Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 294182, 9 pages
http://dx.doi.org/10.1155/2014/294182

Research Article

Nonspectrality of Certain Self-Affine Measures on $\mathbb{R}^3$

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Received 11 September 2013; Accepted 30 November 2013; Published 8 January 2014

Academic Editor: Natig M. Atakishiyev

We will determine the nonspectrality of self-affine measure $\mu_{B,D}$ corresponding to $B = \operatorname{diag}[p_1, p_2, p_3]$ ($p_1 \in (2\mathbb{Z}+1) \setminus \{\pm 1\}$, $p_2 \in 2\mathbb{Z} \setminus \{0\}$), and $D = \{0, e_1, e_2, e_3\}$ in the space $\mathbb{R}^3$ is supported on $T(B, D)$, where e_1, e_2, and e_3 are the standard basis of unit column vectors in $\mathbb{R}^3$, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{B,D})$, where the number 4 is the best. This generalizes the known results on the spectrality of self-affine measures.

1. Introduction

Let $B \in M_n(\mathbb{Z})$ be an expanding integer matrix; that is, all the eigenvalues of the integer matrix B have modulus greater than 1. Associated with a finite subset $D \subset \mathbb{Z}^n$, there exists a unique nonempty compact set $T := T(B, D) \subset \mathbb{R}^n$ such that $T = \cup_{d\in D}\phi_d(T)$. More precisely, $T(B, D)$ is an attractor (or invariant set) of the affine iterated function system (IFS) $\{\phi_d(x)\}_{d\in D}$. Denote by $|D|$ the cardinality of D. Relating to the IFS $\{\phi_d(x)\}_{d\in D}$, there also exists a unique probability measure $\mu := \mu_{B,D}$ satisfying

$$\mu = \frac{1}{|D|}\sum_{d\in D}\mu \circ \phi_d^{-1}. \tag{1}$$

For a given pair (B, D), the spectrality or nonspectrality of $\mu_{B,D}$ is directly connected with the Fourier transform $\widehat{\mu}_{B,D}(\xi)$. From (1), we get

$$\begin{aligned} \widehat{\mu}_{B,D}(\xi) &:= \int e^{2\pi i\langle x,\xi\rangle}\, d\mu_{B,D}(x) \\ &= \prod_{j=1}^{\infty} m_D\left(B^{*-j}\xi\right), \quad (\xi \in \mathbb{R}^n), \end{aligned} \tag{2}$$

where

$$m_D(x) = \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,x\rangle}, \quad (x \in \mathbb{R}^n). \tag{3}$$

The self-affine measure $\mu_{B,D}$ has received much attention in recent years. The previous research on such measure and its Fourier transform revealed some surprising connections with a number of areas in mathematics, such as harmonic analysis, number theory, dynamical systems, and others; see [1, 2] and references cited therein.

The $\mu_{B,D}$ and $T(B, D)$ are all determined by the pair (B, D). So, for $n = 1$, in the way of examples, there are Cantor set and Cantor measure on the line. And for $n = 2$ there is a rich variety of geometries, see Li [3–6], of which the best known example is the Sierpinski gasket. But for $n = 3$, it is more complex.

The problem considered below started with a discovery in an earlier paper of Jorgensen and Pedersen [7] where it was proved that certain IFS fractals have Fourier bases, and furthermore that the question of counting orthogonal Fourier frequencies (or orthogonal exponentials in $L^2(\mu_{B,D})$) for a fixed fractal involves an intrinsic arithmetic of the finite set of functions making up the IFS $\{\phi_d(x)\}_{d\in D}$ under consideration. For example, if $(B, D) = (3, \{0, 2\})$ is the middle-third Cantor example on the line, there cannot be more than two orthogonal Fourier frequencies [7, Theorem 6.1], while a similar Cantor example, using instead a subdivision scale 4 (i.e., $(B, D) = (4, \{0, 2\})$), turns out to have an ONB in $L^2(\mu_{B,D})$ consisting of Fourier frequencies [7, Theorem 3.4].

With the effort of Jorgensen and Pedersen [7, Example 7.1], Strichartz [8], Li [3], and Yuan [9], the related conclusions discussed that the diagonal elements of B are all even or odd, If one of the diagonal elements is even, what about the result? The general case on the spectrality or nonspectrality of the self-affine measure $\mu_{B,D}$ is not known.

The present paper is motivated by these earlier results; we will determine the nonspectrality of self-affine measure $\mu_{B,D}$; the main result of the present paper is the following.

Theorem 1. *Let $\mu_{B,D}$ correspond to $B = \operatorname{diag}[p_1, p_1, p_2]$($p_1 \in (2\mathbb{Z}+1) \setminus \{\pm 1\}$, $p_2 \in 2\mathbb{Z} \setminus \{0\}$) and $D = \{0, e_1, e_2, e_3\}$; then the self-affine measure $\mu_{B,D}$ is a nonspectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{B,D})$, where the number 4 is the best.*

By the proof of Theorem 1, we get a more general case.

Theorem 2. *Let $\mu_{B,D}$ correspond to*

$$B = \begin{bmatrix} p_1 & 0 & p_4 \\ 0 & p_1 & p_3 \\ 0 & 0 & p_2 \end{bmatrix} \quad (p_1 \in (2\mathbb{Z}+1) \setminus \{\pm 1\}; p_2, p_3, p_4 \in \mathbb{Z}),$$
$$D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}; \tag{4}$$

then the self-affine measure $\mu_{B,D}$ is a nonspectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{B,D})$, where the number 4 is the best.

2. Proof of Theorem 1

Firstly, we know from (2) that the zero set $Z(\hat{\mu}_{B,D}(\xi))$ of the Fourier transform $\hat{\mu}_{B,D}(\xi)$ is

$$Z(\hat{\mu}_{B,D}(\xi)) = \bigcup_{j=1}^{\infty} B^{*j} Z(m_D(\xi)) := B_1 \cup B_2 \cup B_3. \tag{5}$$

For the given digit set D in (4), we have

$$Z(m_D(x)) := \left\{ x \in \mathbb{R}^3 : m_D(x) = 0 \right\} = A_1 \cup A_2 \cup A_3, \tag{6}$$

where

$$m_D(x) = \frac{1}{4}\left\{1 + e^{2\pi i x_1} + e^{2\pi i x_2} + e^{2\pi i x_3}\right\}, \quad x = (x_1, x_2, x_3)^t \in \mathbb{R}^3, \tag{7}$$

and A_1, A_2, and A_3 are given by

$$A_1 = \left\{ \begin{pmatrix} \frac{1}{2} + k_1 \\ a + k_2 \\ \frac{1}{2} + a + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3,$$
$$A_2 = \left\{ \begin{pmatrix} \frac{1}{2} + a + k_1 \\ \frac{1}{2} + k_2 \\ a + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3,$$
$$A_3 = \left\{ \begin{pmatrix} a + k_1 \\ \frac{1}{2} + a + k_2 \\ \frac{1}{2} + k_3 \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3. \tag{8}$$

So

$$B_1 = \bigcup_{j=1}^{\infty} B^{*j} A_1 = \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} \left(\frac{1}{2} + k_1\right) p_1^j \\ (a + k_2)\, p_1^j \\ \left(\frac{1}{2} + a + k_3\right) p_2^j \end{pmatrix} : a \in \mathbb{R}, k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^3,$$
$$B_2 = \bigcup_{j=1}^{\infty} B^{*j} A_2 = \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} \left(\frac{1}{2} + a + k_1\right) p_1^j \\ \left(\frac{1}{2} + k_2\right) p_1^j \\ (a + k_3)\, p_2^j \end{pmatrix} : a \in \mathbb{R}, k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^3,$$
$$B_3 = \bigcup_{j=1}^{\infty} B^{*j} A_3 = \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} (a + k_1)\, p_1^j \\ \left(\frac{1}{2} + a + k_2\right) p_1^j \\ \left(\frac{1}{2} + k_3\right) p_2^j \end{pmatrix} : a \in \mathbb{R}, k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^3. \tag{9}$$

Since $p_1 \in (2\mathbb{Z}+1) \setminus \{\pm 1\}$, we can verify directly that the following two lemmas hold.

Lemma 3. *The sets B_j ($j = 1, 2, 3$) given by (9) satisfy the following properties:*

(a) *$\xi \in B_j \Leftrightarrow -\xi \in B_j$ ($j = 1, 2, 3$);*

(b) *$Z(\hat{\mu}_{B,D}(\xi)) \bigcap (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}) = \emptyset$;*

(c) *if $\xi = (\xi_1, \xi_2, \xi_3)^t \in B_1 \pm B_1$, then $\xi_1 \in \mathbb{Z}$;*

(d) *if $\xi = (\xi_1, \xi_2, \xi_3)^t \in B_2 \pm B_2$, then $\xi_2 \in \mathbb{Z}$;*

(e) *if $\xi = (\xi_1, \xi_2, \xi_3)^t \in B_3 \pm B_3$ and $\xi \in Z(\hat{\mu}_{B,D}(\xi))$, then $\xi \in B_1 \cup B_2$ and $\xi_1, \xi_2 \in (1/2) + \mathbb{Z}$.*

By proving (e), the others can be checked directly.

In (e), if $\xi = (\xi_1, \xi_2, \xi_3)^t \in B_3 \pm B_3$, *then*

$$\begin{aligned}
\xi &= (\xi_1, \xi_2, \xi_3)^t \\
&= \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} (a_1 + k_{11})\, p_1^{j_1} \\ \left(\dfrac{1}{2} + a_1 + k_{12}\right) p_1^{j_1} \\ \left(\dfrac{1}{2} + k_{13}\right) p_2^{j} \end{pmatrix} \right. \\
&\qquad \left. \pm \begin{pmatrix} (a_2 + k_{21})\, p_1^{j_2} \\ \left(\dfrac{1}{2} + a_2 + k_{22}\right) p_1^{j_2} \\ \left(\dfrac{1}{2} + k_{23}\right) p_2^{j} \end{pmatrix} \right\} \\
&= \bigcup_{j=1}^{\infty} \left\{ \begin{pmatrix} (a_1 + k_{11})\, p_1^{j_1} \pm (a_2 + k_{21})\, p_1^{j_2} \\ \left(\dfrac{1}{2} + a_1 + k_{12}\right) p_1^{j_1} \pm \left(\dfrac{1}{2} + a_2 + k_{22}\right) p_1^{j_2} \\ \left(\dfrac{1}{2} + k_{13}\right) p_2^{j} \pm \left(\dfrac{1}{2} + k_{23}\right) p_2^{j} \end{pmatrix} \right\},
\end{aligned} \tag{10}$$

where $a_i \in \mathbb{R}$ *and* $k_{i1}, k_{i2} \in \mathbb{Z}$, $i = 1, 2$. *From* (10) *one has* $\xi_2 - \xi_1 = (1/2)(p_1^{j_1} \pm p_1^{j_2}) \pm (k_{12} - k_{11}) p_1^{j_1} \pm (k_{22} - k_{21}) p_1^{j_2} \in \mathbb{Z}$ *(since* $k_{i1}, k_{i2} \in \mathbb{Z}$, $i = 1, 2$ *and* $p_1 \in (2\mathbb{Z} + 1) \setminus \{\pm 1\}$*); then* $\xi \in B_1 \cup B_2$. *In fact, if* $\xi \in Z(\widehat{\mu}_{B,D}(\xi))$ *and* $\xi \in B_3$, *then* $\xi_2 - \xi_1 \in (1/2 + \mathbb{Z})$ *is a contradiction. Since* $\xi \in B_1 \cup B_2$ *and* $\xi_2 - \xi_1 \in \mathbb{Z}$, *then* $\xi_1, \xi_2 \in (1/2) + \mathbb{Z}$.

Lemma 4. *Let* $\xi = (\xi_1, \xi_2, \xi_3)^t \in Z(\widehat{\mu}_{B,D}(\xi)) = B_1 \cup B_2 \cup B_3$. *Then the following statements hold:*

(i) *if* $\xi \in B_j$, *then* $\xi_j \in (1/2) + \mathbb{Z}$, *where* $j = 1, 2$;

(ii) *if* $\xi_1 \in \mathbb{Z}$, *then* $\xi \in B_2 \cup B_3$ *and* $\xi_2 \in (1/2) + \mathbb{Z}$;

(iii) *if* $\xi_2 \in \mathbb{Z}$, *then* $\xi \in B_1 \cup B_3$ *and* $\xi_1 \in (1/2) + \mathbb{Z}$.

Suppose that λ_j $(j = 1, 2, 3, 4, 5) \in \mathbb{R}^3$ are such that the five exponential functions

$$e^{2\pi i \langle \lambda_1, x\rangle}, \quad e^{2\pi i \langle \lambda_2, x\rangle}, \quad e^{2\pi i \langle \lambda_3, x\rangle}, \quad e^{2\pi i \langle \lambda_4, x\rangle}, \quad e^{2\pi i \langle \lambda_5, x\rangle} \tag{11}$$

are mutually orthogonal in $L^2(\mu_{B,D})$, so the differences $\lambda_j - \lambda_k$ $(1 \le j \ne k \le 5)$ are in the zero set $Z(\widehat{\mu}_{B,D}(\xi))$. Then we get

$$\lambda_j - \lambda_k \in Z(\widehat{\mu}_{B,D}(\xi)) = B_1 \cup B_2 \cup B_3 \quad (1 \le j \ne k \le 5). \tag{12}$$

Denote $\lambda_j - \lambda_k$ by

$$\lambda_j - \lambda_k = \left(x_{j,k}, y_{j,k}, z_{j,k}\right)^t \in \mathbb{R}^3 \quad \text{for } 1 \le j \ne k \le 5. \tag{13}$$

We will apply the above two lemmas to get the contradiction below.

The following ten differences

$$\begin{array}{cccc}
\lambda_2 - \lambda_1, & \lambda_3 - \lambda_1, & \lambda_4 - \lambda_1, & \lambda_5 - \lambda_1 \\
& \lambda_3 - \lambda_2, & \lambda_4 - \lambda_2, & \lambda_5 - \lambda_2 \\
& & \lambda_4 - \lambda_3, & \lambda_5 - \lambda_3 \\
& & & \lambda_5 - \lambda_4
\end{array} \tag{14}$$

belong to $B_1 \cup B_2 \cup B_3$. Then

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_1 \cup B_2 \cup B_3. \tag{15}$$

Claim 1. The set B_1 or B_2 or B_3 cannot contain any three differences of the form: $\lambda_{j_1} - \lambda_j$, $\lambda_{j_2} - \lambda_j$, $\lambda_{j_3} - \lambda_j$, where $1 \le j_1 \ne j_2 \ne j_3 \ne j \le 5$.

Claim 1 can be checked easily. For example, if $\lambda_{j_1} - \lambda_j$, $\lambda_{j_2} - \lambda_j$, $\lambda_{j_3} - \lambda_j \in B_1$, so, by applying Lemma 3(c) and (12), we get

$$\begin{aligned}
\lambda_{j_1} - \lambda_{j_3} &= \left(\lambda_{j_1} - \lambda_j\right) - \left(\lambda_{j_3} - \lambda_j\right) \\
&\in (B_1 - B_1) \cap Z(\widehat{\mu}_{M,D}(\xi)), \\
\lambda_{j_2} - \lambda_{j_3} &= \left(\lambda_{j_2} - \lambda_j\right) - \left(\lambda_{j_3} - \lambda_j\right) \\
&\in (B_1 - B_1) \cap Z(\widehat{\mu}_{B,D}(\xi)), \\
\lambda_{j_1} - \lambda_{j_2} &= \left(\lambda_{j_1} - \lambda_j\right) - \left(\lambda_{j_2} - \lambda_j\right) \\
&\in (B_1 - B_1) \cap Z(\widehat{\mu}_{B,D}(\xi)), \\
&x_{j_1,j_3}, x_{j_2,j_3}, x_{j_1,j_2} \in \mathbb{Z},
\end{aligned} \tag{16}$$

and by Lemma 4(ii),

$$\lambda_{j_1} - \lambda_{j_3}, \lambda_{j_2} - \lambda_{j_3}, \lambda_{j_1} - \lambda_{j_2} \in B_2 \cup B_3, \tag{17}$$

which shows that at least one of the two sets B_2 and B_3 contains two differences. If B_2 contains two differences, say $\lambda_{j_2} - \lambda_{j_3}$ and $\lambda_{j_1} - \lambda_{j_2}$, then

$$\lambda_{j_1} - \lambda_{j_3} = \left(\lambda_{j_1} - \lambda_{j_2}\right) + \left(\lambda_{j_2} - \lambda_{j_3}\right) \in B_2 + B_2. \tag{18}$$

This shows (by Lemma 3(d)) that $y_{j_1,j_3} \in \mathbb{Z}$, a contradiction with Lemma 3(b). If B_3 contains two differences, say $\lambda_{j_2} - \lambda_{j_3}$ and $\lambda_{j_1} - \lambda_{j_2}$, then

$$\lambda_{j_1} - \lambda_{j_3} = \left(\lambda_{j_1} - \lambda_{j_2}\right) + \left(\lambda_{j_2} - \lambda_{j_3}\right) \in B_3 + B_3; \tag{19}$$

by Lemma 3(e), $x_{j_1,j_3} \in (1/2) + \mathbb{Z}$, a contradiction with (16).

From (15) and Claim 1, we only need to deal with the following four typical cases.

Case 1. $\lambda_2 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1 \in B_2$, $\lambda_4 - \lambda_1 \in B_3$, and $\lambda_5 - \lambda_1 \in B_1$.

Case 2. $\lambda_2 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1 \in B_2$, $\lambda_4 - \lambda_1 \in B_3$, and $\lambda_5 - \lambda_1 \in B_3$.

Case 3. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_2$.

Case 4. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_3$.

Note that Case 1 denotes the 2−1−1 or 1−2−1 distribution in (15), Case 2 denotes the 1 − 1 − 2 distribution in (15), and Case 3 denotes the 2 − 2 − 0 distribution in (15), while Case 4 denotes the 2 − 0 − 2 (or 0 − 2 − 2) distribution in (15). If the four cases can be proved, then the other cases can be proved similarly.

2.1. Case 1. In this case, we have

$$\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_1) - (\lambda_2 - \lambda_1) \in B_1 - B_1. \quad (20)$$

Applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_5 - \lambda_2 = (x_{5,2}, y_{5,2}, z_{5,2})^t \in B_2 \cup B_3, \quad x_{5,2} \in \mathbb{Z}, y_{5,2} \in \frac{1}{2} + \mathbb{Z}. \quad (21)$$

From $\lambda_5 - \lambda_2 \in B_2 \cup B_3$, Case 1 can be divided into two subcases.

Case 1.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, and $\lambda_4 - \lambda_1 \in B_3$.

Case 1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1 \in B_2$, $\lambda_4 - \lambda_1$, and $\lambda_5 - \lambda_2 \in B_3$.

Step 1. We will prove Case 1.1; since $\lambda_3 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, Case 1.1 can be divided into three cases.

Case 1.1.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, and $\lambda_4 - \lambda_1 \in B_3$.

Case 1.1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2, \lambda_3 - \lambda_2 \in B_2$, and $\lambda_4 - \lambda_1 \in B_3$.

Case 1.1.3. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_3$.

We will give a method to deal with each case by considering the remainder differences in (14), and each case is concluded with a contradiction.

Case 1.1.1. In this case, we have

$$\lambda_3 - \lambda_1 = (\lambda_3 - \lambda_2) + (\lambda_2 - \lambda_1) \in B_1 + B_1; \quad (22)$$

by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1})^t \in B_2, \quad x_{3,1} \in \mathbb{Z}, y_{3,1} \in \frac{1}{2} + \mathbb{Z}. \quad (23)$$

By Lemma 3(a) and Claim 1 we know that $\lambda_4 - \lambda_2 \notin B_1$. So $\lambda_4 - \lambda_2 \in B_2$ or $\lambda_4 - \lambda_2 \in B_3$, so Case 1.1.1 can be divided into two subcases.

Case 1.1.1.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2, \lambda_4 - \lambda_2 \in B_2$, and $\lambda_4 - \lambda_1 \in B_3$.

Case 1.1.1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_1$, $\lambda_3 - \lambda_1, \lambda_5 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1$, and $\lambda_4 - \lambda_2 \in B_3$.

By considering the remainder differences in (14), we apply Lemmas 3 and 4 to deal with each case.

(I) In Case 1.1.1.1, we have

$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_2) - (\lambda_4 - \lambda_2) \in B_2 - B_2; \quad (24)$$

by applying Lemma 3(d), (12) and Lemma 4(iii), we have

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^t \in B_1 \cup B_3, \quad x_{5,4} \in \frac{1}{2} + \mathbb{Z}, \ y_{5,4} \in \mathbb{Z}. \quad (25)$$

(i) If $\lambda_5 - \lambda_4 \in B_1$, so $\lambda_4 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_4) \in B_1 - B_1$; by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^t \in B_3, \quad x_{4,1} \in \mathbb{Z}, \ y_{4,1} \in \frac{1}{2} + \mathbb{Z}. \quad (26)$$

It follows from (23), (26), and $\lambda_4 - \lambda_3 = (\lambda_4 - \lambda_1) - (\lambda_3 - \lambda_1)$ that

$$\lambda_4 - \lambda_3 = (x_{4,3}, y_{4,3}, z_{4,3})^t, \quad x_{4,3} \in \mathbb{Z}, y_{4,3} \in \mathbb{Z}, \quad (27)$$

which shows a contradiction with Lemma 3(b).

(ii) If $\lambda_5 - \lambda_4 \in B_3$, so $\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_1) \in B_3 + B_3$; by applying Lemma 3(e) and (12), we get

$$\lambda_5 - \lambda_1 = (x_{5,1}, y_{5,1}, z_{5,1})^t \in B_1, \quad x_{5,1} \in \frac{1}{2} + \mathbb{Z}, y_{5,1} \in \frac{1}{2} + \mathbb{Z}. \quad (28)$$

Now, consider the remainder difference $\lambda_4 - \lambda_3$ in (14): by Lemma 3(a) and Claim 1, we have $\lambda_4 - \lambda_3 \notin B_3$, so $\lambda_4 - \lambda_3 \in B_1 \cup B_2$. If $\lambda_4 - \lambda_3 \in B_1$, then

$$\lambda_4 - \lambda_2 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_1 + B_1; \quad (29)$$

by applying Lemma 3(c), (12), and Lemma 4(ii) we get

$$\lambda_4 - \lambda_2 = (x_{4,2}, y_{4,2}, z_{4,2})^t \in B_2, \quad x_{4,2} \in \mathbb{Z}, y_{4,2} \in \frac{1}{2} + \mathbb{Z}. \quad (30)$$

It follows from (21), (30), and $\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_2) - (\lambda_4 - \lambda_2)$ that $x_{5,4} \in \mathbb{Z}$, $y_{5,4} \in \mathbb{Z}$, a contradiction with Lemma 3(b). If $\lambda_4 - \lambda_3 \in B_2$, then

$$\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_1) \in B_2 + B_2; \quad (31)$$

by applying Lemma 3(d), (12), and Lemma 4(iii), we get

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^t \in B_3, \quad x_{4,1} \in \frac{1}{2} + \mathbb{Z}, y_{4,1} \in \mathbb{Z}. \quad (32)$$

It follows from (25), (32), and $\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_1)$ that $x_{5,1} \in \mathbb{Z}$ and $y_{5,1} \in \mathbb{Z}$, a contradiction with Lemma 3(b).

From parts (i), (ii), and (25), Case 1.1.1.1 is proved.

(II) In Case 1.1.1.2, we have

$$\lambda_2 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_2) \in B_3 - B_3; \quad (33)$$

by applying Lemma 3(e) and (12), we get

$$\lambda_2 - \lambda_1 = (x_{2,1}, y_{2,1}, z_{2,1})^t \in B_1, \quad x_{2,1} \in \frac{1}{2} + \mathbb{Z}, y_{2,1} \in \frac{1}{2} + \mathbb{Z}. \quad (34)$$

We consider the remainder three differences $\lambda_4-\lambda_3$, $\lambda_5-\lambda_3$, and $\lambda_5-\lambda_4$ in (14). By Claim 1, we have $\lambda_4-\lambda_3\notin B_3$, so $\lambda_4-\lambda_3\in B_1\cup B_2$.

(i) If $\lambda_4-\lambda_3\in B_1$, we have

$$\lambda_4-\lambda_2=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_2)\in B_1+B_1; \quad (35)$$

by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_4-\lambda_2=(x_{4,2},y_{4,2},z_{4,2})^t\in B_3,\quad x_{4,2}\in\mathbb{Z},\,y_{4,2}\in\frac{1}{2}+\mathbb{Z}. \quad (36)$$

It follows from (21), (36), and $\lambda_5-\lambda_4=(\lambda_5-\lambda_2)-(\lambda_4-\lambda_2)$ that

$$\lambda_5-\lambda_4=(x_{5,4},y_{5,4},z_{5,4})^t,\quad x_{5,4}\in\mathbb{Z},\,y_{5,4}\in\mathbb{Z}, \quad (37)$$

a contradiction with Lemma 3(b).

(ii) If $\lambda_4-\lambda_3\in B_2$, we have

$$\lambda_4-\lambda_1=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_1)\in B_2+B_2; \quad (38)$$

by applying Lemma 3(d), (12), and Lemma 4(iii) we have

$$\lambda_4-\lambda_1=(x_{4,1},y_{4,1},z_{4,1})^t\in B_3,\quad x_{4,1}\in\frac{1}{2}+\mathbb{Z},\,y_{4,1}\in\mathbb{Z}. \quad (39)$$

We consider the remainder difference $\lambda_5-\lambda_4$ in (14). By Claim 1, we have $\lambda_5-\lambda_4\notin B_3$, so $\lambda_5-\lambda_4\in B_1\cup B_2$. If $\lambda_5-\lambda_4\in B_1$, we have

$$\lambda_4-\lambda_1=(\lambda_5-\lambda_1)-(\lambda_5-\lambda_4)\in B_1-B_1, \quad (40)$$

by applying Lemma 3(c), (12), and Lemma 4(ii), we get $x_{4,1}\in\mathbb{Z}$, a contradiction with (39). If $\lambda_5-\lambda_4\in B_2$, we have

$$\lambda_4-\lambda_2=(\lambda_5-\lambda_2)-(\lambda_5-\lambda_4)\in B_2-B_2; \quad (41)$$

by applying Lemma 3(d), (12), and Lemma 4(iii), we get

$$\lambda_4-\lambda_2=(x_{4,2},y_{4,2},z_{4,2})^t\in B_3,\quad x_{4,2}\in\frac{1}{2}+\mathbb{Z},\,y_{4,2}\in\mathbb{Z}. \quad (42)$$

It follows from (39), (42), and $\lambda_2-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_2)$ that $x_{2,1}\in\mathbb{Z}$, a contradiction with (34).

The above (i) and (ii) in (II) illustrate that Case 1.1.1.2 is proved. Hence Case 1.1.1 is proved.

Case 1.1.2. In this case, we have

$$\begin{aligned}\lambda_2-\lambda_1&=(\lambda_3-\lambda_1)-(\lambda_3-\lambda_2)\in B_2-B_2,\\ \lambda_5-\lambda_3&=(\lambda_5-\lambda_2)-(\lambda_3-\lambda_2)\in B_2-B_2;\end{aligned} \quad (43)$$

by applying Lemma 3(d), (12), and Lemma 4(iii), we get

$$\lambda_2-\lambda_1=(x_{2,1},y_{2,1},z_{2,1})^t\in B_1,\quad x_{2,1}\in\frac{1}{2}+\mathbb{Z},\,y_{2,1}\in\mathbb{Z}, \quad (44)$$

$$\begin{aligned}\lambda_5-\lambda_3&=(x_{5,3},y_{5,3},z_{5,3})^t\in B_1\cup B_3,\\ &x_{5,3}\in\frac{1}{2}+\mathbb{Z},\,y_{5,3}\in\mathbb{Z}.\end{aligned} \quad (45)$$

By applying Lemma 3(a) and Claim 1, we get

$$\lambda_4-\lambda_2\notin B_2,\qquad \lambda_4-\lambda_3\notin B_2. \quad (46)$$

If $\lambda_4-\lambda_2\in B_3$, then $\lambda_2-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_2)\in B_3-B_3$; by Lemma 3(e), and (12), we get $x_{2,1}\in(1/2)+\mathbb{Z}$, $y_{2,1}\in(1/2)+\mathbb{Z}$, a contradiction with (44); hence

$$\lambda_4-\lambda_2\notin B_3,\qquad \lambda_4-\lambda_2\in B_1. \quad (47)$$

We consider $\lambda_4-\lambda_3$; if $\lambda_4-\lambda_3\in B_1$, we get $\lambda_3-\lambda_2=(\lambda_4-\lambda_2)-(\lambda_4-\lambda_3)\in B_1-B_1$; by Lemma 3(c), (12), and Lemma 4(ii), we have

$$\lambda_3-\lambda_2=(x_{3,2},y_{3,2},z_{3,2})^t\in B_2,\quad x_{3,2}\in\mathbb{Z},\,y_{3,2}\in\frac{1}{2}+\mathbb{Z}, \quad (48)$$

which, combined with (21) and $\lambda_5-\lambda_3=(\lambda_5-\lambda_2)-(\lambda_3-\lambda_2)$, shows that $x_{5,3}\in\mathbb{Z}$, $y_{5,3}\in\mathbb{Z}$, a contradiction with Lemma 3(b); hence

$$\lambda_4-\lambda_3\notin B_1,\qquad \lambda_4-\lambda_3\in B_3. \quad (49)$$

Since $\lambda_3-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_3)\in B_3-B_3$, by Lemma 3(e) and (12), we get

$$\begin{aligned}\lambda_3-\lambda_1&=(x_{3,1},y_{3,1},z_{3,1})^t\in B_2,\\ x_{3,1}&\in\frac{1}{2}+\mathbb{Z},\,y_{3,1}\in\frac{1}{2}+\mathbb{Z}.\end{aligned} \quad (50)$$

Combined with (21), (44), and $\lambda_5-\lambda_1=(\lambda_5-\lambda_2)+(\lambda_2-\lambda_1)$, we get

$$\lambda_5-\lambda_1=(x_{5,1},y_{5,1},z_{5,1})^t,\quad x_{5,1}\in\frac{1}{2}+\mathbb{Z},\,y_{5,1}\in\frac{1}{2}+\mathbb{Z}, \quad (51)$$

which, combined with (50), (51), and $\lambda_5-\lambda_3=(\lambda_5-\lambda_1)-(\lambda_3-\lambda_1)$, shows that $x_{5,3}\in\mathbb{Z}$, $y_{5,3}\in\mathbb{Z}$, a contradiction of (45). This proves Case 1.1.2.

Case 1.1.3. From $\lambda_4-\lambda_2\in B_1\cup B_2\cup B_3$, this Case can be divided into the three cases.

Case 1.1.3.1. $\lambda_2-\lambda_1,\lambda_5-\lambda_1,\lambda_4-\lambda_2\in B_1$, $\lambda_3-\lambda_1,\lambda_5-\lambda_2\in B_2$, $\lambda_4-\lambda_1$, and $\lambda_3-\lambda_2\in B_3$.

Case 1.1.3.2. $\lambda_2-\lambda_1,\lambda_5-\lambda_1\in B_1$, $\lambda_3-\lambda_1,\lambda_5-\lambda_2,\lambda_4-\lambda_2\in B_2$, $\lambda_4-\lambda_1$, and $\lambda_3-\lambda_2\in B_3$.

Case 1.1.3.3. $\lambda_2-\lambda_1$, $\lambda_5-\lambda_1\in B_1$, $\lambda_3-\lambda_1$, $\lambda_5-\lambda_2\in B_2$, $\lambda_4-\lambda_1$, $\lambda_3-\lambda_2$, and $\lambda_4-\lambda_2\in B_3$.

The above three cases denote the $3-2-2$ or $2-3-2$ or $2-2-3$ distribution. In each case, we have (21). The discussion of the first two cases is similar. In the following, we use the above method to deal with Cases 1.1.3.1 and 1.1.3.3.

In Case 1.1.3.1, since

$$\lambda_4-\lambda_1=(\lambda_4-\lambda_2)+(\lambda_2-\lambda_1)\in B_1+B_1, \quad (52)$$

by applying Lemma 3(c), (12), and Lemma 4(ii), we have

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^t \in B_3, \quad x_{4,1} \in \mathbb{Z}, y_{4,1} \in \frac{1}{2} + \mathbb{Z}. \tag{53}$$

We consider the remainder three differences $\lambda_4 - \lambda_3$, $\lambda_5 - \lambda_3$, and $\lambda_5 - \lambda_4$. From (21) and (53) we have

$$\lambda_4 - \lambda_3 \notin B_2 \quad \text{or} \quad \lambda_4 - \lambda_3 \in B_1 \cup B_3, \tag{54}$$

$$\lambda_5 - \lambda_3 \notin B_3 \quad \text{or} \quad \lambda_5 - \lambda_3 \in B_1 \cup B_2. \tag{55}$$

If $\lambda_4 - \lambda_3 \in B_2$, by Lemma 3, so $\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_1) \in B_2 + B_2$; we get $y_{4,1} \in \mathbb{Z}$, a contradiction with (53). If $\lambda_5 - \lambda_3 \in B_3$, so $\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_3 + B_3$ by Lemma 3(e); we get $x_{5,2} \in (1/2) + \mathbb{Z}$, a contradiction with (21). Hence (54) and (55) hold.

According to (54), we deal with the following two cases.

(i) If $\lambda_4 - \lambda_3 \in B_1$, then, from $\lambda_3 - \lambda_2 = (\lambda_4 - \lambda_2) - (\lambda_4 - \lambda_3) \in B_1 - B_1$, by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^t \in B_3, \quad x_{3,2} \in \mathbb{Z}, y_{3,2} \in \frac{1}{2} + \mathbb{Z}. \tag{56}$$

From (21), (56), and $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2)$, we get

$$\lambda_5 - \lambda_3 = (x_{5,3}, y_{5,3}, z_{5,3})^t, \quad x_{5,3} \in \mathbb{Z}, y_{5,3} \in \mathbb{Z}, \tag{57}$$

a contradiction with Lemma 3(b).

(ii) If $\lambda_4 - \lambda_3 \in B_3$, since $\lambda_4 - \lambda_2 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_3 + B_3$ and $\lambda_3 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_3) \in B_3 - B_3$, by applying Lemma 3(e) and (12), we get

$$\begin{gathered}\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1})^t \in B_2, \\ x_{3,1} \in \frac{1}{2} + \mathbb{Z}, y_{3,1} \in \frac{1}{2} + \mathbb{Z}.\end{gathered} \tag{58}$$

We consider (55); if $\lambda_5 - \lambda_3 \in B_1$, by Lemma 3 and $\lambda_3 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_3) \in B_1 - B_1$, we get $x_{3,1} \in \mathbb{Z}$, a contradiction with (58). If $\lambda_5 - \lambda_3 \in B_2$, since $\lambda_3 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_3) \in B_2 - B_2$, by applying Lemma 3(d), (12), and Lemma 4(iii), we get

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^t \in B_3, \quad x_{3,2} \in \frac{1}{2} + \mathbb{Z}, y_{3,2} \in \mathbb{Z}. \tag{59}$$

From (58), (59) and $\lambda_2 - \lambda_1 = (\lambda_3 - \lambda_1) - (\lambda_3 - \lambda_2)$, we have $x_{2,1} \in \mathbb{Z}$, a contradiction with Lemma 4(i) (for $\lambda_2 - \lambda_1 \in B_1$ gives $x_{2,1} \in (1/2) + \mathbb{Z}$).

The above (i), (ii), and (54) indicae that Case 1.1.3.1 is proved.

In Case 1.1.3.3, we get

$$\begin{gathered}\lambda_4 - \lambda_3 = (\lambda_4 - \lambda_2) - (\lambda_3 - \lambda_2) \in B_3 - B_3, \\ \lambda_2 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_2) \in B_3 - B_3;\end{gathered} \tag{60}$$

by applying Lemma 3(e) and (12), we have

$$\begin{gathered}\lambda_4 - \lambda_3 = (x_{4,3}, y_{4,3}, z_{4,3})^t \in B_1 \cup B_2, \\ x_{4,3} \in \frac{1}{2} + \mathbb{Z}, y_{4,3} \in \frac{1}{2} + \mathbb{Z},\end{gathered} \tag{61}$$

$$\begin{gathered}\lambda_2 - \lambda_1 = (x_{2,1}, y_{2,1}, z_{2,1})^t \in B_1, \\ x_{2,1} \in \frac{1}{2} + \mathbb{Z}, y_{2,1} \in \frac{1}{2} + \mathbb{Z}.\end{gathered} \tag{62}$$

By Lemma 3(a) and $\lambda_1 - \lambda_4, \lambda_2 - \lambda_4 \in B_3$, we know from Claim 1 that $\lambda_5 - \lambda_4 \notin B_3$. So $\lambda_5 - \lambda_4 \in B_1$ or $\lambda_5 - \lambda_4 \in B_2$.

From (61), we consider the following two cases.

(i) If $\lambda_4 - \lambda_3 \in B_1$, we consider $\lambda_5 - \lambda_4$, if $\lambda_5 - \lambda_4 \in B_1$. From $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_3) \in B_1 + B_1$, by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\begin{gathered}\lambda_5 - \lambda_3 = (x_{5,3}, y_{5,3}, z_{5,3})^t \in B_2 \cup B_3, \\ x_{5,3} \in \mathbb{Z}, y_{5,3} \in \frac{1}{2} + \mathbb{Z}.\end{gathered} \tag{63}$$

From (21), (63), and $\lambda_3 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_3)$, we get $x_{2,1} \in \mathbb{Z}$, $y_{2,1} \in \mathbb{Z}$, a contradiction with Lemma 3(b). If $\lambda_5 - \lambda_4 \in B_2$, since $\lambda_4 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_4) \in B_2 - B_2$, by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\begin{gathered}\lambda_4 - \lambda_2 = (x_{4,2}, y_{4,2}, z_{4,2})^t \in B_1 \cup B_3, \\ x_{4,2} \in \frac{1}{2} + \mathbb{Z}, y_{4,2} \in \mathbb{Z}.\end{gathered} \tag{64}$$

From (21), (64), and $\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_2) - (\lambda_4 - \lambda_2)$, we get that $x_{5,4} \in (1/2) + \mathbb{Z}$, $y_{5,4} \in (1/2) + \mathbb{Z}$, combined with (61) and $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_3)$, yields $x_{5,4} \in \mathbb{Z}$, $y_{5,4} \in \mathbb{Z}$, a contradiction with Lemma 3(b).

(ii) If $\lambda_4 - \lambda_3 \in B_2$, so $\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_1) \in B_2 + B_2$; by applying Lemma 3(d), (12), and Lemma 4(iii), we get

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^t \in B_3, \quad x_{4,1} \in \frac{1}{2} + \mathbb{Z}, y_{4,1} \in \mathbb{Z}. \tag{65}$$

If $\lambda_5 - \lambda_4 \in B_1$, by Lemma 3(c) and $\lambda_4 - \lambda_1 = (\lambda_5 - \lambda_1) - (\lambda_5 - \lambda_4) \in B_1 - B_1$, we get $x_{4,1} \in \mathbb{Z}$, a contradiction with (65). If $\lambda_5 - \lambda_4 \in B_2$, so $\lambda_4 - \lambda_2 = (\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_4) \in B_2 - B_2$, by applying Lemma 3(d), (12), and Lemma 4(iii), we get

$$\lambda_4 - \lambda_2 = (x_{4,2}, y_{4,2}, z_{4,2})^t \in B_3, \quad x_{4,2} \in \frac{1}{2} + \mathbb{Z}, y_{4,2} \in \mathbb{Z}. \tag{66}$$

From (65), (66), and $\lambda_2 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_2)$, we get $x_{2,1} \in \mathbb{Z}$, a contradiction with (62).

The above (i), (ii), and (61) illustrate that Case 1.1.3.3 is proved and Case 1.1 is proved.

Step 2. In Case 1.2, from $\lambda_3 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, this case can be divided into three cases.

Case 1.2.1. $\lambda_2-\lambda_1, \lambda_5-\lambda_1, \lambda_3-\lambda_2 \in B_1, \lambda_3-\lambda_1 \in B_2, \lambda_4-\lambda_1$, and $\lambda_5 - \lambda_2 \in B_3$.

Case 1.2.2. $\lambda_2-\lambda_1, \lambda_5-\lambda_1 \in B_1, \lambda_3-\lambda_1, \lambda_3-\lambda_2 \in B_2, \lambda_4-\lambda_1$, and $\lambda_5 - \lambda_2 \in B_3$.

Case 1.2.3. $\lambda_2-\lambda_1, \lambda_5-\lambda_1 \in B_1, \lambda_3-\lambda_1 \in B_2, \lambda_4-\lambda_1, \lambda_5-\lambda_2$, and $\lambda_3 - \lambda_2 \in B_3$.

Case 1.2.2 is similar to Case 1.1.3; we will deal with the other two cases by considering the remainder differences in (14), and each case is concluded with a contradiction.

Case 1.2.1. In this case, we get

$$\lambda_3 - \lambda_1 = (\lambda_3 - \lambda_2) + (\lambda_2 - \lambda_1) \in B_1 + B_1; \quad (67)$$

by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_3 - \lambda_1 = (x_{3,1}, y_{3,1}, z_{3,1})^t \in B_2, \quad x_{3,1} \in \mathbb{Z}, y_{3,1} \in \frac{1}{2} + \mathbb{Z}. \quad (68)$$

By Lemma 3(a) and $\lambda_1 - \lambda_2, \lambda_3 - \lambda_2 \in B_1$, we know from Claim 1 that $\lambda_4 - \lambda_2 \notin B_1$. So $\lambda_4 - \lambda_2 \in B_2$ or $\lambda_4 - \lambda_2 \in B_3$, so Case 1.2.1 can be divided into two cases.

Case 1.2.1.1. $\lambda_2-\lambda_1, \lambda_5-\lambda_1, \lambda_3-\lambda_2 \in B_1, \lambda_3-\lambda_1, \lambda_4-\lambda_2 \in B_2$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_2 \in B_3$.

Case 1.2.1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_1, \lambda_3 - \lambda_1 \in B_2$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_2, \lambda_4 - \lambda_2 \in B_3$.

The above two cases denote the $3-2-2$ distribution and $3-1-3$ distribution. Case 1.2.1.1 is similar to Case 1.1.3.1. So we only need to deal with Case 1.2.1.2. By considering the remainder differences in (14), we apply Lemmas 3 and 4 to deal with the case.

In Case 1.2.1.2, we get

$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_2) - (\lambda_4 - \lambda_2) \in B_3 - B_3; \quad (69)$$

by applying Lemma 3(e) and (12), we get

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^t \in B_1 \cup B_2, \quad x_{5,4} \in \frac{1}{2} + \mathbb{Z}, y_{5,4} \in \frac{1}{2} + \mathbb{Z}. \quad (70)$$

By Claim 1, $\lambda_4-\lambda_3 \notin B_3$; then $\lambda_4-\lambda_3 \in B_1$ or $\lambda_4-\lambda_3 \in B_2$. From (70), we consider the following two cases.

(i) If $\lambda_5-\lambda_4 \in B_1$, since $\lambda_4-\lambda_1 = (\lambda_5-\lambda_1)-(\lambda_5-\lambda_4) \in B_1 - B_1$, by applying Lemma 3(c), (12), and Lemma 4(ii) we get

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^t \in B_3, \quad x_{4,1} \in \mathbb{Z}, y_{4,1} \in \frac{1}{2} + \mathbb{Z}. \quad (71)$$

It follows from (68), (71), and $\lambda_4-\lambda_3 = (\lambda_4-\lambda_1)-(\lambda_3-\lambda_1)$ that

$$\lambda_4 - \lambda_3 = (x_{4,3}, y_{4,3}, z_{4,3})^t, \quad x_{4,3} \in \mathbb{Z}, y_{4,3} \in \mathbb{Z}, \quad (72)$$

which shows a contradiction with Lemma 3(b).

(ii) If $\lambda_5 - \lambda_4 \in B_2$, we consider $\lambda_4 - \lambda_3$, if $\lambda_4 - \lambda_3 \in B_1$. Since $\lambda_4 - \lambda_2 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_1 + B_1$, by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_4 - \lambda_2 = (x_{4,2}, y_{4,2}, z_{4,2})^t \in B_2 \cup B_3, \quad x_{4,2} \in \mathbb{Z}, y_{4,2} \in \frac{1}{2} + \mathbb{Z}. \quad (73)$$

From (21), (73), and $\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_2) - (\lambda_4 - \lambda_2)$, we get $x_{5,4} \in \mathbb{Z}$, $y_{5,4} \in \mathbb{Z}$, a contradiction with Lemma 3(b). If $\lambda_4 - \lambda_3 \in B_2$, then $\lambda_4 - \lambda_1 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_1) \in B_2 + B_2$; by applying Lemma 3(d), (12), and Lemma 4(iii), we get

$$\lambda_4 - \lambda_1 = (x_{4,1}, y_{4,1}, z_{4,1})^t \in B_3, \quad x_{4,1} \in \frac{1}{2} + \mathbb{Z}, y_{4,1} \in \mathbb{Z}. \quad (74)$$

It follows from (68), (74), and $\lambda_4-\lambda_3 = (\lambda_4-\lambda_1)-(\lambda_3-\lambda_1)$ that we get

$$\lambda_4 - \lambda_3 = (x_{4,3}, y_{4,3}, z_{4,3})^t \in B_2, \quad x_{4,3} \in \frac{1}{2} + \mathbb{Z}, y_{4,3} \in \frac{1}{2} + \mathbb{Z}. \quad (75)$$

From (70), (75), and $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_3)$, we get $x_{5,4} \in \mathbb{Z}$, $y_{5,4} \in \mathbb{Z}$, a contradiction with Lemma 3(b).

Parts (i), (ii), and (70) indicate that Case 1.2.1.2 is proved and Case 1.2.1 is proved.

Case 1.2.3. In this case, since

$$\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2) \in B_3 - B_3, \quad (76)$$

by applying Lemma 3(e) and (12), we get

$$\lambda_5 - \lambda_3 = (x_{5,3}, y_{5,3}, z_{5,3})^t \in B_1 \cup B_2, \quad x_{5,3} \in \frac{1}{2} + \mathbb{Z}, y_{5,3} \in \frac{1}{2} + \mathbb{Z}. \quad (77)$$

From (77) Case 1.2.3 can be divided into two cases.

Case 1.2.3.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1, \lambda_5 - \lambda_3 \in B_1, \lambda_3 - \lambda_1 \in B_2$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_2$, and $\lambda_3 - \lambda_2 \in B_3$.

Case 1.2.3.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_1 \in B_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_3 \in B_2$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_2$, and $\lambda_3 - \lambda_2 \in B_3$.

The above two cases denote the $3-1-3$ distribution and $2-2-3$ distribution. Case 1.2.3.1 is similar to Case 1.2.1.2. Case 1.2.3.2 is similar to Case 1.1.3.3. So Case 1.2.3 is proved, and so Case 1.2 is proved.

Thus, the proof of Case 1 is completed.

2.2. Case 2. In this case, since

$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_1) \in B_3 - B_3; \quad (78)$$

by applying Lemma 3(e) and (12), we get

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^t \in B_1 \cup B_2, \quad x_{5,4} \in \frac{1}{2} + \mathbb{Z}, y_{5,4} \in \frac{1}{2} + \mathbb{Z}. \quad (79)$$

From $\lambda_5 - \lambda_4 \in B_1 \cup B_2$, the discussion here can be divided into two cases: $\lambda_5 - \lambda_4 \in B_1$ and $\lambda_5 - \lambda_4 \in B_2$. That is, we have the following two subcases.

Case 2.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_4 \in B_1, \lambda_3 - \lambda_1 \in B_2, \lambda_4 - \lambda_1$, and $\lambda_5 - \lambda_1 \in B_3$.

Case 2.2. $\lambda_2 - \lambda_1 \in B_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_2, \lambda_4 - \lambda_1$, and $\lambda_5 - \lambda_1 \in B_3$.

The discussion of Case 2.1 is analogous to Case 2.2; it denotes the $2-1-2$ or $1-2-2$ distribution. So we only need to deal with Case 2.1.

From $\lambda_3 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, Case 2.1 can be divided into three cases.

Case 2.1.1. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_4, \lambda_3 - \lambda_2 \in B_1, \lambda_3 - \lambda_1 \in B_2, \lambda_4 - \lambda_1$, and $\lambda_5 - \lambda_1 \in B_3$.

Case 2.1.2. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_4 \in B_1, \lambda_3 - \lambda_1, \lambda_3 - \lambda_2 \in B_2, \lambda_4 - \lambda_1$, and $\lambda_5 - \lambda_1 \in B_3$.

Case 2.1.3. $\lambda_2 - \lambda_1, \lambda_5 - \lambda_4 \in B_1, \lambda_3 - \lambda_1 \in B_2, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_3$.

The above three cases denote the $3-1-2$ distribution, $2-2-2$ distribution, and $2-1-3$ distribution. Case 2.1.1 is similar to Case 1.2.1, Case 2.1.2 is similar to Case 1.1.3, and Case 2.1.3 is similar to Case 1.2.3.

So the proof of Case 2 is completed.

2.3. Case 3. In this case, since $\lambda_3 - \lambda_2 = (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1) \in B_1 - B_1$, and $\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_1) \in B_2 - B_2$, by Lemmas 3 and 4, we get

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^t \in B_2 \cup B_3, \quad x_{3,2} \in \mathbb{Z}, y_{3,2} \in \frac{1}{2} + \mathbb{Z}, \tag{80}$$

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^t \in B_1 \cup B_3, \quad x_{5,4} \in \frac{1}{2} + \mathbb{Z}, y_{5,4} \in \mathbb{Z}. \tag{81}$$

From $\lambda_3 - \lambda_2 \in B_2 \cup B_3$, Case 3 can be divided into the following two cases.

Case 3.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$.

Case 3.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_2$ and $\lambda_3 - \lambda_2 \in B_3$.

Case 3.2 is similar to Case 1.1, so we only need to deal with Case 3.1.

From $\lambda_5 - \lambda_4 \in B_1 \cup B_3$, the Case 3.1 can be divided into the following two cases.

Case 3.1.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_2$.

Case 3.1.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$, and $\lambda_5 - \lambda_4 \in B_3$.

Case 3.1.2 is similar to Case 1.1.2, so we only need to deal with Case 3.1.1.

Consider the remainder difference $\lambda_4 - \lambda_2$ in (14). From $\lambda_4 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, Case 3.1.1 can be divided into three cases.

Case 3.1.1.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4, \lambda_4 - \lambda_2 \in B_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_2$.

Case 3.1.1.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2$, and $\lambda_4 - \lambda_2 \in B_2$.

Case 3.1.1.3. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_2$, and $\lambda_4 - \lambda_2 \in B_3$.

Case 3.1.1.2 is similar to Case 3.1.1.1, and Case 3.1.1.3 is similar to Case 1.1.1.1. So we only need to deal with Case 3.1.1.1. In this case, since $\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_2) \in B_1 + B_1$, by applying Lemma 3(c), (12), and Lemma 4(ii), we get

$$\lambda_5 - \lambda_2 = (x_{5,2}, y_{5,2}, z_{5,2})^t \in B_2 \cup B_3, \quad x_{5,2} \in \mathbb{Z}, y_{5,2} \in \frac{1}{2} + \mathbb{Z}. \tag{82}$$

Combined with (80), (82) yields

$$\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2) = (x_{5,3}, y_{5,3}, z_{5,3})^t, \quad x_{5,3} \in \mathbb{Z}, y_{5,3} \in \mathbb{Z}, \tag{83}$$

a contradiction with Lemma 3(b). Case 3.1.1.1 is proved.

So the proof of Case 3 is completed.

2.4. Case 4. In this case, since $\lambda_3 - \lambda_2 = (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1) \in B_1 - B_1$, and $\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_1) \in B_3 - B_3$, by Lemmas 3 and 4, we get

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^t \in B_2 \cup B_3, \quad x_{3,2} \in \mathbb{Z}, y_{3,2} \in \frac{1}{2} + \mathbb{Z}, \tag{84}$$

$$\lambda_5 - \lambda_4 = (x_{5,4}, y_{5,4}, z_{5,4})^t \in B_1 \cup B_2, \quad x_{5,4} \in \frac{1}{2} + \mathbb{Z}, y_{5,4} \in \frac{1}{2} + \mathbb{Z}. \tag{85}$$

From $\lambda_3 - \lambda_2 \in B_2 \cup B_3$, Case 4 can be divided into the following two cases.

Case 4.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1, \lambda_3 - \lambda_2 \in B_3$.

Case 4.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1, \lambda_3 - \lambda_2 \in B_2$, and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1 \in B_3$.

Case 4.2 is similar to Case 1.2, so we only need to deal with Case 4.1.

From $\lambda_5 - \lambda_4 \in B_1 \cup B_2$, Case 4.1 can be divided into the following two cases.

Case 4.1.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1$ and $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_3$.

Case 4.1.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in B_1, \lambda_5 - \lambda_4 \in B_2, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_3$.

Case 4.1.2 is similar to Case 1.2.3, so we only need to deal with Case 4.1.1.

Consider the remainder difference $\lambda_4 - \lambda_2$ in (14). According to $\lambda_4 - \lambda_2 \in B_1 \cup B_2 \cup B_3$, Case 4.1.1 can be divided into three cases.

Case 4.1.1.1. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4, \lambda_4 - \lambda_2 \in B_1, \lambda_4 - \lambda_1$, $\lambda_5 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_3$.

Case 4.1.1.2. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1, \lambda_4 - \lambda_2 \in B_2$, $\lambda_4 - \lambda_1, \lambda_5 - \lambda_1$, and $\lambda_3 - \lambda_2 \in B_3$.

Case 4.1.1.3. $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_5 - \lambda_4 \in B_1, \lambda_4 - \lambda_1, \lambda_5 - \lambda_1$, $\lambda_3 - \lambda_2$, and $\lambda_4 - \lambda_2 \in B_3$.

Case 4.1.1.2 is similar to Case 1.2.1.2. We will deal with the other two cases by considering the remainder differences in (14), and each case is concluded with a contradiction.

Case 4.1.1.1. In this case, since $\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_2) \in B_1 + B_1$, by Lemma 3(c), (12), and Lemma 4(ii), we get

$$\begin{gathered} \lambda_5 - \lambda_2 = (x_{5,2}, y_{5,2}, z_{5,2})^t \in B_2 \cup B_3, \\ x_{5,2} \in \mathbb{Z}, y_{5,2} \in \frac{1}{2} + \mathbb{Z}, \end{gathered} \tag{86}$$

combined with (84), (86), and $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2)$ yields $x_{5,3} \in \mathbb{Z}$, $y_{5,3} \in \mathbb{Z}$, a contradiction with Lemma 3(b).

Case 4.1.1.3. In this case, since $\lambda_4 - \lambda_3 = (\lambda_4 - \lambda_2) - (\lambda_3 - \lambda_2) \in B_3 - B_3$, by Lemma 3(e) and (12), we get

$$\begin{gathered} \lambda_4 - \lambda_3 = (x_{4,3}, y_{4,3}, z_{4,3})^t \in B_1 \cup B_2, \\ x_{4,3} \in \frac{1}{2} + \mathbb{Z}, y_{4,3} \in \frac{1}{2} + \mathbb{Z}, \end{gathered} \tag{87}$$

combined with (85), (87), and $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_3)$, yields $x_{5,3} \in \mathbb{Z}$, $y_{5,3} \in \mathbb{Z}$, a contradiction with Lemma 3(b). So Case 4.1.1.3 is proved, and Case 4.1.1 is proved.

So the proof of Case 4 is completed.

Summing up the above discussion, we know that there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{B,D})$. We can find many such orthogonal systems which contain 4 elements, for example, the exponential function system $E(S)$ with S given by

$$S = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{p_1}{2} \\ \frac{p_1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{p_1}{2} \\ \frac{p_1}{2} \end{pmatrix}, \begin{pmatrix} \frac{p_1}{2} \\ 0 \\ \frac{p_1}{2} \end{pmatrix} \right\}. \tag{88}$$

This shows that the number 4 is the best. The proof of Theorem 1 is complete.

The proof of Theorem 2 is similar, it is so omitted.

Corollary 5. *For the self-affine measure $\mu_{B,D}$ corresponding to*

$$\begin{gathered} B = \begin{bmatrix} p & 0 & m \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \quad (p \in \mathbb{Z} \setminus \{0, \pm 1\}), \ m \in \mathbb{Z}, \\ D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \end{gathered} \tag{89}$$

if $p \in (2\mathbb{Z} + 1) \setminus \{\pm 1\}$, then $\mu_{B,D}$ is a nonspectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{B,D})$, where the number 4 is the best.

The corollary improved Yuan [9, Theorem 1].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

Acknowledgments

The author would like to thank the anonymous referees for their valuable suggestions. The present research is supported by the National Natural Science Foundation of China (no. 11171201).

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