), u) \geq 2(2\alpha+1)G(u)$*, where* $(G(u) = \int_0^1 (\mathbf{div}\,\vec{\sigma}(\rho\nabla u), u)d\rho)$*. Finally let* u_0 *satisfy*

$$\begin{aligned}G(u_0) &> \frac{1}{2}(u_0, Au_0) \\ &= \frac{1}{2}\left[\left\|\nabla^3 u_0\right\|^2_{L^2(\Omega)} + \|\nabla u_0\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 u_0\right\|^2_{L^2(\Omega)}\right].\end{aligned} \tag{26}$$

Then the solution u can only exist on a bounded interval $[0, T)$, *and in fact*

$$T \leq \left(4\alpha^2 \beta (u_0, u_0) + \left[(\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1) + \sqrt{[(\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1)]^2 + 4\alpha^2 \beta (u_0, u_0)} \right]^2 \right) \times \left(4\alpha^2 \beta \left([(\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1)]^2 + 4\alpha^2 \beta (u_0, u_0) \right)^{1/2} \right)^{-1}, \tag{27}$$

while also

$$\lim_{t \to T^-} \left[\|u\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)} d\eta \right] = +\infty, \tag{28}$$

and consequently

$$\limsup_{t \to T^-} \|u\|^2_{H^1} = +\infty. \tag{29}$$

Proof. For arbitrary $T_0, \beta, \tau > 0$ and $t \in [0, T_0)$, let

$$F(t) = (u, u) + \int_0^t (u, -\Delta u)\, d\eta + (T_0 - t)(u_0, -\Delta u_0) + \beta (t + \tau)^2. \tag{30}$$

A direct computation yields

$$\begin{aligned} F'(t) &= 2(u, u_t) + (u, -\Delta u) - (u_0, -\Delta u_0) + 2\beta (t + \tau) \\ &= 2(u, u_t) + (\nabla u, \nabla u) - (\nabla u_0, \nabla u_0) + 2\beta (t + \tau) \\ &= 2(u, u_t) + 2\int_0^t (\nabla u, \nabla u_\eta)\, d\eta + 2\beta (t + \tau). \end{aligned} \tag{31}$$

Suppressing the argument t, we see that

$$F''(t) = 2(u, u_{tt}) + 2(u_t, u_t) + 2\int_0^t (\nabla u, \nabla u_\eta)_\eta\, d\eta + 2(\nabla u_1, \nabla u_0) + 2\beta. \tag{32}$$

Hence, from (30), (31), and (32), we find after some algebra that

$$\begin{aligned} F''F - (\alpha + 1)(F'(t))^2 &= 4(\alpha + 1) S^2 + 2F[(u, u_{tt}) - (2\alpha + 1)(u_t, u_t)] \\ &\quad + 2F\left[\int_0^t (\nabla u, \nabla u_\eta)_\eta\, d\eta - 2(\alpha + 1)\int_0^t (\nabla u_\eta, \nabla u_\eta)\, d\eta \right] \\ &\quad + 4(\alpha + 1)(T_0 - t)(\nabla u_0, \nabla u_0) \\ &\quad \times \left[(u_t, u_t) + \int_0^t (\nabla u_\eta, \nabla u_\eta)\, d\eta + \beta \right] \\ &\quad + 2F[(\nabla u_1, \nabla u_0) - (2\alpha + 1)\beta], \end{aligned} \tag{33}$$

where

$$\begin{aligned} S^2 = &- \left[(u, u_t) + \int_0^t (\nabla u, \nabla u_\eta)\, d\eta + \beta (t + \tau) \right]^2 \\ &+ \left[(u, u) + \int_0^t (u, -\Delta u)\, d\eta + \beta (t + \tau)^2 \right] \\ &\times \left[(u_t, u_t) + \int_0^t (\nabla u_\eta, \nabla u_\eta)\, d\eta + \beta \right]. \end{aligned} \tag{34}$$

Using Schwarz's inequality, we have

$$\begin{gathered} (u, u)(u_t, u_t) \geq (u, u_t)^2, \\ \int_0^t (\nabla u, \nabla u)\, d\eta \int_0^t (\nabla u_\eta, \nabla u_\eta)\, d\eta \geq \left[\int_0^t (\nabla u, \nabla u_\eta)\, d\eta \right]^2, \\ \begin{aligned} &(u, u)\int_0^t (\nabla u_\eta, \nabla u_\eta)\, d\eta + (u, u_t)\int_0^t (\nabla u, \nabla u)\, d\eta \\ &\quad \geq 2\sqrt{(u, u)(u_t, u_t)\int_0^t (\nabla u, \nabla u)\, d\eta \int_0^t (\nabla u_\eta, \nabla u_\eta)\, d\eta} \\ &\quad \geq 2(u, u_t)\int_0^t (\nabla u, \nabla u_\eta)\, d\eta, \end{aligned} \\ \begin{aligned} &(u, u)\beta + (u_t, u_t)\beta (t + \tau)^2 \\ &\quad \geq 2\sqrt{(u, u)(u_t, u_t)\beta^2 (t + \tau)^2} \\ &\quad \geq 2(u, u_t)\beta (t + \tau). \end{aligned} \end{gathered} \tag{35}$$

By (35), we have $S^2 \geq 0$. Let

$$\begin{aligned}
H(t) &= (u, u_{tt}) - (2\alpha+1)(u_t, u_t) + (\nabla u_1, \nabla u_0) - (2\alpha+1)\beta \\
&\quad - \int_0^t (\Delta u_\eta, u)_\eta d\eta + 2(\alpha+1)\int_0^t (\Delta u_\eta, u_\eta) d\eta \\
&= (u, u_{tt}) - (2\alpha+1)(u_t, u_t) + (\nabla u_1, \nabla u_0) - (2\alpha+1)\beta \\
&\quad - \int_0^t (u_{\eta\eta}, u)_\eta d\eta + 2(\alpha+1)\int_0^t (u_{\eta\eta}, u_\eta) d\eta \\
&\quad - \int_0^t \left[(Au, u)_\eta - 2(\alpha+1)(Au, u_\eta)\right] d\eta \\
&\quad + \int_0^t \left[(\mathbf{div}\,\vec{\sigma}(\nabla u), u)_\eta - 2(\alpha+1)(\mathbf{div}\,\vec{\sigma}(\nabla u), u_\eta)\right] d\eta.
\end{aligned} \tag{36}$$

Thus

$$\begin{aligned}
H'(t) &= -2\alpha(u_{tt}, u_t) + 2\alpha(u_t, Au) \\
&\quad + (\mathbf{div}\,\vec{\sigma}(\nabla u), u)_t - 2(\alpha+1)\frac{d}{dt}G(u) \\
&= -2\alpha(\Delta u_t, u_t) - 2\alpha(\mathbf{div}\,\vec{\sigma}(\nabla u), u_t) \\
&\quad + 4\alpha(u_t, Au) + (\mathbf{div}\,\vec{\sigma}(\nabla u), u)_t - 2(\alpha+1)\frac{d}{dt}G(u) \\
&= 2\alpha(\nabla u_t, \nabla u_t) + 4\alpha(u_t, Au) \\
&\quad + (\mathbf{div}\,\vec{\sigma}(\nabla u), u)_t - (4\alpha+2)\frac{d}{dt}G(u).
\end{aligned} \tag{37}$$

Using the positive semidefiniteness of A, Lemma 3, and (14), we have

$$\begin{aligned}
H'(t) &= H(0) + 2\alpha\int_0^t (\nabla u_\eta, \nabla u_\eta) d\eta + 4\alpha\int_0^t (u_\eta, Au) d\eta \\
&\quad + \int_0^t \left[(\mathbf{div}\,\vec{\sigma}(\nabla u), u)_\eta - (4\alpha+2)\frac{d}{d\eta}G(u)\right] d\eta \\
&= (4\alpha+2)G(u_0) - (2\alpha+1)(u_0, Au_0) \\
&\quad + (u_1, u_1) + (\nabla u_1, \nabla u_1) - (2\alpha+1)\beta \\
&\quad + 2\alpha\int_0^t (\nabla u_\eta, \nabla u_\eta) d\eta + 2\alpha(u, Au) \\
&\quad + \left[(\mathbf{div}\,\vec{\sigma}(\nabla u), u) - (4\alpha+2)G(u)\right] \\
&\geq (4\alpha+2)G(u_0) - (2\alpha+1)(u_0, Au_0) - (2\alpha+1)\beta \\
&= 2(2\alpha+1)\left[G(u_0) - \frac{1}{2}(Au_0, u_0) - \frac{\beta}{2}\right].
\end{aligned} \tag{38}$$

Thus, from what has been discussed above, we have

$$\begin{aligned}
&FF'' - (\alpha+1)(F')^2 \\
&\quad \geq 4(2\alpha+1)F\left[G(u_0) - \frac{1}{2}(Au_0, u_0) - \frac{\beta}{2}\right].
\end{aligned} \tag{39}$$

Therefore, for any $\beta > 0$ such that

$$\begin{aligned}
\beta &\leq 2G(u_0) - (Au_0, u_0) \\
&= 2\int_0^1 (\mathbf{div}\,\vec{\sigma}(\rho\nabla u_0), u_0) d\rho \\
&\quad - \left[\left\|\nabla^3 u_0\right\|^2_{L^2(\Omega)} + \left\|\nabla u_0\right\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 u_0\right\|^2_{L^2(\Omega)}\right],
\end{aligned} \tag{40}$$

$FF'' - (\alpha+1)(F')^2 \geq 0$ and $(F^{-\alpha}(t))'' \leq 0$. We see that $F(t) > 0$, for all $t \in [0, T_0)$ and $F'(0) = 2(u_0, u_1) + 2\beta\tau > 0$, if τ is sufficiently large. Since a concave function must always lie below any tangent line, so we have

$$F^{-\alpha}(t) \leq F^{-\alpha}(0) + \left[F^{-\alpha}(0)\right]' t \tag{41}$$

or

$$F(t) \geq F^{1+(1/\alpha)}(0)\left[F(0) - \alpha t F'(0)\right]^{-1/\alpha}; \tag{42}$$

we may choose T_0 such that $T_0 \geq (F(0)/\alpha(F'(0))) \equiv T_{\beta\tau}$. Thus, we see that the interval of existence of u must be contained in $[0, F(0)/\alpha(F'(0)))$ and that the finite time blow-up of solution of (9) is proved. Let

$$\begin{aligned}
\beta_0 &= 2G(u_0) - (Au_0, u_0) \\
&= 2\int_0^1 (\mathbf{div}\,\vec{\sigma}(\rho\nabla u_0), u_0) d\rho \\
&\quad - \left[\left\|\nabla^3 u_0\right\|^2_{L^2(\Omega)} + \left\|\nabla u_0\right\|^2_{L^2(\Omega)} - \mu\left\|\nabla^2 u_0\right\|^2_{L^2(\Omega)}\right], \\
T_{\beta\tau} &= \frac{F(0)}{\alpha(F'(0))} = \frac{(u_0, u_0) + T_0(\nabla u_0, \nabla u_0) + \beta\tau^2}{2\alpha(u_0, u_1) + 2\alpha\beta\tau}.
\end{aligned} \tag{43}$$

Since $T_0 \geq T_{\beta\tau}$, we have $(F^{-\alpha}(t))'' \leq 0$. Even if we take $T_0 = T_{\beta\tau}$, we have

$$T_{\beta\tau} = \left[(u_0, u_0) + \beta\tau^2\right]\left[2\alpha(u_0, u_1) + 2\alpha\beta\tau - (\nabla u_0, \nabla u_0)\right]^{-1}; \tag{44}$$

thus we must choose τ so large such that $2\alpha\beta\tau > (\nabla u_0, \nabla u_0) - 2\alpha(u_0, u_1)$. As a function, $T_{\beta\tau}$ has a minimum at

$$\begin{aligned}\tau &= \tau(\beta) \\ &= \Big((\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1) \\ &\quad + \sqrt{[(\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1)]^2 + 4\alpha^2 \beta (u_0, u_0)} \Big) \\ &\quad \times (2\alpha\beta)^{-1},\end{aligned} \tag{45}$$

and this minimum is

$$\begin{aligned}T_{\beta\tau(\beta)} &= \Big(4\alpha^2 \beta (u_0, u_0) \\ &\quad + \Big[(\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1) \\ &\quad + \sqrt{[(\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1)]^2 + 4\alpha^2 \beta (u_0, u_0)} \Big]^2 \Big) \\ &\quad \times \Big(4\alpha^2 \beta \big([(\nabla u_0, \nabla u_0) - 2\alpha (u_0, u_1)]^2 \\ &\quad + 4\alpha^2 \beta (u_0, u_0) \big)^{1/2} \Big)^{-1}.\end{aligned} \tag{46}$$

Since β is restricted to $[0, \beta_0]$, we see that $T_{\beta\tau(\beta)}$ attains its minimum at $\beta = \beta_0$. Thus T cannot exceed $T_{\beta\tau(\beta_0)}$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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