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Research Article

The 2-Pebbling Property of the Middle Graph of Fan Graphs

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A pebbling move on a graph G consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. The pebbling number of a connected graph G, denoted by $f(G)$, is the least n such that any distribution of n pebbles on G allows one pebble to be moved to any specified but arbitrary vertex by a sequence of pebbling moves. This paper determines the pebbling numbers and the 2-pebbling property of the middle graph of fan graphs.

1. Introduction

Pebbling on graphs was first introduced by Chung [1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and the placement of one of those pebbles on an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G, v)$ with the property that from every placement of $f(G, v)$ pebbles on G, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of a graph G, denoted by $f(G)$, is the maximum of $f(G, v)$ over all the vertices of G.

In a graph G, if each vertex (except v) has at most one pebble, then no pebble can be moved to v. Also, if u is of distance d from v and at most $2^d - 1$ pebbles are placed on u (and none elsewhere), then no pebble can be moved from u to v. So it is clear that $f(G) \geq \max\{|V(G)|, 2^D\}$, where $|V(G)|$ is the number of vertices of G and D is the diameter of G.

Throughout this paper, let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a distribution of pebbles on G, denote by $p(H)$ and $p(v)$ the number of pebbles on a subgraph H of G and the number of pebbles on a vertex v of G, respectively. In addition, denote by $\widetilde{p}(H)$ and $\widetilde{p}(v)$ the number of pebbles on H and the number of pebbles on v after a specified sequence of pebbling moves, respectively. For $uv \in E(G)$, $u \xrightarrow{m} v$ refers to taking $2m$ pebbles off u and placing m pebbles on v. Denote by $\langle v_1, v_2, \ldots, v_n \rangle$ the path with vertices $v_1, v_2, \ldots, v_n$ in order.

We now introduce some definitions and give some lemmas, which will be used in subsequent proofs.

Definition 1. A fan graph, denoted by F_n, is a path P_{n-1} plus an extra vertex v_0 connected to all vertices of the path P_{n-1}, where $P_{n-1} = \langle v_1, v_2, \ldots, v_{n-1} \rangle$.

Definition 2. The middle graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G.

Now one creates the middle graph of F_n. Edges $v_1v_2, v_2v_3, \ldots, v_{(n-2)(n-1)}$ of F_n are the inserted new vertices $u_{12}, u_{23}, \ldots, u_{(n-2)(n-1)}$ in the sequence, and edges $v_0v_1, v_0v_2, \ldots, v_0v_{n-1}$ of F_n are the inserted new vertices $u_{01}, u_{02}, \ldots, u_{0(n-1)}$, respectively. By joining by edges those pairs of these inserted vertices which lie on adjacent edges of F_n, this obtains the middle graph of F_n (see Figure 1).

Definition 3. A transmitting subgraph is a path $\langle v_0, v_1, \ldots, v_k \rangle$ such that there are at least two pebbles on v_0, and after a sequence of pebbling moves, one can transmit a pebble from v_0 to v_k.

Lemma 4 (see [2]). *Let $P_{k+1} = \langle v_0, v_1, \ldots, v_k \rangle$. If*

$$p(v_0) + 2p(v_1) + \cdots + 2^i p(v_i) + \cdots + 2^{k-1} p(v_{k-1}) \geq 2^k, \tag{1}$$

then P_{k+1} is a transmitting subgraph.

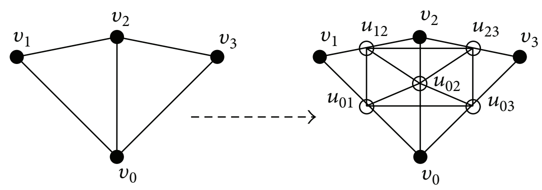

Figure 1: $M(F_4)$.

Definition 5. The t-pebbling number, $f_t(G)$, of a connected graph, G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to a specified target vertex by a sequence of pebbling moves.

Lemma 6 (see [3]). *If K_n is the complete graph with n $(n \geq 2)$ vertices, then $f_t(K_n) = 2t + n - 2$.*

Lemma 7 (see [4]). *Consider $f(M(P_n)) = 2^n + n - 2$.*

Chung found the pebbling numbers of the n-cube Q^n, the complete graph K_n, and the path P_n (see [1]). The pebbling number of C_n was determined in [5]. In [6, 7], Ye et al. gave the number of squares of cycles. Feng and Kim proved that $f(F_n) = n$ and $f(W_n) = n$ (see [8]). Liu et al. determined the pebbling numbers of middle graphs of P_n, K_n, and $K_{1,n-1}$ (see [4]). In [9], Ye et al. proved that $f(M(C_{2n})) = 2^{n+1} + 2n - 2$ $(n \geq 2)$ and $f(M(C_{2n+1})) = \lfloor 2^{n+3}/3 \rfloor + 2n$, where $M(C_n)$ denotes the middle graph of C_n. Motivated by these works, we will determine the value of the pebbling number and the 2-property of middle graphs of F_n.

2. Pebbling Numbers of $M(F_n)$

In this section, we study the pebbling number of $M(F_n)$. Let $S = \{v_0, u_{01}, u_{02}, \ldots, u_{0(n-1)}\}$, and let $A = \{v_1, u_{12}, v_2, u_{23}, \ldots, v_{n-1}\}$. Obviously, the subgraph induced by S is a complete graph with n vertices. For $n = 3$, $M(F_3) \cong M(C_3)$. Hence we have the following theorem.

Theorem 8 (see [9]). *Consider $f(M(F_3)) = 7$.*

Lemma 9. *Let $f(M(F_{n-1})) = p$. If $p + 3$ pebbles are placed on $M(F_n)$, then one pebble can be moved to any specified vertex of S by a sequence of pebbling moves.*

Proof. Let v be our target vertex, and let $p(v) = 0$, where $v \in S$. We may assume that $v \neq u_{01}$ (after relabeling if necessary). Let $B = \{v_1, u_{12}, u_{01}\}$. If $p(B) \geq 5$, then $\tilde{p}(u_{01}) \geq 2$ by Lemma 6, and we can move one pebble to v. If $p(B) = 4$, then $B \xrightarrow{1} u_{02}$. We delete v_1, u_{01}, and u_{12} to obtain the subgraph $M(F_{n-1})$ with p pebbles, thus we can move one pebble to v. If $p(B) \leq 3$, then we delete v_1, u_{01}, and u_{12} to obtain the subgraph $M(F_{n-1})$ with at least p pebbles and we are done. □

Theorem 10. *Consider $f(M(F_4)) = 11$.*

Proof. We place 7 pebbles on v_3 and one pebble on each vertex of the set $\{v_0, u_{02}, v_2\}$, other vertices have no pebble, then no pebble can be moved to v_1. So $p(M(F_4)) \geq 11$. We now place 11 pebbles on $M(F_4)$. We assume that v is our target vertex and $p(v) = 0$. Recall $S = \{v_0, u_{01}, u_{02}, u_{03}\}$ and $A = \{v_1, u_{12}, v_2, u_{23}, v_3\}$.

(1) Consider $v \in S$. By Theorem 8 and Lemma 9, we can move one pebble to v.

(2) Consider $v = v_1$ (or $v = v_3$). Let $A_1 = A - \{v_1\}$, let $A_2 = \{u_{12}, v_2\}$, and let $A_3 = A_1 - A_2$. If $p(S) = t$, then $p(A_1) = 11 - t$. Thus we can move at least $\lfloor (8-t)/2 \rfloor$ pebbles from A_1 to S so that $\tilde{p}(S) = \lfloor (8+t)/2 \rfloor \geq 6$ for $t \geq 4$. By Lemma 6, $\tilde{p}(u_{01}) = 2$ and we can move one pebble to v_1. If $t \leq 2$, then $p(A) \geq 9$. By Lemma 7, we can move one pebble to v_1. If $t = 3$, then at least one of u_{01} and u_{03} can obtain one pebble from every placement of 3 pebbles on S by a sequence of pebbling moves. If $p(A_3) \geq 7$, then $A_3 \xrightarrow{3} u_{03}$. So $\langle u_{03}, u_{01}, v_1 \rangle$ is a transmitting subgraph. If $4 \leq p(A_3) \leq 6$, then $2 \leq p(A_2) \leq 4$. By Lemma 6, $\tilde{p}(u_{23}) \geq 2$ and $\tilde{p}(u_{12}) \geq 1$. So $\langle u_{23}, u_{12}, v_1 \rangle$ is a transmitting subgraph. If $p(A_3) \leq 3$, then $p(A_2) \geq 5$. So $\langle v_2, u_{12}, v_1 \rangle$ is a transmitting subgraph.

(3) Consider $v = v_2$. If $p(S) \geq 4$ or $p(S) \leq 2$, then we are done with (2). If $p(S) = 3$, then $p(v_1) + p(u_{12}) \geq 4$ or $p(u_{23}) + p(v_3) \geq 4$. So $\langle v_1, u_{12}, v_2 \rangle$ or $\langle v_3, u_{23}, v_2 \rangle$ is a transmitting subgraph.

(4) Consider $v = u_{12}$ (or $v = u_{23}$). If $p(S) \geq 4$ or $p(S) \leq 2$, then we are done with (2). If $p(S) = 3$, then $p(v_1) + p(v_2) + p(u_{23}) + p(v_3) = 8$. Obviously, we are done if $p(v_1) \geq 2$ or $p(v_2) \geq 2$. Next suppose that $p(v_1) \leq 1$ and $p(v_2) \leq 1$. Thus $p(u_{23}) + p(v_3) \geq 6$. So $\langle v_3, u_{23}, u_{12} \rangle$ is a transmitting subgraph.

□

Theorem 11. *Consider $f(M(F_n)) = 3n - 1$ $(n \geq 4)$.*

Proof. We place 7 pebbles on v_{n-1} and one pebble on each vertex of $M(F_n)$ except $v_1, u_{01}, u_{12}, u_{(n-2)(n-1)}, u_{0(n-1)}$, and v_{n-1}. In this configuration of pebbles, we cannot move one pebble to v_1. So $f(M(F_n)) \geq 3n - 1$. Next, let us use induction on n to show that $f(M(F_n)) = 3n - 1$. For $n = 4$, our theorem is true by Theorem 10. Suppose that $f(M(F_k)) = 3k - 1$ if $k < n$. Now $3n - 1$ pebbles are placed arbitrarily on the vertices of $M(F_n)$. Suppose that v is our target vertex and $p(v) = 0$.

(1) Consider $v \in S$. By induction and Theorem 8, we can move one pebble to v.

(2) Consider $v = v_1$ (or $v = v_{n-1}$). Obviously, $p(u_{01}) \leq 1$. Otherwise, $p(u_{01}) > 1$. v_1 can obtain one pebble. Let $B_i = \{u_{i(i+1)}, u_{0(i+1)}, v_{i+1}\}$ $(1 \leq i \leq n-2)$.

If $p(B_{n-2}) \leq 3$, then we delete B_{n-2} to obtain the subgraph $M(F_{n-1})$ with at least $3(n-1) - 1$ pebbles. By induction, we can move one pebble to v_1. If $p(B_{n-2}) = 4$, then $B_{n-2} \xrightarrow{1} u_{0(n-2)}$. Thus we delete B_{n-2} to obtain the subgraph $M(F_{n-1})$ with $3(n-1) - 1$ pebbles. By induction, we are done.

Next, suppose that $p(B_{n-2}) \geq 5$. By Lemma 6, $\widetilde{p}(u_{0(n-1)}) \geq 2$. If $p(u_{01}) = 1$, then $\langle u_{0(n-1)}, u_{01}, v_1\rangle$ is a transmitting subgraph. If $p(v_0) \geq 2$, then $v_0 \xrightarrow{1} u_{01}$, and we are done. If there exists some B_i with $p(B_i) \geq 5$ $(i \neq n-2)$, then $B_i \xrightarrow{1} u_{01}$, and we are done. Thus we assume that $p(u_{01}) = 0$, $p(v_0) \leq 1$, and $p(B_i) \leq 4$ for $1 \leq i \leq n-3$.

Now, we consider B_i $(1 \leq i \leq n-3)$. Clearly, if $p(B_1) = 4$, then we are done. Suppose that there exists some B_j with $p(B_j) = 4$ $(j \neq 1)$. It is clear that if one of the three cases ((i) $p(u_{0j}) \geq 1$ $(u_{0j} \in B_{j-1})$, (ii) $p(B_{j-1}) \geq 3$, and (iii) $p(v_j) \geq 2$ $(v_j \in B_{j-1})$) happens, then we can move one pebble to v. Thus we assume that $p(B_i) = 4$ $(2 \leq i \leq n-3)$, $p(B_{i-1}) \leq 2$, $p(u_{0i}) = 0$, and $p(v_i) \leq 1$. If there are r sets $B_{i_1}, B_{i_2}, \ldots, B_{i_r}$ such that $p(B_{i_k}) = 4$ for $1 \leq k \leq r$, then $p(B_{i_k-1}) \leq 2$ for $1 \leq k \leq r$. Let $N_1 = \{i_1, i_2, \ldots, i_r\}$, let $N_2 = \{i_1 - 1, i_2 - 1, \ldots, i_r - 1\}$, and let $N_3 = \{1, 2, \ldots, n-3\} - N_1 - N_2$. If $p(B_j) = 2$ for all $j \in N_2$ and $p(B_k) = 3$ for all $k \in N_3$, then $\widetilde{p}(u_{j(j+1)}) = 1$ and $\widetilde{p}(u_{k(k+1)}) = 1$. Recall that $p(B_i) = 4$ for all $i \in N_1$ and $p(B_{n-2}) \geq 5$. Then $\widetilde{p}(u_{i(i+1)}) = 1$ and $\widetilde{p}(u_{(n-2)(n-1)}) = 2$. Thus $\langle u_{(n-2)(n-1)}, u_{(n-3)(n-2)}, \ldots, u_{12}, v_1\rangle$ is a transmitting subgraph. So there is at least some j in N_2 such that $p(B_j) \leq 1$ or at least some k in N_3 such that $p(B_k) \leq 2$. If there are two j' and j'' in N_2 such that $p(B_{j'}) \leq 1$ and $p(B_{j''}) \leq 1$ or two k' and k'' in N_3 such that $p(B_{k'}) \leq 2$ and $p(B_{k''}) \leq 2$ or some j in N_2 such that $p(B_j) \leq 1$ and some k in N_3 such that $p(B_k) \leq 2$, then $p(B_{n-2}) \geq 9$. By Lemma 6, $\widetilde{p}(u_{0(n-1)}) = 4$. Hence $\langle u_{0(n-1)}, u_{01}, v_1\rangle$ is a transmitting subgraph.

Finally, there are two remaining cases, (i) there is only some j in N_2 such that $p(B_j) \leq 1$, and (ii) there is only some k in N_3 such that $p(B_k) \leq 2$. So $p(B_{n-2}) \geq 8$. If $p(u_{(n-2)(n-1)}) = 0$, then $\langle v_{n-1}, u_{0(n-1)}, u_{01}, v_1\rangle$ is a transmitting subgraph. If $p(u_{(n-2)(n-1)}) \neq 0$, then, in B_{n-2}, $\widetilde{p}(u_{(n-2)(n-1)}) \geq 2$ and $\widetilde{p}(u_{0(n-1)}) \geq 2$. For (i), we have $\widetilde{p}(u_{i(i+1)}) \geq 1$ for $j+2 \leq i \leq n-3$. By the transmitting subgraph $\langle u_{(n-2)(n-1)}, u_{(n-3)(n-2)}, \ldots, u_{(j+1)(j+2)}\rangle$, $\widetilde{p}(B_{j+1}) = 5$ and we are done. Suppose that (ii) holds. If $p(B_k) = 2$, then we can move one pebble from $u_{0(n-1)}$ to $u_{0(k+1)}$ so that $p(B_k) = 3$, and we are done. If $p(B_k) \leq 1$, then $p(B_{n-2}) \geq 9$ and we are done.

(3) Consider $v = u_{12}$ (or $v = u_{(n-2)(n-1)}$). Obviously, $p(u_{01}) \leq 1$ and $p(v_i) \leq 1$ $(i = 1, 2)$. Otherwise, one pebble can be moved to u_{12}. The proof is similar to (2).

(4) Consider $v = v_i (2 \leq i \leq n-2)$ (or $v = u_{j(j+1)} (2 \leq j \leq n-3)$) and $p(v_i) = 0$. Let $B = \{v_1, u_{12}, u_{01}\}$, and let $B' = \{v_{n-1}, u_{(n-2)(n-1)}, u_{0(n-1)}\}$. If $p(B) \leq 3$, then delete B to obtain the subgraph $M(F_{n-1})$ with at least $3(n-1)-1$ pebbles. By induction, we can move one pebble to v. If $p(B) = 4$, then we can move one pebble from B to u_{02}, after deleting B to obtain the subgraph $M(F_{n-1})$ with $3(n-1)-1$ pebbles. Hence we assume that $p(B) \geq 5$. According to symmetry, $p(B') \geq 5$. Therefore we are done. □

3. The 2-Pebbling Property of $M(F_n)$

For a distribution of pebbles on G, let q be the number of vertices with at least one pebble. We say a graph G satisfies the 2-pebbling property if two pebbles can be moved to any specified vertex when the total starting number of pebbles is $2f(G) - q + 1$. Next we will discuss the 2-pebbling property of $M(F_n)$. For the convenience of statement, let $S = \{x_1, x_2, \ldots, x_n\}$, and let $A = \{y_1, y_2, \ldots, y_{2n-3}\}$, where $x_1 = v_0$, $x_2 = u_{01}, \ldots, x_n = u_{0(n-1)}$, $y_1 = v_1$, and $y_2 = u_{12}, \ldots, y_{2n-3} = v_{n-1}$. In this section let $q = q_s + q_a$, where q_s and q_a are the number of vertices with at least one pebble in S and A, respectively.

Lemma 12. *Suppose that $p(M(F_n)) \geq 2(3n-1) - q$ and $q_a = 2n-4$. If $p(S) = q_s + t$ $(t = 0, 1, 2)$ and $p(y_r) = 0$ $(1 \leq r \leq 2n-3)$, then one can move 2 pebbles to y_r.*

Proof. Let $r = 2k-1$ (or $r = 2k$). Since $q_a = 2n-4$ and $p(S) = q_s + t$, so $p(A) \geq 4n + 2 - 2q_s - t$. Without loss of generality, there exists a positive integer j $(j > r)$ such that the corresponding vertex y_j with $p(y_j) \geq 2$ and $p(y_i) = 1$ for $r+1 \leq i \leq j-1$. Thus $y_j \xrightarrow{1} y_{j-1} \xrightarrow{1} \cdots \xrightarrow{1} y_r$. Using the remaining $4n + 2 - t - 2q_s - (j - r + 1)$ pebbles on vertices $y_1, y_2, \ldots, y_{r-1}, y_j, y_{j+1}, \ldots, y_{2n-3}$, we can move at least $n + \lfloor (5-t)/2 \rfloor - q_s$ pebbles to S so that $\widetilde{p}(S) \geq n + \lfloor (5+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 2$. So we can move one additional pebble from x_{k+1} to y_r so that $\widetilde{p}(y_r) = 2$. □

Lemma 13. *Suppose that $p(M(F_n)) = 2(3n-1) - q + 1$ and $q_a = 2n-5$. If $p(S) = q_s + t$ $(t = 0, 1)$ and $p(y_r) = 0$ $(1 \leq r \leq 2n-3)$, then one can move 2 pebbles to y_r.*

Proof. Let $r = 2k-1$ (or $r = 2k$). Since $q_a = 2n-5$, we see that there is only some vertex y_{i_0} $(i_0 \neq r)$ with $p(y_{i_0}) = 0$. Without loss of generality, there exists a positive integer j $(j > r)$ such that the corresponding vertex y_j with $p(y_j) \geq 2$ and $p(y_i) \leq 1$ for $r < i < j$. If $i_0 = 2k_0 - 1$ $(k_0 \neq k)$ or $i_0 \notin \{r+1, r+2, \ldots, j-1\}$, then we can move one pebble to y_r by the transmitting subgraph $\langle y_j, y_{j-2}, \ldots, y_{r+1}, y_r\rangle$ or $\langle y_j, y_{j-1}, y_{j-3}, \ldots, y_{r+1}, y_r\rangle$. Now using the remaining at least $4n + 4 - t - 2q_s - (j - r + 1)$ pebbles on the set $A_1 = \{y_1, y_2, \ldots, y_{r-1}, y_j, y_j, \ldots, y_{2n-3}\}$, we can move $n + \lfloor (7-t)/2 \rfloor - q_s$ pebbles from the A_1 to S so that $\widetilde{p}(S) = n + \lfloor (7+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 2$ and we can move one additional pebble from x_{k+1} to y_r so that $\widetilde{p}(y_r) = 2$.

Suppose that $i_0 = 2k_0$ $(k_0 \geq k)$ and $i_0 \in \{r+1, r+2, \ldots, j-1\}$. If $j = i_0 + 1$, then $y_j \xrightarrow{1} y_{i_0}$. Thus there must exist one vertex $y_{j'}$ $(j' \geq j)$ with $p(y_{j'}) \geq 2$ and $p(y_i) \leq 1$ for $r < i < j'$. Using the transmitting subgraph $\langle y_{j'}, y_{j'-2}, \ldots, y_{r+1}, y_r\rangle$ or $\langle y_{j'}, y_{j'-1}, y_{j'-3}, \ldots, y_{r+1}, y_r\rangle$, we can move one pebble to y_r. Now, using the remaining $4n + 4 - t - 2q_s - (j' - r + 2)$ pebbles on the set $\{y_1, y_2, \ldots, y_{r-1}, y_{j'}, y_{j'+1}, \ldots, y_{2n-3}\}$, we can move $n + \lfloor (6-t)/2 \rfloor - q_s$ pebbles from the set $\{y_1, y_2, \ldots, y_{r-1}, y_{j'}, y_{j'+1}, \ldots, y_{2n-3}\}$ to S so that $\widetilde{p}(S) \geq n + \lfloor (6+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 2$ and we are done. Next, suppose that $j \geq i_0 + 2$.

(1) Consider $p(y_{2k}) = 1$. We divide into three subcases.

(1.1) Consider $p(x_{k+2}) = 0$. We delete vertices $y_r, y_{r+1}, \ldots, y_{2k_0}, x_{k+2}$ to obtain the subgraph with two sets $A_2 = A - \{y_r, y_{r+1}, \ldots, y_{2k_0}\}$ and $S_1 = S - \{x_{k+2}\}$, and $p(A_2) = 4n + 4 - 2q_s - t - (2k_0 - r - 1)$ and $p(S_1) = q_s + t$. Thus we can move $n + \lfloor (10-t)/2 \rfloor - q_s$ pebbles from A_2 to

S_1 so that $\widetilde{p}(S_1) = n + \lfloor (10+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 4$ and we can move two pebbles from x_{k+1} to y_r.

(1.2) Consider $p(x_{k+2}) = 1$. Suppose that $j = 2k'$ or $j = 2k'+1$ $(k' > k)$. Let $A_3 = \{y_{2k'}, y_{2k'+1}\}$. Obviously, $p(A_3) \geq 3$. If $p(A_3) \geq 5$, then

$$A_3 \xrightarrow{2} x_{k'+2} \xrightarrow{1} x_{k+2} \xrightarrow{1} y_{r+1} \xrightarrow{1} y_r. \tag{2}$$

We delete $y_r, y_{r+1}, \ldots, y_{2k_0}, x_{k+2}$ to obtain the subgraph with two sets A_2 and S_1. So $p(A_2) = 4n - 2q_s - t - (2k_0 - r - 1)$ and $\widetilde{p}(S_1) = q_s - 1 + t$. We can move $n + \lfloor (6-t)/2 \rfloor - q_s$ pebbles from A_2 to S_1 so that $\widetilde{p}(S_1) = n + \lfloor (4+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 2$ and we are done. If $p(A_3) = 3, 4$ and $p(x_{k'+2}) \neq 0$, then

$$A_3 \xrightarrow{1} x_{k'+2} \xrightarrow{1} x_{k+2} \xrightarrow{1} y_{r+1} \xrightarrow{1} y_r. \tag{3}$$

We delete $y_r, y_{r+1}, \ldots, y_{2k_0}, x_{k+2}$ to obtain the subgraph with two sets A_2 and S_1. So $p(A_2) = 4n+2-2q_s-t-(2k_0-r-1)$ and $\widetilde{p}(S_1) = q_s-2+t$. We can move $n+\lfloor (8-t)/2 \rfloor - q_s$ pebbles from A_2 to S_1 so that $\widetilde{p}(S_1) = n+\lfloor (4+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 2$ and we are done. If $p(A_3) = 3, 4$ and $p(x_{k'+2}) = 0$, then $A_3 \xrightarrow{1} x_{k'+1}$. We delete $y_r, y_{r+1}, \ldots, y_{2k_0}, y_{2k'}, y_{2k'+1}, x_{k'+2}$ to obtain the subgraph with two sets $A_4 = A_2 - A_3$ and $S_2 = S - \{x_{2k'+2}\}$. So $p(A_4) \geq 4n - 2q_s - t - (2k_0 - r - 1)$ and $\widetilde{p}(S_2) = q_s + 1 + t$. We can move $n + \lfloor (8-t)/2 \rfloor - q_s$ pebbles from A_4 to S_2 so that $\widetilde{p}(S_2) = n + \lfloor (10+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 4$.

(1.3) Consider $p(x_{k+2}) = 2$ for $t = 1$. Thus $x_{k+2} \xrightarrow{1} y_{2k} \xrightarrow{1} y_r$. We delete $y_r, y_{r+1}, \ldots, y_{2k_0}, x_{k+2}$ to obtain the subgraph with two sets A_2 and S_1. So $p(A_2) = 4n+3-2q_s-(2k_0-r-1)$ and $\widetilde{p}(S_1) = q_s - 1$. $n + 4 - q_s$ pebbles can be moved from A_2 to S_1 so that $\widetilde{p}(S_1) = n + 3$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 3$. So we can move one additional pebble from x_{k+1} to y_r.

(2) Consider $p(y_{2k}) = 0$; that is, $k = k_0$. We divide into three subcases.

(2.1) Consider $p(x_{2k+2}) = 0$. We delete $y_r, y_{r+1}, y_{r+2}, x_{2k+2}$ to obtain the subgraph with two sets $A_5 = A - \{y_r, y_{r+1}, y_{r+2}\}$ and S_1. One has $p(A_5) = 4n + 3 - 2q_s - t$ and $p(S_1) = q_s + t$. We can move $n + \lfloor (10-t)/2 \rfloor - q_s$ pebbles from A_5 to S_1 so that $\widetilde{p}(S_1) = n + \lfloor (10+t)/2 \rfloor$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 4$ and we can move two pebbles from x_{k+1} to y_r.

(2.2) Consider $p(x_{k+2}) = 1$. We have

$$y_j \xrightarrow{1} y_{j-1} \xrightarrow{1} \cdots \xrightarrow{1} y_{r+2} \xrightarrow{1} x_{k+2} \xrightarrow{1} x_{k+1}. \tag{4}$$

We delete vertices $y_r, y_{r+1}, \ldots, y_{j-1}, x_{k+2}$ to obtain the subgraph with two sets A_1 and S_1. So $p(A_1) = 4n + 4 - 2q_s - t - (j - r)$ and $\widetilde{p}(S_1) = q_s + t - 1$ (except one moved pebble on x_{k+1}). We can move $n + \lfloor (8-t)/2 \rfloor - q_s$ pebbles from A_5 to S_1 so that $\widetilde{p}(S_1) = n + \lfloor (6+t)/2 \rfloor$ (except one moved pebble on x_{k+1}). By Lemma 6, we can move 3 additional pebbles to x_{k+1} so that $\widetilde{p}(x_{k+1}) = 4$.

(2.3) $p(x_{k+2}) = 2$ for $t = 1$. Thus $x_{k+2} \xrightarrow{1} x_{k+1}$. Deleting $y_r, y_{r+1}, y_{r+2}, x_{k+2}$ to obtain the subgraph with two sets A_5 and S_1. One has $p(A_5) = 4n+2-2q_s$ and $\widetilde{p}(S_1) = q_s-1$ (except one moved pebble on x_{k+1}). We can move $n + 4 - q_s$ pebbles from A_4 to S_1 so that $\widetilde{p}(S_1) = n+3$ (except one moved pebble on x_{k+1}). By Lemma 6, we can move 3 additional pebbles to x_{k+1} so that $\widetilde{p}(x_{k+1}) = 4$. □

Theorem 14. *$M(F_n)$ has the 2-pebbling property.*

Proof. Suppose that v is our target vertex. If $p(v) = 1$, then the number of pebbles on $M(F_n)$ except one pebble on v is $2(3n-1)+1-q-1$ $(> 3n-1)$. By Theorem 11, we can move one additional pebble to v so that $\widetilde{p}(v) = 2$. Next we assume that $p(v) = 0$.

(1) Consider $v = x_r$ $(1 \leq r \leq n)$. If there exists some vertex x_i with $p(x_i) \geq 2$ $(i \neq r)$, then $x_i \xrightarrow{1} x_r$. Using the remaining $2(3n-1)+1-q-2 > 3n-1$ pebbles, we can move one additional pebble to x_r so that $\widetilde{p}(x_r) = 2$. If $p(x_i) \leq 1$ for $1 \leq i \leq n$, then $p(A) = 2(3n-1)-q+1-q_s = 6n-1-q_a-2q_s \geq 4n+2-2q_s$. Thus we can move at least $n + 2 - q_s$ pebbles from A to S so that $\widetilde{p}(S) = n + 2$. By Lemma 6, we can move two pebbles to x_r.

(2) Consider $v = y_r$ $(1 \leq r \leq 2n-3)$. Let $r = 2k-1$ (or $r = 2k$). If $p(x_{k+1}) \geq 2$, then we can put one pebble on y_r. After that, the remaining $2(3n-1)-q+1-2$ $(> 3n-1)$ pebbles on $M(F_n)$ suffice to put one additional pebble on y_r by Theorem 11. Next we assume $p(x_{k+1}) \leq 1$.

(2.1) Suppose that $p(x_{k+1}) = 1$. If there is some vertex x_i with $p(x_i) \geq 2$ $(i \neq k+1)$, then $x_i \xrightarrow{1} x_{k+1} \xrightarrow{1} y_r$. The remaining $2(3n-1)-q+1-3$ $(> 3n-1)$ pebbles on $M(F_n)$ will suffice to put one additional pebble on y_r so that $\widetilde{p}(y_r) = 2$. Next we assume that $p(x_i) \leq 1$ for $1 \leq i \leq n$. Obviously, $p(S) = q_s$ and $p(A) = 2(3n-1)-q+1-q_s = 6n-1-q_a-2q_s$. If $q_a \leq 2n-5$, then $p(A) \geq 4n+4-2q_s$. Thus we can move at least $n+5-q_s$ pebbles from A to S so that $\widetilde{p}(S) = n+5$. By Lemma 6, we can move 3 additional pebbles to x_{k+1} so that $\widetilde{p}(x_{k+1}) = 4$ and we are done. If $q_a = 2n-4$, then, by Lemma 12, we are done.

(2.2) Suppose that $p(x_{k+1}) = 0$. If we can find some vertex x_i with $p(x_i) \geq 4$ or find two vertices x_j with $p(v_j) \geq 2$ and $x_{j'}$ with $p(x_{j'}) \geq 2$, then using 4 pebbles on x_i or two pebbles on x_j and two pebbles on $x_{j'}$ we can move one pebble to y_r. Then the remaining $2(3n-1)-q+1-4$ $(> 3n-1)$ pebbles on $M(F_n)$ will suffice to put one additional pebble to y_r so that $\widetilde{p}(y_r) = 2$.

Consider only some vertex x_i with $2 \leq p(x_i) \leq 3$. If $p(x_i) = 3$, then $x_i \xrightarrow{1} x_{k+1}$, $\widetilde{p}(S) = q_s$, and $p(A) = 2(3n-1) - q_s - q_a + 1 - (q_s+2) = 6n-3-2q_s-q_a$. When $q_a \leq 2n-5$ and $p(A) \geq 4n+2-2q_s$, we can move at least $n+4-q_s$ pebbles from A to S so that $\widetilde{p}(S) \geq n+4$ except for one pebble on x_{k+1}. By Lemma 6, we can put 3 additional pebbles on x_{k+1} so that $\widetilde{p}(x_{k+1}) = 4$. When $q_a = 2n-4$, we are done with Lemma 12. If $p(x_i) = 2$, then $x_i \xrightarrow{1} x_{k+1}$, $\widetilde{p}(S) = q_s - 1$, and $p(A) = 2(3n-1) - q_s - q_a + 1 - (q_s+1) = 6n-2-2q_s-q_a$. When $q_a \leq 2n-6$ and $p(A) \geq 4n+4-2q_s$, we can move at least $n+5-q_s$ pebbles from A to S so that $\widetilde{p}(S) \geq n+4$ except for one pebble on x_{k+1}. By Lemma 6, we can put 3 additional pebbles on x_{k+1} so that $\widetilde{p}(x_{k+1}) = 4$. When $q_a = 2n-4$ and $q_a = 2n-5$, we are done with Lemmas 12 and 13.

Consider $p(x_i) \leq 1$ for $1 \leq i \leq n$. Obviously, $p(S) = q_s$ and $p(A) = 6n-1-q_a-2q_s$. When $q_a \leq 2n-6$

and $p(A) \geq 4n+5-2q_s$, we can move at least $n+6-q_s$ pebbles from A to S so that $\widetilde{p}(S) \geq n+6$. By Lemma 6, $\widetilde{p}(x_{k+1}) = 4$ and we are done. When $q_a = 2n-4$ and $q_a = 2n-5$, we are done with Lemmas 12 and 13. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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