Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 304867, 6 pages
http://dx.doi.org/10.1155/2014/304867

Research Article

Hankel Operators on the Weighted L^p-Bergman Spaces with Exponential Type Weights

Hong Rae Cho and Jeong Wan Seo

Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea

Correspondence should be addressed to Hong Rae Cho; chohr@pusan.ac.kr

Received 5 November 2013; Accepted 20 December 2013; Published 12 February 2014

Academic Editor: Abdelghani Bellouquid

We characterize the boundedness and compactness of the Hankel operator with conjugate analytic symbols on the weighted L^p-Bergman spaces with exponential type weights.

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and $dA(z)$ the area measure on $\mathbb{D}$, and denote by $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$. For $0 < p \le \infty$, the weighted Bergman space A^p_φ is the space of functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{p,\varphi} = \int_{\mathbb{D}} \left|f(z)e^{-\varphi(z)}\right|^p dA(z) < \infty \quad \text{for } 0 < p < \infty,$$
$$\|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{D}} |f(z)| e^{-\varphi(z)}. \tag{1}$$

Note that A^p_φ is the closed subspace of $L^p_\varphi := L^p(\mathbb{D}, e^{-p\varphi}dA)$ consisting of analytic functions. Since the space A^2_φ is a reproducing kernel Hilbert space, for each $z \in \mathbb{D}$, there are functions $K_z \in A^2_\varphi$ with $f(z) = \langle f, K_z\rangle$, where $\langle\cdot,\cdot\rangle$ is the usual inner product in L^2_φ. The orthogonal projection from L^2_φ to A^2_φ is given by

$$Pf(z) = \int_{\mathbb{D}} f(w) K(w,z) e^{2\varphi} dA(w), \tag{2}$$

where $K(w,z) = \overline{K_z(w)}$.

Given $\sigma \in C^1(\mathbb{D})$ so that there exists a dense subset $\mathscr{D}$ of A^2_φ with $\sigma f \in L^2_\varphi$ for $f \in \mathscr{D}$, the big Hankel operator H_σ with symbol σ is densely defined by

$$H_\sigma f = \sigma f - P(\sigma f), \quad f \in \mathscr{D}, \tag{3}$$

where P is the orthogonal projection of L^2_φ onto A^2_φ.

We write $\overline{\partial} = \partial/\partial\overline{z}$. Then the $\overline{\partial}$-equation can be written by

$$\overline{\partial}u = f. \tag{4}$$

For $f \in A^2_\varphi$, we look for a solution $v \in L^2_\varphi$ of minimal L^2_φ-norm. Notice that the solution of minimal norm is the one that is orthogonal to the kernel of $\overline{\partial}$ on L^2_φ; that is, $v \perp A^2_\varphi$. Then, if $u \in L^2_\varphi$ solves (4), we get

$$v = (I-P)u. \tag{5}$$

The linear operator $N : A^2_\varphi \to L^2_\varphi$ given by

$$N(f) = v \tag{6}$$

is called the canonical solution operator to $\overline{\partial}$ on A^2_φ.

For any $f \in A^2_\varphi$, obviously $\overline{\partial}(\overline{z}f) = f$ and

$$N(f) = (I-P)(\overline{z}f) = H_{\overline{z}}f. \tag{7}$$

That is, the canonical solution operator coincides with the big Hankel operator acting on A^2_φ with symbol $\overline{z}$. Motivated by this fact, we now consider Hankel operators with conjugate analytic symbols on A^2_φ. For $f, g \in A^2_\varphi$, we do not necessarily have $\overline{g}f \in L^2_\varphi$. Let $\mathcal{D} := \text{Span}\{K_z : z \in \mathbb{D}\}$. Then $\mathcal{D}$ is dense in A^2_φ. For symbol $g \in A^2_\varphi$ such that

$$\overline{g}K_z \in L^2_\varphi \quad \forall z \in \mathbb{D}, \tag{8}$$

we consider the densely defined big Hankel operator on A^2_φ given by

$$H_{\overline{g}}f = (I - P)(\overline{g}f). \tag{9}$$

A positive function τ on $\mathbb{D}$ is said to belong to the class $\mathscr{E}$ if it satisfies the following three properties.

(a) There exists a constant $C_1 > 0$ such that

$$\tau(z) \le C_1(1 - |z|), \quad \text{for } z \in \mathbb{D}. \tag{10}$$

(b) There exists a constant $C_2 > 0$ such that

$$|\tau(z) - \tau(w)| \le C_2|z - w|, \quad \text{for } z, w \in \mathbb{D}. \tag{11}$$

(c) For each $m \ge 1$, there are constants $b_m > 0$ and $0 < t_m < 1/m$ such that

$$\tau(z) \le \tau(w) + t_m|z - w|, \quad \text{for } |z - w| > b_m\tau(w). \tag{12}$$

In this paper, we characterize the boundedness and compactness of the Hankel operator with conjugate analytic symbols on the weighted L^p-Bergman spaces with exponential type weights as follows.

Theorem 1. *Let $1 \le p < \infty$ and $g \in A^p_\varphi$. Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, and the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ is in the class $\mathscr{E}$. Then $H_{\overline{g}}$ extends to a bounded linear operator on A^p_φ if and only if*

$$\sup_{z\in\mathbb{D}} \tau(z)\left|g'(z)\right| < \infty. \tag{13}$$

Theorem 2. *Let $1 \le p < \infty$ and $g \in A^p_\varphi$. Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, and the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ is in the class $\mathscr{E}$. Then $H_{\overline{g}}$ extends to a compact linear operator on A^p_φ if and only if*

$$\lim_{|z|\to 1^-} \tau(z)\left|g'(z)\right| = 0. \tag{14}$$

In [1], Luecking firstly proved the same results in the context of the ordinary L^2-Bergman spaces. For L^2-Bergman spaces with exponential type weights, the same results were proved in [2–4]. Moreover, Schatten-class Hankel operators are also indicated in their papers.

The expression $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \le Cg$, and $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$.

2. Preliminaries

From now on we assume that $\varphi \in C^2(\mathbb{D})$, $\Delta\varphi > 0$, and the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ is in the class $\mathscr{E}$. The following notations will be frequently used:

$$m_\tau = \frac{\min\left(1, C_1^{-1}, C_2^{-1}\right)}{4}, \tag{15}$$

where C_1 and C_2 are the constants in the conditions (a) and (b) in Section 1 and

$$d_\tau(z, w) = \frac{|z - w|}{\min[\tau(z), \tau(w)]}. \tag{16}$$

Lemma 3 (see [5]). *For each $M \ge 1$, there exists a constant $C > 0$ (depending on M) such that for $z, w \in \mathbb{D}$, one has*

$$|K(w, z)| \le C\frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)} \min\left[1, \frac{1}{d_\tau(z, w)}\right]^M. \tag{17}$$

By using the upper estimates for $K(w, z)$ in Lemma 3, Arroussi and Pau [5] proved that the orthogonal projection P projects L^p_φ boundedly onto A^p_φ for $1 \le p \le \infty$.

Lemma 4 (see [6]). *Let $0 < \rho \le m_\tau$ and $w \in \mathbb{D}$. Then,*

$$\frac{3}{4}\tau(w) \le \tau(z) \le \frac{5}{4}\tau(w), \quad z \in D^\rho(w). \tag{18}$$

By using the third Green formula, we get the following two approximation results.

Lemma 5 (see [7]). *For small $K > 0$, there exists $A = A(K) > 0$ such that*

$$\sup_{w\in D^K(z)} \left|\varphi(w) - \varphi(z) - h_z(w)\right| \le A, \tag{19}$$

where h_z is a harmonic function in $D^K(z)$ with $h_z(z) = 0$.

Lemma 6 (see [7]). *For small $K > 0$, one has the estimate*

$$\left|\frac{\partial\varphi}{\partial w}(w) - \frac{\partial h_z}{\partial w}(w)\right| \lesssim \frac{1}{\tau(w)} \quad \text{for } w \in D^K(z), \tag{20}$$

where h_z is defined in Lemma 5.

The following is a certain submean value property of $|f(z)|e^{-\varphi(z)}$. We follow the proof of ([8], Lemma 19).

Lemma 7. *Let $0 < p < \infty$. For any small $r > 0$, there exists $C = C(r) > 0$ such that for any $f \in A^p_\varphi$ and $z \in \mathbb{D}$*

(a) $|f(z)|e^{-\varphi(z)} \le C((1/\tau(z)^2)\int_{D^r(z)} |f|^p e^{-p\varphi}\, dA)^{1/p}$;

(b) $|\nabla(|f|e^{-\varphi})(z)| \le C(1/\tau(z))((1/\tau(z)^2)\int_{D^r(z)} |f|^p e^{-p\varphi}\, dA)^{1/p}$, *provided* $f(z) \ne 0$.

Proof. (a) By Lemma 5, there exists some constant $A = A(K) > 0$ such that

$$\sup_{w\in D^K(z)} \left|\varphi(w) - \varphi(z) - h_z(w)\right| \le A, \tag{21}$$

for $z \in \mathbb{D}$. Since h_z is harmonic, there is an analytic function Φ_z on $D^K(z)$ such that $\Phi_z(z) = 0$ and $\operatorname{Re}\Phi_z = h_z$ on $D^K(z)$. Thus we have $|e^{\Phi_z}| = e^{h_z}$. Hence by the submean value property together with Lemma 5, we get

$$\begin{aligned} |f(z)|^p &= \left|f(z)e^{-(1/p)\Phi_z(z)}\right|^p \\ &\leq \frac{1}{(\pi K\tau(z))^2}\int_{D^K(z)}\left|fe^{-(1/p)\Phi_z}\right|^p dA \\ &= \frac{1}{(\pi K\tau(z))^2}\int_{D^K(z)}\left|fe^{-(1/p)h_z}\right|^p dA \\ &\lesssim \frac{e^{\varphi(z)}}{\tau(z)^2}\int_{D^K(z)}\left|fe^{-(1/p)\varphi}\right|^p dA. \end{aligned} \tag{22}$$

(b) We begin as follows:

$$\begin{aligned} \left|\nabla\left(|f|e^{-\varphi}\right)\right| &= \left|\frac{1}{2}\frac{1}{|f|}f'\overline{f}e^{-\varphi} - |f|e^{-\varphi}\frac{\partial\varphi}{\partial w}\right| \\ &\lesssim \left|f'e^{-\varphi} - 2fe^{-\varphi}\frac{\partial\varphi}{\partial w}\right| \\ &\lesssim \left|f'e^{-\varphi} - 2fe^{-\varphi}\frac{\partial h_z}{\partial w}\right| + \left|2fe^{-\varphi}\frac{\partial h_z}{\partial w} - 2fe^{-\varphi}\frac{\partial\varphi}{\partial w}\right| \\ &\lesssim \left|f' - 2f\frac{\partial h_z}{\partial w}\right|e^{-\varphi} + |f|e^{-\varphi}\left|\frac{\partial h_z}{\partial w} - \frac{\partial\varphi}{\partial w}\right|. \end{aligned} \tag{23}$$

Since h_z is harmonic, there is an analytic function $\Phi_z \in H(D^K(z))$ such that

$$\Phi_z(z) = 0, \qquad \Phi_z' = 2\frac{\partial h_z}{\partial w}, \qquad \left|e^{\Phi_z}\right| = e^{h_z}. \tag{24}$$

Note that

$$\begin{aligned} \left|\nabla\left(fe^{-\Phi_z}\right)(z)\right| &= \left|f'(z) - f(z)\Phi_z'(z)\right| \\ &= \left|f'(z) - 2f(z)\frac{\partial h_z}{\partial w}(z)\right|. \end{aligned} \tag{25}$$

On the other hand,

$$\begin{aligned} \left|\nabla\left(fe^{-\Phi_z}\right)(z)\right| &\lesssim \left|\int_{|z-\zeta|=\tau(z)}\frac{f(\zeta)e^{-\Phi_z(\zeta)}}{(z-\zeta)^2}d\zeta\right| \\ &\lesssim \frac{1}{\tau(z)^2}\int_{|z-\zeta|=\tau(z)}|f(\zeta)|e^{-h_z(\zeta)}|d\zeta|. \end{aligned} \tag{26}$$

For $|z-\zeta| = \tau(z)$, we have

$$\begin{aligned} |f(\zeta)|e^{-h_z(\zeta)} &\lesssim |f(\zeta)|e^{-\varphi(\zeta)+\varphi(z)} \\ &\lesssim \left(\frac{1}{\tau(z)^2}\int_{D^K(z)}|f|^p e^{-p\varphi}dA\right)^{1/p}e^{\varphi(z)}. \end{aligned} \tag{27}$$

Hence we have

$$\left|\nabla\left(fe^{-\Phi_z}\right)(z)\right| \lesssim \frac{1}{\tau(z)}\left(\frac{1}{\tau(z)^2}\int_{D^K(z)}|f|^p e^{-p\varphi}dA\right)^{1/p}e^{\varphi(z)}. \tag{28}$$

Thus

$$\left|\nabla\left(|f|e^{-\varphi}\right)(z)\right| \lesssim \frac{1}{\tau(z)}\left(\frac{1}{\tau(z)^2}\int_{D^K(z)}|f|^p e^{-p\varphi}dA\right)^{1/p}. \tag{29}$$

□

Despite that the next result was proved in [4], we give the proof of different method by using Lemma 7.

Proposition 8. *There is an $r > 0$ independent of z such that*

$$|K(w,z)| \gtrsim \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)}, \quad w \in D^r(z). \tag{30}$$

Proof. Let $r > 0$. By (b) of Lemma 7, we have

$$\begin{aligned} &\left||K(w,z)|e^{-\varphi(w)} - |K(z,z)|e^{-\varphi(z)}\right| \\ &\quad\lesssim \frac{|w-z|}{\tau(z)}\left(\frac{1}{\tau(z)^2}\int_{D^r(z)}|K(\zeta,z)|^2e^{-2\varphi(\zeta)}dA(\zeta)\right)^{1/2} \\ &\quad\lesssim \frac{r}{\tau(z)}K(z,z)^{1/2}, \quad w \in D^r(z). \end{aligned} \tag{31}$$

Hence it follows that

$$\begin{aligned} |K(w,z)|e^{-\varphi(w)} &\gtrsim K(z,z)e^{-\varphi(z)} - \frac{r}{\tau(z)}K(z,z)^{1/2}, \\ &\qquad w \in D^r(z). \end{aligned} \tag{32}$$

Note that by Lemma 3 and (a) of Lemma 7, we have

$$K(z,z) \approx \tau(z)^{-2}e^{2\varphi(z)}, \quad z \in \mathbb{D}. \tag{33}$$

Thus we have

$$\begin{aligned} |K(w,z)|e^{-\varphi(w)} &\gtrsim \tau(z)^{-2}e^{\varphi(z)} - r\tau(z)^{-2}e^{\varphi(z)} \\ &\geq (1-r)\tau(z)^{-2}e^{\varphi(z)}, \\ &\qquad w \in D^r(z), \end{aligned} \tag{34}$$

if we choose small $r > 0$. □

3. Hankel Operators on A^p_φ

For the proof of boundedness of Hankel operator on L^2-Bergman spaces with exponential type weights in [2–4], they used Hörmander's L^2-estimates for $\overline{\partial}$. However, for L^p-Bergman spaces, we need the following L^p-estimates for $\overline{\partial}$.

Theorem 9 (see [6]). *Let $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$. Suppose that the function $\tau(z) = (\Delta\varphi(z))^{-1/2}$ satisfies conditions (a) and (b) in Section 1. Let $1 \leq p \leq \infty$. Then there is a solution u to the equation $\overline{\partial}u = f$ such that*

$$\int_{\mathbb{D}}\left|ue^{-\varphi}\right|^p dA(z) \lesssim \int_{\mathbb{D}}\left|\frac{fe^{-\varphi}}{\sqrt{\Delta\varphi}}\right|^p dA(z), \tag{35}$$

provided the right hand side integral is finite.

Let g be an analytic function in $\mathbb{D}$ satisfying the condition (8). Let

$$k_\zeta(z) = \frac{K(\zeta, z)}{\sqrt{K(\zeta,\zeta)}}, \quad \zeta, z \in \mathbb{D}. \tag{36}$$

By the reproducing formula in A^2_φ we get

$$H_{\overline{g}} k_\zeta(z) = \left(\overline{g(z)} - \overline{g(\zeta)}\right) k_\zeta(z), \quad \zeta, z \in \mathbb{D}. \tag{37}$$

Lemma 10. *Let $0 < p < \infty$. Then*

$$\|k_z\|^p_{p,\varphi} \approx \tau(z)^{2-p}. \tag{38}$$

Proof. Consider

$$\|k_z\|^p_{p,\varphi} = \int_{|z-w|\le\delta_0\tau(z)} + \int_{|z-w|>\delta_0\tau(z)} |k_z(w)|^p e^{p\varphi(w)} dA(w). \tag{39}$$

First,

$$\begin{aligned}\int_{|z-w|\le\delta_0\tau(z)} &|k_z(w)|^p e^{p\varphi(w)} dA(w) \\ &\approx \int_{|z-w|\le\delta_0\tau(z)} \frac{dA(w)}{\tau(w)^p} \approx \tau(z)^{2-p}.\end{aligned} \tag{40}$$

Now,

$$\begin{aligned}\int_{|z-w|>\delta_0\tau(z)} &|k_z(w)|^p e^{p\varphi(w)} dA(w) \\ &\lesssim \int_{|z-w|>\delta_0\tau(z)} \frac{1}{\tau(w)^p}\left(\frac{\min[\tau(z),\tau(w)]}{|z-w|}\right)^{pM} dA(w).\end{aligned} \tag{41}$$

We take a large constant $M > 1$ so that $pM > 2$. Then

$$\begin{aligned}\int_{|z-w|>\delta_0\tau(z)} &\frac{1}{\tau(w)^p}\left(\frac{\min[\tau(z),\tau(w)]}{|z-w|}\right)^{pM} dA(w) \\ &\lesssim \tau(z)^{pM-p}\int_{\delta_0\tau(z)<|z-w|<2} \frac{1}{|z-w|^{pM}} dA(w) \\ &\lesssim \tau(z)^{pM-p}\int_{\delta_0\tau(z)}^2 \frac{1}{r^{pM-1}}\, dr \\ &\lesssim \tau(z)^{2-p}.\end{aligned} \tag{42}$$

Thus we get the result. □

Theorem 11. *Let $1 \le p < \infty$. Let $g \in A^p_\varphi$. Then $H_{\overline{g}}$ extends to a bounded linear operator on A^p_φ if and only if*

$$\sup_{z\in\mathbb{D}} \tau(z)\left|g'(z)\right| < \infty. \tag{43}$$

Proof. Assume first that

$$\sup_{z\in\mathbb{D}} \tau(z)\left|g'(z)\right| < \infty. \tag{44}$$

By Theorem 9, there is a solution u of the equation $\overline{\partial}u = f\overline{\partial g}$ such that

$$\|u\|_{p,\varphi} \le C\left\|\tau f \overline{\partial g}\right\|_{p,\varphi}. \tag{45}$$

Since $H_{\overline{g}}f$ is the minimal L^2_φ-norm solution of the $\overline{\partial}$-equation, we have $H_{\overline{g}}f = (I-P)u$. In [5], Arroussi and Pau proved that the orthogonal projection P projects L^p_φ boundedly onto A^p_φ for $1 \le p \le \infty$. Thus we have

$$\left\|H_{\overline{g}}f\right\|_{p,\varphi} = \|(I-P)u\|_{p,\varphi} \lesssim \|u\|_{p,\varphi}. \tag{46}$$

By (45) and (46), we have

$$\left\|H_{\overline{g}}f\right\|_{p,\varphi} \lesssim \left\|\tau f\overline{\partial g}\right\|_{p,\varphi} \le \sup_{z\in\mathbb{D}} \tau(z)\left|g'(z)\right| \|f\|_{p,\varphi}, \tag{47}$$

which shows that $H_{\overline{g}}$ can be extended to a bounded linear operator on A^p_φ.

Conversely, assume that $H_{\overline{g}}$ is bounded on A^p_φ. Then we have

$$\left\|H_{\overline{g}}\left(\frac{k_z}{\|k_z\|_{p,\varphi}}\right)\right\|_{p,\varphi} < M \quad \text{for } z \in \mathbb{D}. \tag{48}$$

Using Proposition 8 and Lemma 3, there exists $r > 0$ such that

$$\left|k_\zeta(z)\right| = \frac{|K(\zeta,z)|}{\sqrt{K(\zeta,\zeta)}} \gtrsim \frac{e^{\varphi(z)}}{\tau(z)} \quad \text{on } z \in D^r(\zeta). \tag{49}$$

Hence we have

$$\begin{aligned}M^p &> \frac{1}{\|k_z\|^p_{p,\varphi}}\left\|H_{\overline{g}}k_\zeta\right\|^p_{p,\varphi} \\ &\approx \tau(z)^{p-2}\int_{\mathbb{D}} |g(z)-g(\zeta)|^p \left|k_\zeta(z)\right|^p e^{-p\varphi(z)} dA(z) \\ &\ge \tau(z)^{p-2}\int_{D^r(\zeta)} |g(z)-g(\zeta)|^p \left|k_\zeta(z)\right|^p e^{-p\varphi(z)} dA(z) \\ &\gtrsim \frac{1}{\tau(z)^2}\int_{D^r(\zeta)} |g(z)-g(\zeta)|^p dA(z).\end{aligned} \tag{50}$$

Since g is an analytic function in $\mathbb{D}$, by the Cauchy estimates applied to $g_\zeta(z) := g(z) - g(\zeta)$, we can now conclude

$$\begin{aligned}\tau(\zeta)^p&\left|g'(\zeta)\right|^p \\ &\lesssim \frac{1}{\tau(\zeta)^2}\int_{D^r(\zeta)} |g(z)-g(\zeta)|^p dA(z) \lesssim M^p, \\ &\qquad \zeta \in \mathbb{D}.\end{aligned} \tag{51}$$

Thus we get the result. □

Lemma 12. *Let $0 < p < \infty$. Then*

$$\frac{k_z}{\|k_z\|_{p,\varphi}} \longrightarrow 0 \textit{ uniformly on compact subsets}, \quad |z| \longrightarrow 1^-. \tag{52}$$

Proof. Let K be a compact subset of $\mathbb{D}$. We choose a large constant M so that $M-(2/p)-1>0$. Then we have for $w\in K$

$$\begin{aligned}\frac{|k_z(w)|}{\|k_z\|_{p,\varphi}} &\lesssim \tau(z)^{(2/p)-1}\frac{e^{\varphi(w)}}{\tau(w)}\frac{\min[\tau(z),\tau(w)]^M}{|z-w|}\\ &\lesssim \tau(z)^{M-(2/p)-2}\frac{1}{\operatorname{dist}(K,z)}\sup_{w\in K}e^{\varphi(w)}\\ &\lesssim (1-|z|)^{M-(2/p)-2}\frac{1}{\operatorname{dist}(K,z)}\sup_{w\in K}e^{\varphi(w)}\longrightarrow 0,\\ &\qquad |z|\longrightarrow 1^-,\end{aligned} \tag{53}$$

where $\operatorname{dist}(K,z)=\min\{|z-w|: w\in K\}$. □

Theorem 13. *Let $1\le p<\infty$. Let $g\in A^p_\varphi$. Then $H_{\overline{g}}$ extends to a compact linear operator on A^p_φ if and only if*

$$\lim_{|z|\to 1^-}\tau(z)\left|g'(z)\right|=0. \tag{54}$$

Proof. Suppose now that $H_{\overline{g}}$ is compact on A^p_φ. Then by Riesz-Tamarkin compactness theorem, we have

$$\lim_{r\to 1^-}\frac{1}{\|k_\zeta\|^p_{p,\varphi}}\int_{r<|z|<1}\left|H_{\overline{g}}k_\zeta(z)\right|^p e^{-p\varphi(z)}dA(z)=0, \tag{55}$$

uniformly in $\zeta\in\mathbb{D}$. Now, by Lemma 12,

$$\begin{aligned}&\frac{1}{\|k_\zeta\|^p_{p,\varphi}}\int_{|z|\le r}\left|H_{\overline{g}}k_\zeta(z)\right|^p e^{-p\varphi(z)}dA(z)\\ &\le \sup_{|z|\le r}\left(\frac{|k_\zeta(z)|}{\|k_\zeta\|_{p,\varphi}}\right)^p\\ &\times\int_{|z|\le r}|g(z)-g(\zeta)|^p e^{-p\varphi(z)}dA(z)\longrightarrow 0,\end{aligned} \tag{56}$$

as $|\zeta|\to 1^-$. Thus we have

$$\begin{aligned}&\left\|H_{\overline{g}}\left(\frac{k_z}{\|k_z\|_{p,\varphi}}\right)\right\|_{p,\varphi}\\ &=\frac{1}{\|k_\zeta\|_{p,\varphi}}\left\|H_{\overline{g}}k_\zeta\right\|_{p,\varphi}\longrightarrow 0,\quad |\zeta|\longrightarrow 1^-.\end{aligned} \tag{57}$$

We choose $\rho>0$ so that

$$\left|k_\zeta(z)\right|=\frac{|K(\zeta,z)|}{\sqrt{K(\zeta,\zeta)}}\gtrsim\frac{e^{\varphi(z)}}{\tau(z)}\quad\text{on } z\in D^r(\zeta). \tag{58}$$

Then

$$\begin{aligned}&\frac{1}{\|k_\zeta\|^p_{p,\varphi}}\int_{\mathbb{D}}\left|H_{\overline{g}}k_\zeta(z)\right|^p e^{-p\varphi(z)}dA(z)\\ &\gtrsim\frac{1}{\tau(z)^2}\int_{D^r(\zeta)}|g(z)-g(\zeta)|^p dA(z)\\ &\gtrsim \tau(\zeta)^p\left|g'(\zeta)\right|^p.\end{aligned} \tag{59}$$

This implies that

$$\lim_{|z|\to 1^-}\tau(z)\left|g'(z)\right|=0. \tag{60}$$

For $|\zeta|>r+\rho$, the inclusion $D^\rho(\zeta)\subset\{r<|z|<1\}$ holds, and

$$\begin{aligned}&\frac{1}{\|k_\zeta\|^p_{p,\varphi}}\int_{r<|z|<1}\left|H_{\overline{g}}k_\zeta(z)\right|^p e^{-p\varphi(z)}dA(z)\\ &=\frac{1}{\|k_\zeta\|^p_{p,\varphi}}\int_{r<|z|<1}|g(z)-g(\zeta)|^p\left|k_\zeta(z)\right|^p e^{-p\varphi(z)}dA(z)\\ &\gtrsim\frac{1}{\tau(z)^2}\int_{D^r(\zeta)}|g(z)-g(\zeta)|^p dA(z)\\ &\gtrsim\left(\tau(\zeta)\left|g'(\zeta)\right|\right)^p.\end{aligned} \tag{61}$$

This implies that

$$\lim_{|z|\to 1^-}\tau(z)\left|g'(z)\right|=0. \tag{62}$$

Assume now that

$$\lim_{|z|\to 1^-}\tau(z)\left|g'(z)\right|=0. \tag{63}$$

It is enough to show that for any sequence $\{f_n\}$ that is bounded in norm and converges uniformly to zero on compact subsets, we have $\|H_{\overline{g}}f_n\|_{p,\varphi}\to 0$ as $n\to\infty$. As in relation (46), we have

$$\left\|H_{\overline{g}}f_n\right\|^p_{p,\varphi}\lesssim\int_{\mathbb{D}}\left|g'\right|^p\tau^p\left|f_n\right|^p e^{-p\varphi}dA. \tag{64}$$

Now

$$\int_{\mathbb{D}}\left|g'\right|^p\tau^p\left|f_n\right|^p e^{-p\varphi}dA=\int_{|z|\le r}+\int_{r<|z|<1}\left|g'\right|^p\tau^p\left|f_n\right|^p e^{-p\varphi}dA. \tag{65}$$

Since $\{f_n\}$ converges uniformly to zero on compact subsets,

$$\begin{aligned}&\int_{|z|\le r}\left|g'\right|^p\tau^p\left|f_n\right|^p e^{-p\varphi}dA\\ &\le\sup_{|z|\le r}\left|f_n\right|^p\int_{|z|\le r}\left|g'\right|^p\tau^p e^{-p\varphi}dA\longrightarrow 0,\\ &\qquad n\longrightarrow\infty.\end{aligned} \tag{66}$$

Now

$$\begin{aligned}&\int_{r<|z|<1}\left|g'\right|^p\tau^p\left|f_n\right|^p e^{-p\varphi}dA\\ &\le\sup_{r<|z|<1}\left|g'\right|^p\tau^p\int_{\mathbb{D}}\left|f_n\right|^p e^{-p\varphi}dA\longrightarrow 0,\\ &\qquad r\longrightarrow 1^-.\end{aligned} \tag{67}$$

Hence $H_{\overline{g}}$ is compact. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This is supported by the National Research Foundation of Korea (NRF) Grant funded by the Korea government (MEST) (NRF-2011-0013740).

References

[1] D. H. Luecking, "Characterizations of certain classes of Hankel operators on the Bergman spaces of the unit disk," *Journal of Functional Analysis*, vol. 110, no. 2, pp. 247–271, 1992.

[2] P. Galanopoulos and J. Pau, "Hankel operators on large weighted Bergman spaces," *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 37, no. 2, pp. 635–648, 2012.

[3] P. Lin and R. Rochberg, "Hankel operators on the weighted Bergman spaces with exponential type weights," *Integral Equations and Operator Theory*, vol. 21, no. 4, pp. 460–483, 1995.

[4] P. Lin and R. Rochberg, "Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights," *Pacific Journal of Mathematics*, vol. 173, no. 1, pp. 127–146, 1996.

[5] H. Arroussi and J. Pau, "Reproducing kernel estimates, bounded projections and duality on largeweighted Bergman spaces," http://arxiv.org/abs/1309.6072.

[6] H. R. Cho and S. K. Han, "Exponentially weighted L^p-estimates for $\bar{\partial}$ on the unit disc," *Journal of Mathematical Analysis and Applications*, vol. 404, no. 1, pp. 129–134, 2013.

[7] H. R. Cho, S. K. Han, and I. Park, "Peak functions for Bergman spaces with exponential type weight on the unit disc," preprint.

[8] N. Marco, X. Massaneda, and J. Ortega-Cerdà, "Interpolating and sampling sequences for entire functions," *Geometric and Functional Analysis*, vol. 13, no. 4, pp. 862–914, 2003.